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EXISTENCE OF PERIODIC SOLUTION FOR FOURTH-ORDER LIÉNARD TYPE P-LAPLACIAN GENERALIZED NEUTRAL DIFFERENTIAL EQUATION WITH VARIABLE PARAMETER*

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Abstract In this paper, we consider the following fourth-order Liénard type p-Laplacian generalized neutral differential equation with variable parameter

$$(\varphi_p(x(t)-c(t)x(t-\delta(t)))'')''+f(x(t))x'(t)+g(t,x(t),x(t-\tau(t)),x'(t))=e(t).$$

By applications of coincidence degree theory and some analysis skills, sufficient conditions for the existence of periodic solutions are established.

Keywords Periodic solution, p-Laplacian, fourth-order, neutral operator, liénard type.

MSC(2010) 34C25, 34K13, 34K40.

1. Introduction

In this paper, we consider the following fourth-order Liénard type p-Laplacian neutral differential equation with variable parameter

$$(\varphi_p(x(t)-c(t)x(t-\delta(t)))'')''+f(x(t))x'(t)+g(t,x(t),x(t-\tau(t)),x'(t))=e(t), \quad (1.1)$$

where $p \geq 2$, $\varphi_p(x)=|x|^{p-2}x$ for $x \neq 0$ and $\varphi_p(0)=0$; $|c(t)| \neq 1$, c, $\delta \in C^2(\mathbb{R},\mathbb{R})$ and c, δ are T-periodic functions for some $T>0$; f is continuous function; g is continuous function defined on $\mathbb{R}^4$ and periodic in t with $g(t,\cdot)=g(t+T,\cdot,\cdot,\cdot)$, e, $\tau:\mathbb{R}\to\mathbb{R}$ are continuous periodic functions with $e(t+T)\equiv e(t)$, $\int_0^T e(t)dt=0$ and $\tau(t+T)\equiv\tau(t)$.

In recent years, there is a good amount of work on periodic solutions for Liénard type p-Laplacian differential equations (see [1–3,5–10,12,13,15,16] and the references cited therein.) For example, in [10], applying Mawhin's continuation theorem, Shan & Lu study the existence of periodic solution for a kind of fourth-order p-Laplacian functional differential equation with a deviating argument as follows

$$[\varphi_p(u''(t))]''+f(u(t))u'(t)+g(t,u(t),u(t-\tau(t)))=e(t). \quad (1.2)$$

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Afterwards, by means of Mawhin's continuation theorem, Wang & Zhu [13] study a kind of fourth-order p-Laplacian neutral functional differential equation

$$[\varphi_p(x(t)-cx(t-\delta))'']''+f(x(t))x'(t)+g(t,x(t-\tau(t,|x|_\infty)))=e(t). \tag{1.3}$$

Some sufficient criteria to guarantee the existence of periodic solutions are obtained.

However, the fourth-order p-Laplacian neutral differential equation (1.1), in which there are the p-Laplacian neutral differential equation, has not attracted much in the literature. There are not so many existence results for (1.1) even when the neutral operator with variable parameter. In this paper, we try to fill gap and establish the existence of periodic solution of (1.1) using some Mawhin's continuation theory. Our new results generalize in several aspects some recent results contained in [10, 13].

Here $A=x(t)-c(t)x(t-\delta(t))$ is a natural generalization of the operator $A_1 = x(t)-cx(t-\delta)$, which typically possesses a more complicated nonlinearity than A_1. For example, A_1 is homogeneous in the following sense $(A_1x)'(t)=(A_1x')(t)$, whereas A in general is inhomogeneous. As a consequence many of the new results for differential equations with the neutral operator A will not be a direct extension of known theorems for neutral differential equations.

The remaining part of the paper is organized as follows, in Section 2, we first give qualitative properties of the neutral operator A which will be helpful for further studies of differential equations with this neutral operator; in Section 3, by applying Mawhin's continuation theory and some new inequalities, we obtain sufficient conditions for the existence of periodic solutions for (1.1); in Section 4, some examples are also given to illustrate our results.

2. Preparation

Let

$$c_\infty=\max_{t\in[0,T]}|c(t)|, \qquad c_0=\min_{t\in[0,T]}|c(t)|.$$

Define operators $A: C_T\to C_T$ by

$$(Ax)(t)=x(t)-c(t)x(t-\delta(t)).$$

Lemma 2.1 (see [14]). *If $|c(t)|\neq 1$, then the operator A has a continuous inverse A^{-1} on C_T, satisfying*

(1) $$\left(A^{-1}f\right)(t)=\begin{cases} f(t)+\sum\limits_{j=1}^{\infty}\prod\limits_{i=1}^{j}c(D_i)x\Big(t-\sum\limits_{i=1}^{j}\delta(D_i)\Big), \text{for } |c(t)|<1,\ \forall\ f\in C_T,\\ -\frac{f(t+\delta(t))}{c(t+\delta(t))}-\sum\limits_{j=1}^{\infty}\frac{f\Big(t+\delta(t)+\sum\limits_{i=1}^{j}\delta(D_i')\Big)}{c(t+\delta(t))\prod\limits_{i=1}^{j}c(D_i')}, \ \text{for } |c(t)|>1,\ \forall\ f\in C_T. \end{cases}$$

(2) $$\left|\left(A^{-1}f\right)(t)\right|\le\begin{cases}\frac{|f|_\infty}{1-c_\infty}, & \text{for } \ c_\infty<1\ \ \forall\ f\in C_T,\\ \frac{|f|_\infty}{c_0-1}, & \text{for }\ c_0>1\ \ \forall\ f\in C_T.\end{cases}$$

(3) $$\int_0^T\left|\left(A^{-1}f\right)(t)\right|dt\le\begin{cases}\frac{1}{1-c_\infty}\int_0^T|f(t)|dt, & \text{for }\ c_\infty<1\ \ \forall\ f\in C_T,\\ \frac{1}{c_0-1}\int_0^T|f(t)|dt, & \text{for }\ c_0>1\ \ \forall\ f\in C_T.\end{cases}$$

where $D_1 = t$ *and* $D_j = t - \sum_{i=1}^{j} \delta(D_i)$, $j = 1, 2, \ldots$; $D'_1 = t$, $D'_j = t + \sum_{i=1}^{j} \delta(D'_i)$, $j = 1, 2, \cdots$.

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $Im\ L$ is closed in Y and $\dim Ker\ L = \dim(Y/Im\ L) < +\infty$. Consider supplementary subspaces X_1, Y_1 of X, Y respectively, such that $X = Ker\ L \oplus X_1$, $Y = Im\ L \oplus Y_1$. Let $P : X \to Ker\ L$ and $Q : Y \to Y_1$ denote the natural projections. Clearly, $Ker\ L \cap (D(L) \cap X_1) = \{0\}$ and so the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Let K denote the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.2 (Gaines and Mawhin [4])**.** *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(1) $Lx \neq \lambda N x, \forall\ x \in \partial\Omega \cap D(L),\ \lambda \in (0, 1)$*;*

(2) $Nx \notin Im\ L, \forall\ x \in \partial\Omega \cap Ker\ L$*;*

(3) $\deg\{JQN, \Omega \cap Ker\ L, 0\} \neq 0$*, where $J : Im\ Q \to Ker\ L$ is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

In order to apply Mawhin's continuation degree theorem to study the existence of periodic solution for (1.1), we rewrite (1.1) in the form:

$$\begin{cases} (Ax_1)''(t) = \varphi_q(x_2(t)), \\ x_2''(t) = -f(x_1(t))x_1'(t) - g(t, x_1(t), x_1(t - \tau(t)), x_1'(t)) + e(t), \end{cases} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is an T-periodic solution to (2.1), then $x_1(t)$ must be an T-periodic solution to (1.1). Thus, the problem of finding an T-periodic solution for (1.1) reduces to finding one for (2.1).

Now, Set $X = \{x = (x_1(t), x_2(t)) \in C^2(\mathbb{R}, \mathbb{R}^2) : x(t + T) \equiv x(t)\}$ with the norm $|x|_\infty = \max\{|x_1|_\infty, |x_2|_\infty\}$; $Y = \{x = (x_1(t), x_2(t)) \in C^2(\mathbb{R}, \mathbb{R}^2) : x(t + T) \equiv x(t)\}$ with the norm $||x|| = \max\{|x|_\infty, |x'|_\infty\}$. Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) = \{x \in C^2(\mathbb{R}, \mathbb{R}^2) :\ x(t + T) = x(t), t \in \mathbb{R}\} \subset X \to Y$$

by

$$(Lx)(t) = \begin{pmatrix} (Ax_1)''(t) \\ x_2''(t) \end{pmatrix}$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1(t))x_1'(t) - g(t, x_1(t), x_1(t - \tau(t)), x_1'(t)) + e(t) \end{pmatrix}. \tag{2.2}$$

Then (2.1) can be converted to the abstract equation $Lx = Nx$. From the definition of L, one can easily see that

$$Ker\ L \cong \mathbb{R}^2, \qquad Im\ L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P : X \to Ker\ L$ and $Q : Y \to Im\ Q \subset \mathbb{R}^2$ be defined by

$$Px = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix}, \qquad Qy = \frac{1}{T} \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

then $Im\ P$=$Ker\ L$, $Ker\ Q$=$Im\ L$. let K denote the inverse of $L|_{Kerp \cap D(L)}$. It is easy to see that $Ker\ L$ =Im$Q = \mathbb{R}^2$ and

$$[Ky](t) = \int_0^T G(t,s) y(s) ds,$$

where

$$G(t,s) = \begin{cases} \frac{-s(T-t)}{T}, & 0 \le s \le t \le T. \\ \frac{-t(T-s)}{T}, & 0 \le t < s \le T. \end{cases} \tag{2.3}$$

From (2.2) and (2.3), it is clearly that QN and $K(I-Q)N$ are continuous, $QN(\overline{\Omega})$ is bounded and then $K(I-Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$ which means N is L-compact on $\bar{\Omega}$.

3. Main results

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

(H_1) There exists a constant $D > 0$ such that

$$v_1 g(t, v_1, v_2, v_3) > 0 \ \ \forall\ (t, v_1, v_2, v_3) \in [0,T] \times \mathbb{R}^3 \ \text{ with } \ |v_1| > D.$$

(H_2) There exist positive constants a, b such that $|f(x)| \le a|x|^{p-2} + b, \ \ x \in \mathbb{R}$.

(H_3) Tthere exist positive constants α_1, α_2 such that $|f(x)| \le \alpha_1 |x|^{p-1} + \alpha_2$, $x \in \mathbb{R}$.

(H_4) There exist positive constants β_1, β_2, β_3, m such that $|g(t, v_1, v_2, v_3)| \le \beta_1 |v_1|^{p-1} + \beta_2 |v_2|^{p-1} + \beta_3 |v_3|^{p-1} + m$, for $(t, v_1, v_2, v_3) \in [0,T] \times \mathbb{R}^3$.

Theorem 3.1. *Assume that conditions (H_1)-(H_2), (H_4) hold. Suppose the following one of conditions is satisfied*

(i) If $c_\infty < 1$ and

$$0 < \frac{2^{p-1}(\beta_1 + \beta_2) T^{2p} + (4\pi)^{p-1} T^{p+1} \beta_3 + 2^p a T^{2p-1}}{2^{3p-1} \pi^{p-1} \left(1 - c_\infty - \left(\frac{T^2}{4\pi} c_2 + T c_1 + T c_1 \delta_1 + \frac{T}{2} c_\infty \delta_2 + c_\infty \delta_1^2 + 2 c_\infty \delta_1\right)\right)^{p-1}} < 1,$$

(ii) If $c_0 > 1$ and

$$0 < \frac{2^{p-1}(\beta_1 + \beta_2) T^{2p} + (4\pi)^{p-1} T^{p+1} \beta_3 + 2^p a T^{2p-1}}{2^{3p-1} \pi^{p-1} \left(c_0 - 1 - \left(\frac{T^2}{4\pi} c_2 + T c_1 + T c_1 \delta_1 + \frac{T}{2} c_\infty \delta_2 + c_\infty \delta_1^2 + 2 c_\infty \delta_1\right)\right)^{p-1}} < 1,$$

where $\delta_i = \max\limits_{t\in[0,\omega]} |\delta^{(i)}(t)|,\ c_i = \max\limits_{t\in[0,\omega]} |c^{(i)}(t)|,\ i = 1, 2.$

Then (1.1) *has at least non-constant* T*-periodic solution.*

Proof. Consider the equation

$$Lx = \lambda N x, \quad \lambda \in (0,1).$$

Set $\Omega_1 = \{x : Lx = \lambda Nx, \lambda \in (0,1)\}$. If $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} (Ax_1)''(t) = \lambda\varphi_q(x_2(t)), \\ x_2''(t) = -\lambda f(x_1(t))x_1'(t) - \lambda g(t, x_1(t), x_1(t-\tau(t)), x_1'(t)) + \lambda e(t). \end{cases} \tag{3.1}$$

Substituting $x_2(t) = \lambda^{1-p}\varphi_p[(Ax_1)''(t)]$ into the the second equation of (3.1)

$$(\varphi_p(Ax_1)''(t))'' + \lambda^p f(x_1(t))x_1'(t) + \lambda^p g(t, x_1(t), x_1(t-\tau(t)), x_1'(t)) = \lambda^p e(t). \tag{3.2}$$

Integrating both side of (3.2) over $[0, T]$, we have

$$\int_0^T g(t, x_1(t), x_1(t-\tau(t)), x_1'(t))dt = 0.$$

From the integral mean value theorem, there is a constant $\xi \in [0, T]$ such that

$$g(\xi, x_1(\xi), x_1(\xi - \tau(\xi)), x_1'(\xi)) = 0.$$

In view of (H_1), we obtain

$$|x_1(\xi)| \le D.$$

Then, we have

$$|x_1(t)| = \left| x_1(\xi) + \int_\xi^t x_1'(s)ds \right| \le D + \int_\xi^t |x_1'(s)|ds, \qquad t \in [\xi, \xi+T],$$

and

$$|x_1(t)| = |x_1(t-T)| = \left| x_1(\xi) - \int_{t-T}^\xi x_1'(s)ds \right| \le D + \int_{t-T}^\xi |x_1'(s)|ds, \quad t \in [\xi, \xi+T].$$

Combing the above two inequalities, we obtain

$$\begin{aligned} |x_1|_\infty &= \max_{t\in[0,T]} |x_1(t)| = \max_{t\in[\xi,\xi+T]} |x_1(t)| \\ &\le \max_{t\in[\xi,\xi+T]} \left\{ D + \frac{1}{2}\left(\int_\xi^t |x_1'(s)|ds + \int_{t-T}^\xi |x_1'(s)|ds \right) \right\} \\ &\le D + \frac{1}{2}\int_0^T |x_1'(s)|ds. \end{aligned} \tag{3.3}$$

Since $(Ax_1)(t) = x_1(t) - c(t)x_1(t-\delta(t))$, we have

$$\begin{aligned}(Ax_1)'(t) &= (x_1(t) - c(t)x_1(t-\delta(t)))' \\ &= x_1'(t) - c'(t)x_1(t-\delta(t)) - c(t)x_1'(t-\delta(t)) + c(t)x_1'(t-\delta(t))\delta'(t). \\ (Ax_1)''(t) &= (x_1'(t) - c'(t)x_1(t-\delta(t)) - c(t)x_1'(t-\delta(t)) + c(t)x_1'(t-\delta(t))\delta'(t))' \\ &= x_1''(t) - [c''(t)x(t-\delta(t)) + c'(t)x'(t-\delta(t))(1-\delta'(t)) \\ &\quad + c'(t)x'(t-\delta(t)) + c(t)x''(t-\delta(t))\cdot(1-\delta'(t)) - c'(t)x'(t-\delta(t))\delta'(t) \\ &\quad - c(t)x''(t-\delta(t))(1-\delta'(t))\delta'(t) - c(t)x'(t-\delta(t))\delta''(t)] \\ &= x_1''(t) - c(t)x_1''(t-\delta(t)) - [c''(t)x(t-\delta(t)) + (2c'(t) - 2c'(t)\delta'(t) \\ &\quad - c(t)\delta''(t))x_1'(t-\delta(t)) + (c(t)(\delta'(t))^2 - 2c(t)\delta'(t))x_1''(t-\delta(t))]\end{aligned}$$

and

$$\begin{aligned}(Ax_1'')(t) =&(Ax_1)''(t) + c''(t)x(t-\delta(t)) + (2c'(t) - 2c'(t)\delta'(t) \\ &- c(t)\delta''(t))x_1'(t-\delta(t)) + (c(t)(\delta'(t))^2 - 2c(t)\delta'(t))x_1''(t-\delta(t)).\end{aligned} \tag{3.4}$$

Case (I): If $|c(t)| \le c_\infty < 1$, by applying Lemma 2.1, we have

$$\begin{aligned}|x_1''|_\infty &= \max_{t\in[0,T]} \left|A^{-1}Ax_1''(t)\right| \le \frac{\max\limits_{t\in[0,T]} |Ax_1''(t)|}{1-c_\infty} \\ &\le \frac{\varphi_q(|x_2|_\infty) + c_2|x_1|_\infty + (2c_1 + 2c_1\delta_1 + c_\infty\delta_2)|x_1'|_\infty + (c_\infty\delta_1^2 + 2c_\infty\delta_1)|x_1''|_\infty}{1-c_\infty}.\end{aligned}$$

From (3.3) and Wirtinger inequality ([11]), we have

$$\begin{aligned}|x_1|_\infty \le& D + \frac{1}{2}\int_0^T |x_1'(t)|dt \le D + \frac{T^{\frac{1}{2}}}{2}\left(\int_0^T |x_1'(t)|^2 dt\right)^{\frac{1}{2}} \\ \le& D + \frac{T^{\frac{1}{2}}}{2}\frac{T}{2\pi}\left(\int_0^T |x_1''(t)|^2 dt\right)^{\frac{1}{2}} \le D + \frac{T^2}{4\pi}|x_1''|_\infty.\end{aligned} \tag{3.5}$$

From $x_1(0) = x_1(T)$, there exists a point $t_1 \in [0,T]$ such that $x_1'(t_1) = 0$, then we have

$$|x_1'|_\infty \le x_1'(t_1) + \frac{1}{2}\int_0^T |x_1''(t)|dt \le \frac{T}{2}|x_1''|_\infty. \tag{3.6}$$

Therefore, we have

$$\begin{aligned}&|x_1''|_\infty \\ \le&\Big(\varphi_q(|x_2|_\infty) + c_2(D + \frac{T^2}{4\pi}|x_1''|_\infty) + \frac{T}{2}(2c_1 + 2c_1\delta_1 + c_\infty\delta_2)|x_1''|_\infty + (c_\infty\delta_1^2 \\ &+ 2c_\infty\delta_1)|x_1''|_\infty\Big)\Big/\Big(1-c_\infty\Big) \\ \le&\frac{\varphi_q(|x_2|_\infty) + (\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1)|x_1''|_\infty + c_2D}{1-c_\infty}.\end{aligned}$$

Since $1-c_\infty-\Big(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\Big) > 0$, so, we have

$$|x_1''|_\infty \le \frac{\varphi_q(|x_2|_\infty) + c_2D}{1-c_\infty-\big(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\big)}. \tag{3.7}$$

On the other hand, from $x_2(0)=x_2(T)$, there exists a point $t_2\in[0,T]$ such that $x_2'(t_2)=0$, which together with the integration of the second (3.1) on interval $[0,T]$ gives

$$\begin{aligned}
&2\,|x_2'(t)|\le 2\left(x_2'(t_2)+\frac{1}{2}\int_0^T|x_2''(t)|\,dt\right)\\
=&\lambda\int_0^T|-f(x_1(t))x_1'(t)-g(t,x_1(t),x_1(t-\tau(t)),x_1'(t))+e(t)|\,dt \qquad (3.8)\\
\le&\int_0^T|f(x_1(t))||x_1'(t)|dt+\int_0^T|g(t,x_1(t),x_1(t-\tau(t)),x_1'(t))|dt+\int_0^T|e(t)|dt.
\end{aligned}$$

From (H_2) and (H_4), we have

$$\begin{aligned}
2\,|x_2'(t)|\le{}& a\int_0^T|x_1(t)|^{p-2}|x_1'(t)|dt+b\int_0^T|x_1'(t)|dt+\beta_1\int_0^T|x_1(t)|^{p-1}dt\\
&+\beta_2\int_0^T|x_1(t-\tau(t))|^{p-1}dt+\beta_3\int_0^T|x_1'(t)|^{p-1}dt+(m+|e|_\infty)T \qquad (3.9)\\
\le{}& a|x_1|_\infty^{p-2}\int_0^T|x_1'(t)|dt+b\int_0^T|x_1'(t)|dt+(\beta_1+\beta_2)T|x_1|_\infty\\
&+\beta_3T|x_1'|_\infty^{p-1}+(m+|e|_\infty)T.
\end{aligned}$$

From Wirtinger inequality, we have

$$\int_0^T|x_1'(t)|dt\le T^{\frac{1}{2}}\left(\int_0^T|x_1'(t)|^2dt\right)^{\frac{1}{2}}\le T^{\frac{1}{2}}\frac{T}{2\pi}\left(\int_0^T|x_1''(t)|^2dt\right)^{\frac{1}{2}}\le\frac{T^2}{2\pi}|x_1''|_\infty. \qquad (3.10)$$

Substituting (3.5), (3.6) and (3.10) into (3.9)

$$\begin{aligned}
&2\,|x_2'(t)|\le(\beta_1+\beta_2)T\left(D+\frac{T^2}{4\pi}|x_1''|_\infty\right)^{p-1}+\beta_3\frac{T^p}{2^{p-1}}|x_1''|_\infty^{p-1}\\
&+a\left(D+\frac{T^2}{4\pi}|x_1''|_\infty\right)^{p-2}\left(\frac{T^2}{2\pi}\right)|x_1''|_\infty+b\left(\frac{T^2}{2\pi}\right)|x_1''|_\infty+(m+|e|_\infty)\,T\\
=&(\beta_1+\beta_2)T\left(\frac{T^2}{4\pi}\right)^{p-1}\left(1+\frac{4\pi D}{T^2|x_1''|_\infty}\right)^{p-1}|x_1''|_\infty^{p-1}+\beta_3\frac{T^p}{2^{p-1}}|x_1''|_\infty^{p-1} \qquad (3.11)\\
&+a\left(\frac{T^2}{4\pi}\right)^{p-2}\left(\frac{T^2}{2\pi}\right)\left(1+\frac{4\pi D}{T^2|x_1''|_\infty}\right)^{p-2}|x_1''|_\infty^{p-1}+b\frac{T^2}{2\pi}|x_1''|_\infty+(m+|e|_\infty)T.
\end{aligned}$$

For constant $\delta>0$, which is only dependent on $k>0$, we have

$$(1+x)^k\le1+(1+k)x\quad\text{ for }\ x\in[0,\delta].$$

From (3.7) and (3.11), we have

$$\begin{aligned}
2\,|x_2'(t)|\le{}&(\beta_1+\beta_2)T\left(\frac{T^2}{4\pi}\right)^{p-1}\left(1+\frac{4\pi Dp}{T^2|x_1''|_\infty}\right)|x_1''|_\infty^{p-1}+\beta_3\frac{T^p}{2^{p-1}}|x_1''|_\infty^{p-1}\\
&+a\left(\frac{T^2}{4\pi}\right)^{p-2}\left(\frac{T^2}{2\pi}\right)\left(1+\frac{4\pi D(p-1)}{T^2|x_1''|_\infty}\right)|x_1''|_\infty^{p-1}+b\frac{T^2}{2\pi}|x_1'|_\infty+(m+|e|_\infty)T
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{(\beta_1+\beta_2)T^{2p-1}}{(4\pi)^{p-1}} + \beta_3 \frac{T^p}{2^{p-1}} + \frac{2aT^{2p-2}}{(4\pi)^{p-1}} \right) |x_1''|_\infty^{p-1} + \left(\frac{(\beta_1+\beta_2)T^{2p-3}Dp}{(4\pi)^{p-2}} \right. \\
&\quad \left. + aD(p-1) \left(\frac{T^2}{2\pi} \right) \left(\frac{T^2}{4\pi} \right)^{p-3} \right) |x_1''|_\infty^{p-2} + b\frac{T^2}{2\pi}|x_1''|_\infty + (m+|e|_\infty)T \\
&\le \left(\frac{(\beta_1+\beta_2)T^{2p-1}}{(4\pi)^{p-1}} + \beta_3 \frac{T^p}{2^{p-1}} + \frac{2aT^{2p-2}}{(4\pi)^{p-1}} \right) \\
&\quad \cdot \frac{(\varphi_q(|x_2|_\infty)+c_2D)^{p-1}}{\left(1-c_\infty - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)\right)^{p-1}} \\
&\quad + \left(\frac{(\beta_1+\beta_2)T^{2p-3}Dp}{(4\pi)^{p-2}} + aD(p-1)\left(\frac{T^2}{2\pi}\right)\left(\frac{T^2}{4\pi}\right)^{p-3} \right) \\
&\quad \cdot \frac{(\varphi_q(|x_2|_\infty)+c_2D)^{p-2}}{\left(1-c_\infty - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)\right)^{p-2}} \\
&\quad + b\frac{T^2}{2\pi} \frac{\varphi_q(|x_2|_\infty)+c_2D}{1-c_\infty - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)} + (m+|e|_\infty)T.
\end{aligned}
\tag{3.12}
$$

Since $\int_0^T \varphi_p(x_2(t))dt = \int_0^T (Ax_1)''(t)dt = 0$, then there exists a point $t_3 \in [0,T]$ such that $x_2(t_3) = 0$. So, we have

$$
|x_2|_\infty \le \frac{1}{2}\int_0^T |x_2'(t)|dt \le \frac{T}{2}|x_2'|_\infty. \tag{3.13}
$$

Combination of (3.12) and (3.13) implies

$$
\begin{aligned}
|x_2|_\infty \le \frac{T}{4} \Bigg[& \left(\frac{(\beta_1+\beta_2)T^{2p-1}}{(4\pi)^{p-1}} + \beta_3 \frac{T^p}{2^{p-1}} + \frac{2aT^{2p-2}}{(4\pi)^{p-1}} \right) \\
&\cdot \frac{(\varphi_q(|x_2|_\infty)+c_2D)^{p-1}}{\left(1-c_\infty - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)\right)^{p-1}} \\
&+ \left(\frac{(\beta_1+\beta_2)T^{2p-3}Dp}{(4\pi)^{p-2}} + aD(p-1)\left(\frac{T^2}{2\pi}\right)\left(\frac{T^2}{4\pi}\right)^{p-3} \right) \\
&\cdot \frac{(\varphi_q(|x_2|_\infty)+c_2D)^{p-2}}{\left(1-c_\infty - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)\right)^{p-2}} \\
&+ b\frac{T^2}{2\pi} \frac{\varphi_q(|x_2|_\infty)+c_2D}{1-c_\infty - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)} + (m+|e|_\infty)T \Bigg] \\
= \frac{T}{4} \Bigg[& \left(\frac{(\beta_1+\beta_2)T^{2p-1}}{(4\pi)^{p-1}} + \beta_3 \frac{T^p}{2^{p-1}} + \frac{2aT^{2p-2}}{(4\pi)^{p-1}} \right) \\
&\cdot \frac{|x_2|_\infty + (p-1)|x_2|_\infty^{2-q}c_2D + \cdots + (c_2D)^{p-1}}{\left(1-c_\infty - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)\right)^{p-1}} \\
&+ \left(\frac{(\beta_1+\beta_2)T^{2p-3}Dp}{(4\pi)^{p-2}} + aD(p-1)\left(\frac{T^2}{2\pi}\right)\left(\frac{T^2}{4\pi}\right)^{p-3} \right) \\
&\cdot \frac{(\varphi_q(|x_2|_\infty)+c_2D)^{p-2}}{\left(1-c_\infty - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)\right)^{p-2}}
\end{aligned}
$$

$$+ b\frac{T^2}{2\pi}\frac{\varphi_q(|x_2|_\infty) + c_2 D}{1 - c_\infty - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)}$$
$$+ (m + |e|_\infty)T\Bigg].$$

Since $p \geq 2$ and $\frac{2^{p-1}(\beta_1+\beta_2)T^{2p}+(4\pi)^{p-1}T^{p+1}\beta_3+2^p aT^{2p-1}}{2^{3p-1}\pi^{p-1}\left(1-c_\infty-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\delta_1+\frac{T}{2}c_\infty\delta_2+c_\infty\delta_1^2+2c_\infty\delta_1\right)\right)^{p-1}} < 1$, then there exists a positive constant M_1 (independent of λ) such that

$$|x_2|_\infty \leq M_1. \tag{3.14}$$

Case (ii): If $c_0 > 1$, we have

$$|x_1''|_\infty = \max_{t\in[0,T]} \left|A^{-1}Ax_1''(t)\right| \leq \frac{\max_{t\in[0,T]}|Ax_1''(t)|}{c_0 - 1}$$
$$\leq \frac{\varphi_q(|x_2|_\infty) + \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)|x_1''|_\infty + c_2 D}{c_0 - 1}.$$

Since $c_0 - 1 - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right) > 0$, so, we have

$$|x_1''|_\infty \leq \frac{\varphi_q(|x_2|_\infty) + c_2 D}{c_0 - 1 - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\delta_1 + \frac{T}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)}.$$

Similarly, we can get $|x_2|_\infty \leq M_1$.

From (3.7) and (3.14), we obtain that

$$|x_1''|_\infty \leq \frac{\varphi_q(|x_2|_\infty) + c_2 D}{1 - c_\infty - \left(\frac{T^2}{4\pi}c_2 + \sqrt{T}c_1 + \sqrt{T}c_1\delta_1 + \frac{\sqrt{T}}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)}$$
$$\leq \frac{M_1^{q-1} + c_2 D}{1 - c_\infty - \left(\frac{T^2}{4\pi}c_2 + \sqrt{T}c_1 + \sqrt{T}c_1\delta_1 + \frac{\sqrt{T}}{2}c_\infty\delta_2 + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)} := M_2^*.$$

It follows from (3.5) that

$$|x_1|_\infty \leq D + \frac{T^2}{4\pi}|x_1''|_\infty \leq D + \frac{T^2}{4\pi}M_2^* := M_3,$$

by (3.6)

$$|x_1'|_\infty \leq \frac{T}{2}|x_1''|_\infty \leq \frac{T}{2}M_2^* := M_2.$$

From (3.8), (H_2) and (H_4), we know

$$|x_2'|_\infty \leq \frac{1}{2}\max\left|\int_0^T x_2''(t)dt\right|$$
$$\leq \frac{1}{2}\int_0^T |-f(x_1(t))x_1'(t) - g(t, x_1(t), x_1(t-\tau(t)), x_1'(t)) + e(t)|dt$$
$$\leq \frac{1}{2}\left[aT|x_1|_\infty^{p-1}|x_1'|_\infty + bT|x_1'|_\infty + (\beta_1+\beta_2)T|x_1|_\infty^{p-1} + \beta_3 T|x_1'|_\infty^{p-1} + (m + |e|_\infty)T\right]$$
$$\leq \frac{1}{2}\left[aTM_3^{p-1}M_2 + bTM_2 + (\beta_1+\beta_2)TM_3^{p-1} + \beta_3 TM_2^{p-1} + (m + |e|_\infty)T\right] := M_4.$$

Let $M = \max\{M_1, M_2, M_3, M_4\} + 1$, $\Omega = \{x = (x_1, x_2)^\top : ||x|| < M\}$ and $\Omega_2 = \{x : x \in \partial\Omega \cap Ker\ L\}$ then $\forall\ x \in \partial\Omega \cap Ker\ L$

$$QNx = \frac{1}{T}\int_0^T \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(x_1(t))x_1'(t) - g(t, x_1(t), x_1(t-\tau(t)), x_1'(t)) + e(t) \end{pmatrix} dt.$$

If $QNx = 0$, then $x_2(t) = 0$, $\ x_1 = M$ or $-M$. But if $x_1(t) = M$, we know

$$0 = \int_0^T \{g(t, M, M, 0) - e(t)\}dt.$$

From assumption (H_1), we have $x_1(t) \leq D \leq M$, which yields a contradiction. Similarly if $x_1 = -M$. We also have $QNx \neq 0$, i.e., $\forall\ x \in \partial\Omega \cap Ker\ L,\ x \notin Im\ L$, so conditions (1) and (2) of Lemma 2.2 are both satisfied. Define the isomorphism $J : Im\ Q \to Ker\ L$ as follows:

$$J(x_1, x_2)^\top = (x_2, -x_1)^\top.$$

Let $H(\mu, x) = -\mu x + (1-\mu)JQNx$, $\ (\mu, x) \in [0,1] \times \Omega$, then $\forall\ (\mu, x) \in (0,1) \times (\partial\Omega \cap Ker\ L)$,

$$H(\mu, x) = \begin{pmatrix} -\mu x_1(t) - \frac{1-\mu}{T}\int_0^T [g(t, x_1, x_1, 0) - e(t)]dt \\ -\mu x_2(t) - (1-\mu)\varphi_q(x_2(t)) \end{pmatrix}.$$

We have $\int_0^T e(t)dt = 0$. So, we can get

$$H(\mu, x) = \begin{pmatrix} -\mu x_1(t) - \frac{1-\mu}{T}\int_0^T g(t, x_1, x_1, 0)dt \\ -\mu x_2(t) - (1-\mu))\varphi_q(x_2(t)) \end{pmatrix},$$

$$\forall\ (\mu, x) \in (0,1) \times (\partial\Omega \cap Ker\ L).$$

From (H_1), it is obvious that $x^\top H(\mu, x) < 0$, $\forall\ (\mu, x) \in (0,1) \times (\partial\Omega \cap Ker\ L)$. Hence

$$\begin{aligned} \deg\{JQN, \Omega \cap Ker\ L, 0\} &= \deg\{H(0, x), \Omega \cap Ker\ L, 0\} \\ &= \deg\{H(1, x), \Omega \cap Ker\ L, 0\} \\ &= \deg\{I, \Omega \cap Ker\ L, 0\} \neq 0. \end{aligned}$$

So condition (3) of Lemma 2.2 is satisfied. By applying Lemma 2.2, we conclude that equation $Lx = Nx$ has a solution $x = (x_1, x_2)^\top$ on $\bar{\Omega} \cap D(L)$, i.e., (2.1) has an T-periodic solution $x_1(t)$.

Finally, observe that $y_1^*(t)$ is not constant. For if $y_1^* \equiv a$ (constant), then from (1.1) we have $g(t, a, a, 0) - e(t) \equiv 0$, which contradicts to the assumption that $g(t, a, a, 0) - e(t) \not\equiv 0$. The proof is complete. $\square$

Theorem 3.2. *Assume that conditions (H_1), (H_3), (H_4) hold. Suppose the following one of conditions is satisfied*

(i) If $c_\infty < 1$ and $0 < \frac{T^{\frac{1}{q}}\left((c_\infty\alpha_1+(1+c_\infty)(\beta_1+\beta_2))T^{p+1}+2^{p-1}T^2(1+c_\infty)\beta_3\right)^{\frac{1}{p}}}{2^2\left(1-c_\infty-\left(\frac{T^2}{4}c_2+T(c_1+c_1\delta_1+\frac{1}{2}c_\infty\delta_2)+c_\infty\delta_1^2+2c_\infty\delta_1\right)\right)} < 1$;

(ii) If $c_0 > 1$ and $0 < \frac{T^{\frac{1}{q}}\left((c_\infty\alpha_1+(1+c_\infty)(\beta_1+\beta_2))T^{p+1}+2^{p-1}T^2(1+c_\infty)\beta_3\right)^{\frac{1}{p}}}{2^2\left(c_0-1-\left(\frac{T^2}{4}c_2+T(c_1+c_1\delta_1+\frac{1}{2}c_\infty\delta_2)+c_\infty\delta_1^2+2c_\infty\delta_1\right)\right)} < 1$.

Then (1.1) has at least non-constant T-periodic solution.

Proof. Let Ω_1 be defined as in Theorem 3.1. The proof of (3.3) is the same strategy and notations as the proof of Theorem 3.1.

Next, Multiplying the both sides of (3.2) by $(Ax_1)(t)$ and integrating over $[0,T]$, we get

$$\begin{aligned}\int_0^T(\varphi_p(Ax_1)''(t))''(Ax_1(t))dt =& -\lambda^p\int_0^T f(x_1(t))x_1'(t)(Ax_1)(t)dt\\ &-\lambda^p\int_0^T g(t,x_1(t),x_1(t-\tau(t)),x_1'(t))(Ax_1)(t)dt\\ &+\lambda^p\int_0^T e(t)(Ax_1)(t)dt. \end{aligned}\tag{3.15}$$

Substituting $\int_0^T(\varphi_p(Ax_1)''(t))''(Ax_1(t))dt=\int_0^T|(Ax_1)''(t)|^pdt$, $\int_0^T f(x_1(t))x_1'(t)x_1(t)dt=0$ into (3.15), in view of (H_3) and (H_4), we have

$$\begin{aligned}&\int_0^T|(Ax_1)''(t)|^pdt\\ \le&\int_0^T|f(x_1(t))||c(t)x_1(t-\delta(t))|dt+\int_0^T|g(t,x_1(t),x_1(t-\tau(t)),x_1(t))|\\ &\cdot|(Ax_1)(t)|dt+\int_0^T|e(t)||(Ax_1)(t)|dt \qquad (3.16)\\ \le& c_\infty|x_1|_\infty(\alpha_1\int_0^T|x_1(t)|^{p-1}dt+\alpha_2)+(1+c_\infty)|x_1|_\infty(\beta_1\int_0^T|x_1(t)|^{p-1}dt\\ &+\beta_2\int_0^T|x_1(t-\tau(t))|^{p-1}dt+\beta_3\int_0^T|x_1'(t)|^{p-1}dt+mT)+(1+c_\infty)T|e|_\infty|x_1|_\infty.\end{aligned}$$

Substituting (3.3) into (3.16), we have

$$\begin{aligned}&\int_0^T|(Ax_1)''(t)|^pdt\\ \le& c_\infty(D+\frac{1}{2}\int_0^T|x_1'(t)|dt)(\alpha_1\int_0^T|x_1(t)|^{p-1}dt+\alpha_2)\\ &+(1+c_\infty)(D+\frac{1}{2}\int_0^T|x_1'(t)|dt)(\beta_1\int_0^T|x_1(t)|^{p-1}dt+\beta_2\int_0^T|x_1(t-\tau(t))|^{p-1}dt\\ &+\beta_3\int_0^T|x_1'(t)|^{p-1}dt+mT)+(1+c_\infty)T|e|_\infty(D+\frac{1}{2}\int_0^T|x_1'(t)|dt)\\ =&\left(\frac{1}{2}c_\infty\alpha_1+\frac{1}{2}(1+c_\infty)\beta_1\right)\int_0^T|x_1(t)|^{p-1}dt\int_0^T|x_1'(t)|dt \qquad (3.17)\\ &+\frac{1}{2}(1+c_\infty)\beta_2\int_0^T|x_1(t-\tau)|^{p-1}dt\int_0^T|x_1'(t)|dt\\ &+\frac{1}{2}(1+c_\infty)\beta_3\int_0^T|x_1'(t)|^{p-1}dt\int_0^T|x_1'(t)|dt\\ &+(c_\infty D\alpha_1+(1+c_\infty)D\beta_1)\int_0^T|x_1(t)|^{p-1}dt\end{aligned}$$

$$
\begin{aligned}
&+ (1+c_\infty)D\beta_2 \int_0^T |x_1(t-\tau(t))|^{p-1}dt \\
&+ (1+c_\infty)D\beta_3 \int_0^T |x_1'(t)|^{p-1}dt + N_1 \int_0^T |x_1'(t)|dt + N_2 \\
\le & \frac{T^2}{2}(c_\infty\alpha_1 + (1+c_\infty)\beta_1 + (1+c_\infty)\beta_2)|x_1|_\infty^{p-1}|x_1'|_\infty + \frac{T^2}{2}(1+c_\infty)\beta_3|x_1'|_\infty^p \\
&+ N_3|x_1|_\infty^{p-1} + (1+c_\infty)D\beta_3 T|x_1'|_\infty^{p-1} + N_1T|x_1'|_\infty + N_2,
\end{aligned}
$$

where $N_1 = \frac{1}{2}T(c_\infty\alpha_2 + (1+c_\infty)m + (1+c_\infty)|e|_\infty)$, $N_2 = DT(c_\infty\alpha_2 + (1+c_\infty)m + (1+c_\infty)|e|_\infty)$ and $N_3 = (c_\infty\alpha_1 + (1+c_\infty)(\beta_1+\beta_2))DT$.

From $x_1(0) = x_1(T)$, there exists a point $t_4 \in [0,T]$ such that $x_1'(t_4) = 0$, then we have

$$
|x_1'|_\infty \le x_1'(t_3) + \frac{1}{2}\int_0^T |x_1''(t)|dt = \frac{1}{2}\int_0^T |x_1''(t)|dt. \tag{3.18}
$$

From (3.3), we have

$$
\begin{aligned}
|x_1|_\infty \le & D + \frac{1}{2}\int_0^T |x_1'(t)|dt \\
\le & D + \frac{T}{2}|x_1'|_\infty \\
\le & D + \frac{T}{4}\int_0^T |x_1''(t)|dt.
\end{aligned} \tag{3.19}
$$

Substituting (3.18) and (3.19) into (3.17), we have

$$
\begin{aligned}
&\int_0^T |(Ax_1)''(t)|^p dt \\
\le & \frac{T^2}{2}(c_\infty\alpha_1 + (1+c_\infty)(\beta_1+\beta_2))(D + \frac{T}{4}\int_0^T |x_1''(t)|dt)^{p-1}\frac{1}{2}\int_0^T |x_1''(t)|dt \\
&+ \frac{T^2}{2}(1+c_\infty)\beta_3\frac{1}{2^p}\Big(\int_0^T |x_1''(t)|dt\Big)^p \\
&+ N_3(D + \frac{T}{4}\int_0^T |x_1''(t)|dt)^{p-1} + N_4(\frac{1}{2}\int_0^T |x_1''(t)|dt)^{p-1} \\
&+ \frac{N_1T}{2}\int_0^T |x_1''(t)|dt + N_2 \\
= & \frac{T^2}{2}((c_\infty\alpha_1 + (1+c_\infty)(\beta_1+\beta_2)) \cdot \frac{T^{p-1}}{2^{2p-1}} + \frac{(1+c_\infty)\beta_3}{2^p})(\int_0^T |x_1''(t)|dt)^p \\
&+ \frac{T^2}{2}(c_\infty\alpha_1 + (1+c_\infty)(\beta_1+\beta_2)) \cdot \frac{T^{p-2}}{2^{2p-3}}D(p-1)(\int_0^T |x_1''(t)|dt)^{p-1} + \cdots \\
&+ \frac{T^2}{4}(c_\infty\alpha_1 + (1+c_\infty)(\beta_1+\beta_2))D^{p-1}\int_0^T |x_1''(t)|dt \\
&+ N_3(D + \frac{T}{4}\int_0^T |x_1''(t)|dt)^{p-1} + N_4(\frac{1}{2}\int_0^T |x_1''(t)|dt)^{p-1} \\
&+ \frac{N_1T}{2}\int_0^T |x_1''(t)|dt + N_2,
\end{aligned} \tag{3.20}
$$

where $N_4 = \frac{(1+c_\infty)D\beta_3 T}{2^{p-1}}$.

Case (I): If $|c(t)| \leq c_\infty < 1$, by applying Lemma 2.1 and (3.4), we have

$$\begin{aligned}
&\int_0^T |x_1''(t)|dt \\
&= \int_0^T \left|A^{-1}Ax_1''(t)\right| dt \leq \frac{\int_0^T |Ax_1''(t)|\, dt}{1-c_\infty} \\
&\leq \Big(\int_0^T |(Ax_1)''(t)|dt + c_2 T|x_1|_\infty + T(2c_1 + 2c_1\delta_1 + c_\infty\delta_2)|x_1'|_\infty \\
&\quad + (c_\infty\delta_1^2 + 2c_\infty\delta_1) \int_0^T |x''(t)|dt\Big)/(1-c_\infty).
\end{aligned}$$

From (3.18) and (3.19), we have

$$\begin{aligned}
&\int_0^T |x_1''(t)|dt \\
&\leq \frac{\int_0^T |(Ax_1)''(t)|dt + c_2 T\left(D + \frac{T}{4}\int_0^T |x_1''(t)|dt\right)}{1-c_\infty} \\
&\quad + \frac{\frac{T}{2}(2c_1 + 2c_1\delta_1 + c_\infty\delta_2)\int_0^T |x_1''(t)|dt + (c_\infty\delta_1^2 + 2c_\infty\delta_1)\int_0^T |x''(t)|dt}{1-c_\infty} \\
&= \Big(\int_0^T |(Ax_1)''(t)|dt + (\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1) \\
&\quad \cdot \int_0^T |x_1''(t)|dt + Tc_2 D\Big)/\big(1-c_\infty\big).
\end{aligned}$$

Since $1-c_\infty-\left(\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1\right) > 0$, so, we have

$$\begin{aligned}
\int_0^T |x_1''(t)|dt &\leq \frac{\int_0^T |(Ax_1)''(t)|dt + Tc_2 D}{1-c_\infty - \left(\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)} \\
&\leq \frac{T^{\frac{1}{q}}\left(\int_0^T |(Ax_1)''(t)|^p dt\right)^{\frac{1}{p}} + Tc_2 D}{1-c_\infty - \left(\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)}.
\end{aligned} \tag{3.21}$$

Applying the inequality $(a+b)^k \leq a^k + b^k$ for $a,\ b > 0,\ 0 < k < 1$, it follow from (3.20) and (3.21)

$$\begin{aligned}
&\int_0^T |x_1''(t)|dt \\
&\leq \frac{T^{\frac{1}{q}}\left[\frac{T^2}{2}\left((c_\infty\alpha_1 + (1+c_\infty)(\beta_1+\beta_2))\cdot\frac{T^{p-1}}{2^{2p-1}} + \frac{(1+c_\infty)\beta_3}{2^p}\right)\right]^{\frac{1}{p}}\int_0^T |x_1''(t)|dt}{1-c_\infty - \left(\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)} \\
&\quad + \frac{T^{\frac{1}{q}}\left[\frac{T^2}{2}(c_\infty\alpha_1 + (1+c_\infty)(\beta_1+\beta_2))\cdot\frac{T^{p-2}}{2^{2p-3}}D(p-1)\right]^{\frac{1}{p}}\left(\int_0^T |x_1''(t)|dt\right)^{\frac{p-1}{p}}}{1-c_\infty - \left(\frac{T^2}{4}c_2 + T(c_1 + c_1\delta_1 + \frac{1}{2}c_\infty\delta_2) + c_\infty\delta_1^2 + 2c_\infty\delta_1\right)}
\end{aligned}$$

$$+\cdots+\frac{T^{\frac{1}{q}}\left[\frac{T^2}{4}(c_\infty\alpha_1+(1+c_\infty)(\beta_1+\beta_2))D^{p-1}\right]^{\frac{1}{p}}\left(\int_0^T|x_1''(t)|dt\right)^{\frac{1}{p}}}{1-c_\infty-\left(\frac{T^2}{4}c_2+T(c_1+c_1\delta_1+\frac{1}{2}c_\infty\delta_2)+c_\infty\delta_1^2+2c_\infty\delta_1\right)}$$

$$+\frac{T^{\frac{1}{q}}N_3^{\frac{1}{p}}\left(D+\frac{T}{4}\int_0^T|x_1''(t)|dt\right)^{\frac{p-1}{p}}+T^{\frac{1}{q}}N_4^{\frac{1}{p}}\left(\frac{1}{2}\int_0^T|x_1''(t)|dt\right)^{\frac{p-1}{p}}}{1-c_\infty-\left(\frac{T^2}{4}c_2+T(c_1+c_1\delta_1+\frac{1}{2}c_\infty\delta_2)+c_\infty\delta_1^2+2c_\infty\delta_1\right)}$$

$$+\frac{\left(\frac{1}{2}N_1\right)^{\frac{1}{p}}T\left(\int_0^T|x_1''(t)|dt\right)^{\frac{1}{p}}+T^{\frac{1}{q}}N_2^{\frac{1}{p}}+c_2DT}{1-c_\infty-\left(\frac{T^2}{4}c_2+T(c_1+c_1\delta_1+\frac{1}{2}c_\infty\delta_2)+c_\infty\delta_1^2+2c_\infty\delta_1\right)}.$$

Since $\frac{T^{\frac{1}{q}}\left((c_\infty\alpha_1+(1+c_\infty)(\beta_1+\beta_2))T^{p+1}+2^{p-1}T^2(1+c_\infty)\beta_3\right)^{\frac{1}{p}}}{2^2\left(1-c_\infty-\left(\frac{T^2}{4}c_2+T(c_1+c_1\delta_1+\frac{1}{2}c_\infty\delta_2)+c_\infty\delta_1^2+2c_\infty\delta_1\right)\right)}<1$, then there exists a positive constant M^* (independent of λ) such that

$$\int_0^T|x_1''(t)|dt\le M^*.$$

It follow from (3.19) that

$$|x_1|_\infty\le D+\frac{T}{4}\int_0^T|x_1''(t)|dt\le D+\frac{T}{4}M^*:=M_3.$$

By (3.18)

$$|x_1'|_\infty\le\frac{1}{2}\int_0^T|x_1''(t)|dt\le\frac{1}{2}M^*:=M_2.$$

On the other hand, form $x_2(0)=x_2(T)$, we know that there is a point $t_5\in[0,T]$ such that $x_2'(t_5)=0$; then by the second equation of (3.1), (H_3) and (H_4)

$$\begin{aligned}&|x_2'|_\infty\le\frac{1}{2}\int_0^T|x_2''(t)|dt\\&\le\int_0^T(|f(x_1(t))||x_1'(t)|+|g(t,x_1(t),x_1(t-\tau(t))x_1'(t))|+|e(t)|)dt\\&\le\alpha_1TM_3^{p-1}M_2+\alpha_2TM_2+T((\beta_1+\beta_2)M_3^{p-1}+\beta_3M_2^{p-1})+T(m+|e|_\infty):=M_4.\end{aligned}$$

Integrating the first equation of over $[0,T]$, we have $\int_0^T|x_2(t)|^{q-2}x_2(t)dt=0$, which implies that there is a point $t_6\in[0,T]$ such that $x_2(t_6)=0$, so

$$|x_2|_\infty\le\frac{1}{2}\int_0^T|x_2'(t)|dt\le T|x_2'|_\infty\le TM_4:=M_1.$$

This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 3.1. □

If $c(t)\equiv c$ and $|c|\ne1$, $\delta(t)\equiv\delta$, then (1.1) translate into the follows form:

$$(\varphi_p(x(t)-cx(t-\delta))'')''+f(x(t))x'(t)+g(t,x(t),x(t-\tau(t))x'(t))=e(t).\quad(3.22)$$

Similarly, we can get the following result:

Theorem 3.3. *Assume that conditions (H_1), (H_2), (H_4) hold. Then* (3.22) *has at least non-constant T-periodic solution if*

$$\frac{2^{p-1}(\beta_1+\beta_2)T^{2p}+(4\pi)^{p-1}T^{p+1}\beta_3+2^p aT^{2p-1}}{2^{3p-1}\pi^{p-1}|1-|c||^{p-1}}<1.$$

Theorem 3.4. *Assume that conditions (H_1), (H_3), (H_4) hold. Then* (3.22) *has at least non-constant T-periodic solution if*

$$\frac{T^{\frac{1}{q}}\left((|c|\alpha_1+(1+|c|)(\beta_1+\beta_2))T^{p+1}+2^{p-1}T^2(1+|c|)\beta_3\right)^{\frac{1}{p}}}{2^2|1-|c||}<1.$$

4. Application

We illustrate our results with some examples.

Example 4.1. Consider the following fourth order Liénard type p-laplacian generality neutral functional differential

$$\left(\varphi_p\left(x(t)-\frac{1}{16}\sin(4t)x\left(t-\frac{1}{32}\cos 4t\right)\right)''\right)''+\frac{1}{4}x^2(t)x'(t)+\frac{1}{3\pi}x^3(t)$$
$$+\frac{1}{6\pi}\sin x(t-\cos 4t)+\frac{1}{8\pi}\cos 4t\sin x'(t)=\sin 4t. \tag{4.1}$$

Here $p=4$ is a constant. It is clear that $T=\frac{\pi}{2}$, $c(t)=\frac{1}{16}\sin 4t$, $\delta(t)=\frac{1}{32}\cos 4t$, $\tau(t)=\cos 4t$, $e(t)=\sin 4t$, $c_1=\max_{t\in[0,T]}|\frac{1}{4}\cos 4t|=\frac{1}{4}$, $c_2=\max_{t\in[0,T]}|-\sin 4t|=1$, $\delta_1=\max_{t\in[0,T]}|-\frac{1}{8}\sin 4t|=\frac{1}{8}$, $\delta_2=\max_{t\in[0,T]}|-\frac{1}{2}\cos 4t|=\frac{1}{2}$. $f(x)=\frac{1}{4}x^2$, take $a=\frac{1}{4}$, $b=1$ such that (H_2) holds; $g(t,v_1,v_2,v_3)=\frac{1}{3\pi}v_1^3+\frac{1}{6\pi}\sin v_2+\frac{1}{8\pi}\cos 4t\sin v_3$, and $g(t,a,a,0)-e(t)=\frac{1}{3\pi}a^3+\frac{1}{6\pi}\sin a-\sin 4t\not\equiv 0$. Choose $D=\frac{1}{3\pi}$ such that (H_1) holds. Now we consider the assumption (H_4), it is easy to see

$$|g(t,v_1,v_2,v_3)|\le\frac{1}{3\pi}|v_1|^3+1,$$

which mean (H_3) holds with $\beta_1=\frac{1}{3\pi}$, $\beta_2=0$, $\beta_3=0$, $m=1$. Obviously

$$\begin{aligned}&\frac{2^{p-1}(\beta_1+\beta_2)T^{2p}+(4\pi)^{p-1}T^{p+1}\beta_3+2^p aT^{2p-1}}{2^{3p-1}\pi^{p-1}\left(1-c_\infty-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\delta_1+\frac{T}{2}c_\infty\delta_2+c_\infty\delta_1^2+2c_\infty\delta_1\right)\right)^{p-1}}\\ =&\frac{2^3\times\frac{1}{3\pi}\times\left(\frac{\pi}{2}\right)^8+2^4\times\frac{1}{4}\times\left(\frac{\pi}{2}\right)^7}{2^11\times\pi^3\times\left(1-\frac{1}{16}-\left(\frac{\pi}{16}+\frac{\pi}{8}+\frac{\pi}{64}+\frac{\pi}{128}+\frac{1}{1024}+\frac{1}{64}\right)\right)^3}\\ \approx&\frac{5^3(\pi^5+3\pi^3)}{51\times 2^{13}}\approx 0.1790<1.\end{aligned}$$

So by Theorem 3.1, (4.1) has at least one nonconstant $\frac{\pi}{2}$ -periodic solution.

Example 4.2. Consider the following a kind of fourth order p-Laplacian neutral functional differential

$$\left(\varphi_p\left(x(t)-5x\left(t-\delta\right)\right)''\right)''+\frac{1}{5}x^4(t)x'(t)\qquad+\frac{1}{6\pi}x^4(t)+\frac{1}{8\pi}\cos x(t-\sin 2t)$$
$$+\frac{1}{10\pi}\sin 2t\cos x'(t)=\cos 2t. \tag{4.2}$$

Here $p = 5$ and δ is a constant. It is clear that $T = \pi, c = 5,\ \tau(t) = \sin 2t,\ e(t) = \cos 2t$. $f(x) = \frac{1}{5}x^4$, take $a = \frac{1}{5},\ b = 1$ such that (H_3) holds; $g(t, v_1, v_2, v_3) = \frac{1}{6\pi}v_1^4 + \frac{1}{8\pi}\cos v_2 + \frac{1}{10\pi}\sin 2t \cos v_3$, and $g(t, a, a, 0) - e(t) = \frac{1}{6\pi}a^4 + \frac{1}{8\pi}\cos a + \frac{1}{10\pi}\sin 2t - \cos 2t \not\equiv 0$. Choose $D = \frac{1}{6\pi}$ such that (H_1) holds. Now we consider the assumption (H_4), it is easy to see

$$|g(t, v_1, v_2, v_3)| \leq \frac{1}{6\pi}|v_1|^4 + 1,$$

which mean (H_3) holds with $\beta_1 = \frac{1}{3\pi},\ \beta_2 = 0,\ \beta_3 = 0,\ m = 1$. Obviously

$$\begin{aligned}&\frac{T^{\frac{1}{q}}\left((|c|\alpha_1 + (1+|c|)(\beta_1+\beta_2))T^{p+1} + 2^{p-1}T^2(1+|c|)\beta_3\right)^{\frac{1}{p}}}{2^2|1-|c||}\\ =\ &\frac{\pi^{\frac{5}{4}}(1+\pi^5)^{\frac{1}{5}}}{16} \approx 0.8217 < 1.\end{aligned}$$

So by Theorem 3.4, (4.2) has at least one nonconstant π -periodic solution.

References

[1] Z. Cheng and J. Ren, *Periodic solution for high-order differential system*, J. Appl. Anal. Comput., 3 (2013), 239–249.

[2] W. Cheung and J. Ren, *On the existence of periodic solutions for p-Laplacian generalized Liénard equation*, Nonlinear Anal. TMA, 60 (2005), 65–75.

[3] B. Du, L. Guo, W. Ge and S. Lu, *Periodic solutions for generalized Liénard neutral equation with variable parameter*, Nonlinear Anal. TMA, 70 (2009), 2387–2394.

[4] R. Gaines and J. Mawhin, *Coincidence Degree and Nonlinear Differential Equation*, Springer, Berlin, 1977.

[5] W. Liu, *Existence and uniqueness of periodic solutions for a kind of Liénard type p-Laplacian equation*, Nonlinear Anal. TMA, 69 (2008), 724–729.

[6] W. Liu, J. Liu, H. Zhang, Z. Hu and Y. Wu, *Existence of periodic solutions for Liénard-type p-Laplacian systems with variable coefficients*, Ann. Polon. Math., 109 (2013), 109–119.

[7] S. Lu and J. Shan, *Existence of periodic solutions for a fourth-order p-Laplacian equation with a deviating argument*, J. Comput. Appl. Math., 230 (2009), 513–520.

[8] H. Meng and F. Long, *Periodic solutions for a Liénard type p-Laplacian differential equation*, J. Comput. Appl. Math., 224 (2009), 696–701.

[9] J. Meng, *Positive periodic solutions for Liénard Type p-Laplacian Equations*, Electronic J. Differential Equations, 39 (2009), 1–7.

[10] J. Shan and S. Lu, *Periodic solutions for a fourth-order p-Laplacian differential equation with a deviating argument*, Nonlinear Anal. TMA, 69 (2008), 1710–1718.

[11] P. Torres, Z. Cheng and J. Ren, *Non-degeneracy and uniqueness of periodic solutions for 2n-order differential equation*, Discrete Contin. Dyn. Syst. A, 33 (2013), 2155–2168.

[12] Y. Wang, X. Dai and X. Xia, *On the existence of a unique periodic solution to a Liénard type p-Laplacian non-autonomous equation*, Nonlinear Anal. TMA, 71 (2009), 275–280.

[13] K. Wang, and Y. Zhu, *Periodic solutions for a fourth-order p-Laplacian neutral functional differential equation*, J. Franklin Inst., 347 (2010), 1158–1170.

[14] Y. Xin, and Z. Cheng, *Neutral operator with variable parameter and third-order neutral differential*, Adv. Difference Equations, 2014:173 (2014), 1–18.

[15] Y. Xin, X. Han and Z. Cheng, *Existence and uniqueness of positive periodic solution for ϕ-Laplacian Liénard equation*, Boundary Value Problems, 2014:244 (2014), 1–11.

[16] L. Yin and Z. Zhang, *Existence of a positive solution for a first-order p-Laplacian BVP with impulsive on time scales*, J. Appl. Anal. Comput., (2) (2012), 103–109.