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A CHARACTERIZATION OF GENERALIZED EXPONENTIAL DICHOTOMY*

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Abstract This paper studies some important properties of the notion *generalized exponential dichotomy*. A new notion called *generalized bounded growth* is introduced to describe the characterization of generalized exponential dichotomy. The relations between *generalized bounded growth* and *generalized exponential dichotomy* are established.

Keywords Generalized exponential dichotomy, linear system.

MSC(2010) 93B18, 34D09, 34D10, 34D20.

1. Introduction and Motivation

1.1. History

Consider the following linear system

$$x' = A(t)x, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $A(t)$ is a $n \times n$ continuous matrix defined on $\mathbb{R}$. Let $X(t)$ be a fundamental matrix of (1.1).

Definition 1.1. System (1.1) is said to possess an exponential dichotomy on $\mathbb{R}$ (Coppel [4]), if there exists a projection P and strictly positive constants K, α such that

$$\begin{cases} \|X(t)PX^{-1}(s)\| \le K \exp\{-\alpha(t-s)\}, & \text{for } t \ge s,\ t,\ s \in \mathbb{R}, \\ \|X(t)(I-P)X^{-1}(s)\| \le K \exp\{\alpha(t-s)\}, & \text{for } t \le s,\ s,\ t \in \mathbb{R}, \end{cases} \tag{1.2}$$

hold.

The properties and applications of exponential dichotomy have been well studied. For examples, one can refer to [4–28]. However, Lin [13] argued that the notion of exponential dichotomy considerably restricts the dynamics. It is thus important to look for more general types of hyperbolic behavior. He proposed the notion of *generalized exponential dichotomy* which is more general than the classical notion

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of *exponential dichotomy*. Jiang [7–10] also thought that the notion of *generalized exponential dichotomy* was very important and he had applied it to improve the Palmer linearization theorem.

Definition 1.2. System (1.1) is said to possess a generalized exponential dichotomy on $\mathbb{R}$ (shortly for GED), if there exists a projection P and a strictly positive constant K such that

$$\begin{cases} \|X(t)PX^{-1}(s)\| \leq K\exp\{-\int_s^t \alpha(\tau)d\tau\}, & \text{for } t \geq s,\ s,\ t \in \mathbb{R}, \\ \|X(t)(I-P)X^{-1}(s)\| \leq K\exp\{\int_s^t \alpha(\tau)d\tau\}, & \text{for } t \leq s,\ s,\ t \in \mathbb{R}, \end{cases} \tag{1.3}$$

hold, where $\alpha(t)$ is a nonnegative continuous function, satisfying

$$\lim_{t\to+\infty}\int_0^t \alpha(\xi)d\xi = +\infty, \quad \lim_{t\to-\infty}\int_t^0 \alpha(\xi)d\xi = +\infty \text{ (see Lin [14]).}$$

Remark 1.1. When $\alpha(\xi) = \alpha$, Definition 1.2 reduces to Definition 1.1.

Example 1.1. Consider the system

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{|t|+1}} & 0 \\ 0 & \frac{1}{\sqrt{|t|+1}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.4}$$

Then system (1.4) has a GED, but the classical exponential dichotomy can not be satisfied.

1.2. Motivation and comparison with previous works

Lin [13] has obtained a characterization of exponential dichotomy in terms of Lyapunov function. Different from his consideration, some new criteria are established for the existence of GED based on proposing a new notion of *generalized bounded growth*. Moreover, motivated by the work [4, 12, 23], we obtain a set of new properties of GED by discussing the relations between the n independent solutions and GED. Our results generalize some previously known results in [4, 12, 23]. Recently, another kind of generalization of the dichotomy is so-called the nonuniform hyperbolicity (e.g see Chu [1–3]). Zhang [29] also proposed a generalized notion of exponential dichotomy in Banach space in order to find finer invariant manifolds based on nonhyperbolic or pseudohyperbolic systems. It should be noted that our notion of *generalized exponential dichotomy* is not a kind of nonuniform hyperbolicity. Our notion still belongs to a kind of uniform hyperbolicity. It is more general than the classical notion of exponential dichotomy. So our consideration is different from those in [1–3, 29].

1.3. Outline of the paper

In next section, some definitions and lemmas are introduced. In Section 3, some criteria for the existence of the generalized exponential dichotomy are established. In Section 4, some properties on characterization of the generalized exponential dichotomy are presented.

2. Some definitions and lemmas

In this section, we recall some known results which will play role in our proofs. Consider the following two linear systems

$$x' = A(t)x, \tag{2.1}$$

and

$$y' = B(t)y, \tag{2.2}$$

where $x,\ y\in\mathbb{R}^n$, $A(t),\ B(t)$ are continuous and bounded matrix functions on $\mathbb{R}$.

Definition 2.1. Suppose that $S(t)$ is a non-degenerate square matrix defined on $\mathbb{R}$ or $\mathbb{R}^+$, $S(t)$ is said to be a Lyapunov square matrix, if $S(t)$ is differentiable and $\|S(t)\|$, $\|S^{-1}(t)\|$ are bounded.

Definition 2.2. System (2.1) is kinematically similar to system (2.2), if there exists a Lyapunov square matrix $S(t)$ such that

$$S'(t) = A(t)S(t) - S(t)B(t) \text{ or } B(t) = S^{-1}(t)A(t)S(t) - S^{-1}(t)S'(t),$$

hold.

Definition 2.3. System (2.1) is said to be a diagonal block, if system (2.1) is kinematically similar to system (2.2). Moreover, $B(t)$ has a diagonal block of the form $\begin{pmatrix} B_1(t) & \\ & B_2(t) \end{pmatrix}$, where the ranks of $B_1(t), B_2(t)$ are lower than $B(t)$ (see [4, Chap.5]).

Definition 2.4. Linear system (2.1) is said to be of bounded growth (see [4, Chap.5]), if there exist constants $C \geq 1, h > 0$, such that any solution of system (2.1) $x(t)$ satisfies

$$\|x(t)\| \leq C\|x(s)\|, \ \ (s \leq t \leq s+h).$$

Lemma 2.1. *System (2.1) is kinematically similar to system (2.2), if and only if there exists a Lyapunov transformation $y = S(t)x$ which can send system (2.1) into system (2.2).*

Lemma 2.2. *Let $X(t)$ be an invertible matrix and P be an orthogonal projection, then there exists a continuous and differentiable non-degenerate square matrix $S(t)$, such that*

$$\begin{cases} S(t)PS^{-1}(t) = X(t)PX^{-1}(t), \\ S(t)(I-P)S^{-1}(t) = X(t)(I-P)X^{-1}(t), \end{cases}$$

and

$$\|S(t)\| \leq \sqrt{2}, \tag{2.3}$$

$$\|S^{-1}(t)\| \leq [\|X(t)PX^{-1}(t)\|^2 + \|X(t)(I-P)X^{-1}(t)\|^2]^{\frac{1}{2}} \tag{2.4}$$

hold, where $S(t) = X(t)R^{-1}(t)$, $R(t)$ is an uniqueness positive square root of $G(t)$, $G(t) = PX^T(t)X(t)P + (I-P)X^T(t)X(t)(I-P)$, $X^T(t)$ denotes the transpose of $X(t)$ (see [4, Chap.5]).

Remark 2.1. It should be noted that Lin [14] has given an equivalent definition of GED as follows.

$$\begin{cases} \|X(t)P\xi\| \le K\exp\{-\int_s^t \alpha(\tau)d\tau\}\|X(s)P\xi\| & \text{for } s \le t,\ s,\ t \in \mathbb{R}, \\ \|X(t)(I-P)\xi\| \le K\exp\{\int_s^t \alpha(\tau)d\tau\}\|X(s)(I-P)\xi\| & \text{for } t \le s,\ s,\ t \in \mathbb{R}. \end{cases} \tag{2.5}$$

Moreover, for arbitrary $t \in \mathbb{R}$, he proved that

$$\|X(t)PX^{-1}(t)\| \le M, \tag{2.6}$$

and

$$\|X(t)(I-P)X^{-1}(t)\| \le M, \tag{2.7}$$

where M is a positive constant.

3. Criteria for the existence of GED

To continue our work, we should introduce a new definition here.

Definition 3.1. Linear system (2.1) is said to be of *generalized bounded growth*, if for some fixed $h > 0$, there exists a nonnegative continuous function $\varrho(t)$, such that any solution of system (2.1) $x(t)$ satisfies

$$\|x(t)\| \le c(s)\|x(s)\|, \qquad s \le t \le s+h, \tag{3.1}$$

where $c(s) = \mu\exp\{\int_s^{s+h} \varrho(\tau)d\tau\}$.

It is easy to see that $\mu \ge 1$ and $c(s)$ is non-increasing for s. Next theorem gives an equivalent definition of generalized bounded growth.

Theorem 3.1. *Linear system (2.1) is of generalized bounded growth, if and only if there exists a nonnegative nonincreasing continuous function $\varpi(t)$ and a constant $\mu \ge 1$, such that*

$$\|X(t)X^{-1}(s)\| \le \mu\exp\{\int_s^t \varpi(\tau)d\tau\}, \qquad t \ge s.$$

Proof. First, we show the necessity. Since linear system (2.1) is of generalized bounded growth, there exists a nonnegative continuous function $\varrho(t)$, such that for arbitrary $\xi\in\mathbb{R}^n$, we have

$$\|X(t)\xi\| \le c(s)\|X(s)\xi\|, \quad s \le t \le s+h,$$

where $c(s)$ and μ have been defined in Definition 3.1. Taking $s, t\in\mathbb{R}$, $s \le t$, if

$s+kh \le t < s+(k+1)h$, then

$$\begin{aligned}\|X(t)\xi\| &\le c(t-h)\|X(t-h)\xi\| \\ &\le c(t-h)c(t-2h)\|X(t-2h)\xi\| \\ &\le ... \le c(t-h)c(t-2h)...c(t-kh)c(s)\|X(s)\xi\| \\ &\le ... \le c^{k+1}(t-h)\|X(s)\xi\| \\ &= \mu^{k+1}\exp\{(k+1)\int_{t-h}^{t}\varrho(\tau)d\tau\}\|X(s)\xi\| \\ &\le \mu^{k+1}\exp\{(k+1)\int_{t-kh}^{t}\varrho(\tau)d\tau\}\|X(s)\xi\| \\ &\le \mu^{k+1}\exp\{(k+1)\int_{s}^{t}\varrho(\tau)d\tau\}\|X(s)\xi\|.\end{aligned}$$

Then we can choose a nonnegative continuous function $\varpi(t)$ such that

$$(k+1)\int_s^t \varrho(\tau)d\tau \le \int_s^t \varpi(\tau)d\tau.$$

Let $\mu_1 = \mu^{k+1}$, that is

$$\|X(t)\xi\| \le \mu_1 \exp\{\int_s^t \varpi(\tau)d\tau\}\|X(s)\xi\|.$$

Set $\xi = X^{-1}(s)y$, for $t \ge s$, we have

$$\begin{aligned}\|X(t)X^{-1}(s)y\| &\le \mu_1\|X(s)X^{-1}(s)y\|\exp\{\int_s^t \varpi(\tau)d\tau\} \\ &= \mu_1\|y\|\exp\{\int_s^t \varpi(\tau)d\tau\}.\end{aligned}$$

Since ξ is arbitrary and $X(s)$ is reversible, y is arbitrary. Thus, the following inequality follows

$$\|X(t)X^{-1}(s)\| \le \mu\exp\{\int_s^t \varpi(\tau)d\tau\}, \quad t \ge s.$$

Next, we show the sufficiency. Assume that there exists a nonnegative continuous function $\varpi(t)$ and a constant $\mu \ge 1$, such that

$$\|X(t)X^{-1}(s)\| \le \mu\exp\{\int_s^t \varpi(\tau)d\tau\}, \quad t \ge s.$$

So for arbitrary $\xi\in\mathbb{R}^n$, we have

$$\begin{aligned}\|X(t)\xi\| &= \|X(t)X^{-1}(s)X(s)\xi\| \\ &\le \|X(t)X^{-1}(s)\|\cdot\|X(s)\xi\| \\ &\le \mu\exp\{\int_s^t \varpi(\tau)d\tau\}\|X(s)\xi\| \\ &\le \mu\exp\{\int_s^{s+h} \varpi(\tau)d\tau\}\|X(s)\xi\|.\end{aligned}$$

Let $c(s) = \mu \exp\big(\int_s^{s+h} \varpi(\tau)d\tau\big)$, that is

$$\|X(t)\xi\| \le c(s)\|X(s)\xi\| s \le t \le s+h.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Linear system (2.1) is of generalized bounded growth if* $\int_t^{t+h} \|A(r)\|dr$ *is nonincreasing.*

Proof. For any solution of linear system (2.1) $x(t)$ satisfying

$$x(t) = x(s) + \int_s^t A(r)x(r)dr,$$

consider $s \le t \le s+h$, then

$$\|x(t)\| \le \|x(s)\| + \int_s^t \|A(r)\| \cdot \|x(r)\|dr.$$

Using the Bellman inequality, we get

$$\|x(t)\| \le \|x(s)\| \exp\{\int_s^t \|A(r)\|dr\} \le \|x(s)\| \exp\{\int_s^{s+h} \|A(r)\|dr\}.$$

Let $\|A(r)\| = \mu\varrho(r)$, $\mu \ge 1$, then

$$\|x(t)\| \le \mu \exp\{\int_s^{s+h} \varrho(r)dr\}\|x(s)\|.$$

If we denote $c(s) = \mu \exp\{\int_s^{s+h} \varrho(r)dr\}$, then

$$\|x(t)\| \le c(s)\|x(s)\|,$$

where $s \le t \le s+h$. This completes the proof of Theorem 3.2. □

Next theorem is to give a sufficient and necessary condition for GED based on the relationship between independent solutions and GED.

Theorem 3.3. *Suppose that system (2.1) is of generalized bounded growth. Linear system (2.1) possesses a GED, if and only if there exist n linearly independent solutions* $x_1(t), x_2(t), ..., x_n(t)$ *satisfying*

$$\begin{cases} \|\sum_{i=1}^{r} a_i x_i(t)\| \le K\|\sum_{i=1}^{r} a_i x_i(s)\| \exp\{-\int_s^t \alpha(\tau)d\tau\}, \quad t \ge s, \\ \|\sum_{i=r+1}^{n} a_i x_i(t)\| \le K\|\sum_{i=r+1}^{n} a_i x_i(s)\| \exp\{\int_s^t \alpha(\tau)d\tau\}, \quad t \le s, \end{cases}$$

where the constant $K > 0$*,* $\alpha(t)$ *is a nonnegative continuous function with*

$$\lim_{t\to+\infty} \int_0^t \alpha(\xi)d\xi = +\infty, \lim_{t\to-\infty} \int_t^0 \alpha(\xi)d\xi = +\infty,$$

r *is the rank of projection* P*,* $a_1, a_2, ..., a_n$ *are arbitrary real numbers.*

Proof. First, we prove the necessity. Suppose that linear system (2.1) has a GED, then for arbitrary $\xi \in \mathbb{R}^n$,

$$\begin{aligned}\|\widetilde{X}(t)P\xi\| &= \|\widetilde{X}(t)P\widetilde{X}^{-1}(s)\widetilde{X}(s)P\xi\| \\ &\le \|\widetilde{X}(t)P\widetilde{X}^{-1}(s)\| \cdot \|\widetilde{X}(s)P\xi\| \\ &\le \overline{K}\|\widetilde{X}(s)P\xi\| \exp\{-\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad (t \ge s),\end{aligned}$$

and

$$\begin{aligned}\|\widetilde{X}(t)(I-P)\xi\| &= \|\widetilde{X}(t)(I-P)\widetilde{X}^{-1}(s)\widetilde{X}(s)(I-P)\xi\| \\ &\le \|\widetilde{X}(t)(I-P)\widetilde{X}^{-1}(s)\| \cdot \|\widetilde{X}(s)(I-P)\xi\| \\ &\le \overline{K}\|\widetilde{X}(s)(I-P)\xi\| \exp\{\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad (t \le s),\end{aligned}$$

where $\widetilde{X}(t)$ is a standard matrix of linear system (2.1). Let r be the rank of projection P. Taking vectors $\xi_1, ..., \xi_n \in \mathbb{R}^n$, such that $P\xi_1, ..., P\xi_r, (I-P)\xi_{r+1}, ..., (I-P)\xi_n$ are linearly independent, set

$$\begin{aligned}&x_i(t) = \widetilde{X}(t)P\xi_i, \quad i = 1, 2, ..., r, \\ &x_i(t) = \widetilde{X}(t)(I-P)\xi_i, \quad i = r+1, ..., n,\end{aligned}$$

then $x_1(t), x_2(t), ..., x_n(t)$ is n linearly independent solutions of system (2.1), and

$$\begin{cases} \|\sum\limits_{i=1}^{r} a_i x_i(t)\| \le \overline{K}\|\sum\limits_{i=1}^{r} a_i x_i(s)\| \exp\{-\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad t \ge s, \\ \|\sum\limits_{i=r+1}^{n} a_i x_i(t)\| \le \overline{K}\|\sum\limits_{i=r+1}^{n} a_i x_i(s)\| \exp\{\int_s^t \overline{\alpha}(\tau)d\tau\}, \ t \le s. \end{cases}$$

Next we prove sufficiency. Since there exist n linearly independent solutions $x_1(t), x_2(t), ..., x_n(t)$ such that

$$\begin{cases} \|\sum\limits_{i=1}^{r} a_i x_i(t)\| \le K\|\sum\limits_{i=1}^{r} a_i x_i(s)\| \exp\{-\int_s^t \alpha(\tau)d\tau\}, \quad t \ge s, \\ \|\sum\limits_{i=r+1}^{n} a_i x_i(t)\| \le K\|\sum\limits_{i=r+1}^{n} a_i x_i(s)\| \exp\{\int_s^t \alpha(\tau)d\tau\}, \quad t \le s. \end{cases}$$

Let $\widetilde{X}(t)$ be a standard matrix of linear system(2.1), then there exists a real invertible matrix Q, such that $\big(x_1(t), x_2(t), ..., x_n(t)\big) = \widetilde{X}(t)Q$. Let $P = QE_kQ^{-1}$, $\overline{K} = K$, $\overline{\alpha} = \alpha$, then r is the rank of projection P, and for arbitrary $\xi \in \mathbb{R}^n$, we have

$$\begin{aligned}&\|\widetilde{X}(t)P\xi\| \le \overline{K}\|\widetilde{X}(s)P\xi\| \exp\{-\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad t \ge s, \\ &\|\widetilde{X}(t)(I-P)\xi\| \le \overline{K}\|\widetilde{X}(s)(I-P)\xi\| \exp\{\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad t \le s.\end{aligned}$$

Let $\xi = \widetilde{X}^{-1}(s)y$, then

$$\begin{aligned}\|\widetilde{X}(t)P\widetilde{X}^{-1}(s)y\| &\le \overline{K}\|\widetilde{X}(s)P\widetilde{X}^{-1}(s)y\| \exp\{-\int_s^t \overline{\alpha}(\tau)d\tau\} \\ &\le \overline{K}\|\widetilde{X}(s)P\widetilde{X}^{-1}(s)\| \cdot \|y\| \exp\{-\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad t \ge s,\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\|\widetilde{X}(t)(I-P)\widetilde{X}^{-1}(s)y\| \\
\leq\ &\overline{K}\|\widetilde{X}(s)(I-P)\widetilde{X}^{-1}(s)y\|\exp\{\int_s^t \overline{\alpha}(\tau)d\tau\} \\
\leq\ &\overline{K}\|\widetilde{X}(s)(I-P)\widetilde{X}^{-1}(s)\|\cdot\|y\|\exp\{\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad t\leq s.
\end{aligned} \tag{3.3}$$

Since $\xi\in\mathbb{R}^n$ is arbitrary, we conclude that for arbitrary $y\in\mathbb{R}^n$, the inequalities (3.2)and (3.3) hold.

$$\begin{cases}
\|\widetilde{X}(t)P\widetilde{X}^{-1}(s)\|\leq\overline{K}\|\widetilde{X}(s)P\widetilde{X}^{-1}(s)\|\exp\{-\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad t\geq s, \\
\|\widetilde{X}(t)(I-P)\widetilde{X}^{-1}(s)\|\leq\overline{K}\|\widetilde{X}(s)(I-P)\widetilde{X}^{-1}(s)\|\exp\{\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad t\leq s.
\end{cases} \tag{3.4}$$

From inequalities (2.6) and (2.7), we conclude that for arbitrary $s\in\mathbb{R}$, there exists a constant $M>0$, such that

$$\|\widetilde{X}(s)P\widetilde{X}^{-1}(s)\|\leq M, \|\widetilde{X}(s)(I-P)\widetilde{X}^{-1}(s)\|\leq M.$$

Then (3.4) can written as

$$\begin{aligned}
&\|\widetilde{X}(t)P\widetilde{X}^{-1}(s)\|\leq\overline{K}M\exp\{-\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad t\geq s, \\
&\|\widetilde{X}(t)(I-P)\widetilde{X}^{-1}(s)\|\leq\overline{K}M\exp\{\int_s^t \overline{\alpha}(\tau)d\tau\}, \quad t\leq s.
\end{aligned}$$

Taking $\widetilde{K}=\overline{K}M$, then system (2.1) has a GED.
This completes the proof of Theorem 3.4. □

Now we introduce an interesting lemma.

Lemma 3.1. *(i) If $x(t)$ is a continuous real vector function defined on $\mathbb{R}^+(\mathbb{R}^+ = [0,+\infty))$, $x(t)\neq 0$, and there exist nonnegative continuous functions $\varrho(t)$ and $\kappa(t)$, a constant $h>0$, such that the following inequalities*

$$\|x(t)\|\leq c(s)\|x(s)\|, \qquad 0\leq s\leq t\leq s+h,$$

$$\|x(t)\|\leq\theta(u)\sup_{|u-t|\leq h}\|x(u)\|, \qquad t\geq h,$$

hold, where $c(s)$ and μ have been defined in Definition 3.1, $\theta(u)=\exp\{-\int_u^{u+h}\kappa(\tau)d\tau\}$ and $\kappa(\tau+u)$ is non-increasing for u. Then there exists a nonnegative continuous function $\alpha(t)$ and a constant $K\geq 1$, such that one of the following inequalities

$$\|x(t)\|\leq K\exp\{-\int_s^t \alpha(\tau)d\tau\}, \quad t\geq s\geq 0,$$

$$\|x(t)\|\leq K\exp\{\int_s^t \alpha(\tau)d\tau\}, \quad s\geq t\geq 0.$$

hold.

(ii) If $x(t)$ is a continuous real vector function defined on $\mathbb{R}^-(\mathbb{R}^- = (-\infty,0))$,

$x(t) \neq 0$, there exist nonnegative continuous functions $\varrho(t)$ and $\kappa(t)$, a constant $h > 0$, such that the following inequalities

$$\|x(t)\| \leq c(s)\|x(s)\|, \quad s \leq t \leq s+h \leq 0,$$

$$\|x(t)\| \leq \theta(u) \sup_{|u-t|\leq h} \|x(u)\|, \quad t \leq -h,$$

hold, where $c(s)$ and μ have been defined in Definition 3.1, $\theta(u) = \exp\{-\int_u^{u+h} \kappa(\tau)d\tau\}$ and $\kappa(\tau+u)$ is non-increasing for s. Then there exists a nonnegative continuous function $\alpha(t) \geq 0$ and a constant $K \geq 1$ such that one of the following inequalities

$$\|x(t)\| \leq K \exp\{-\int_s^t \alpha(\tau)d\tau\}, \quad 0 \geq t \geq s,$$

$$\|x(t)\| \leq K \exp\{\int_s^t \alpha(\tau)d\tau\}, \quad 0 \geq s \geq t,$$

hold.

Proof. we show the proof of (i), and the proof of (ii) can use the similar methods. If

$$\sup_{t\geq 0} \|x(u)\| < +\infty,$$

let

$$\mu(s) = \sup_{u\geq s} \|x(u)\|,$$

so for the arbitrary $s \in \mathbb{R}^+$, there exists $u_m \geq s$, such that

$$\lim_{m\to+\infty} \|x(u_m)\| = \mu(s), \tag{3.5}$$

hold.

Now, we proof that there exists a natural number N, as $m \geq N$, we have $u_m \leq s+h$.

If not, there exists a subsequence $\{u_{m_k}\}$ of $\{u_m\}$, such that $u_{m_k} > s+h$, then

$$\begin{aligned}\|x(u_{m_k})\| &\leq \theta(u) \sup_{|u-u_{m_k}|\leq h} \|x(u)\| \\ &\leq \theta(u) \sup_{u\geq u_{m_k}-h} \|x(u)\| \\ &= \theta(u)\mu(u_{m_k}-h) \\ &\leq \theta(u)\mu(s),\end{aligned}$$

as $k\to\infty$, we have

$$\|x(u_{m_k})\|\to\mu(s),$$

that is, $\mu(s) \leq \theta(u)\mu(s)$. Hence, $\theta(u) \geq 1$, which is a contradiction to $\theta(u) < 1$ due to the definition $\theta(u) = \exp\{-\int_u^{u+h} \kappa(\tau)d\tau\}$. Then we have

$$\mu(s) = \sup_{s\leq u\leq s+h} \|x(u)\|.$$

Therefore, for $t \geq s \geq 0$,

$$\|x(t)\| \leq \mu(s) = \sup_{s\leq u\leq s+h} \|x(u)\| \leq c(s)\|x(s)\|.$$

As $t \geq s$, we can take $s + mh \leq t < s + (m+1)h$, then

$$\begin{aligned}
\|x(t)\| &\leq \theta(u) \sup_{|u-t|\leq h} \|x(u)\| \\
&\leq \theta(u)\theta(u-h) \sup_{|u-t|\leq 2h} \|x(u)\| \\
&\leq ... \\
&\leq \theta(u)\theta(u-h)...\theta[u-(m-1)h] \sup_{|u-t|\leq mh} \|x(u)\| \\
&\leq \theta^m(u) \sup_{|u-t|\leq mh} \|x(u)\| \\
&\leq \theta^m(u)c(s)\|x(s)\| \\
&= \exp\{-m\int_u^{u+h} \kappa(\tau)d\tau\}\mu \exp\{\int_s^{s+h} \varrho(\tau)d\tau\}\|x(s)\| \\
&= \mu \exp\{\int_s^{s+h} \varrho(\tau)d\tau - m\int_u^{u+h} \kappa(\tau)d\tau\}\|x(s)\|.
\end{aligned}$$

Then there exists a nonnegative continuous function $\widetilde{\varrho}(t)$ such that

$$\int_s^{s+h} \varrho(\tau)d\tau - m\int_u^{u+h} \kappa(\tau)d\tau \leq -\int_s^t \widetilde{\varrho}(\tau)d\tau,$$

hold. Take $K = \mu \geq 1$, then

$$\|x(t)\| \leq K \exp\{-\int_s^t \widetilde{\varrho}(\tau)d\tau\}\|x(s)\|, \quad t \geq s \geq 0.$$

If

$$\sup_{t\geq 0} \|x(u)\| = +\infty,$$

because of the continuity of $x(t)$, we can take a subsequence $\{t_m\}$ satisfying

$$\begin{cases} \|x(t_m)\| = \theta(u)^{-m}c(s)\|x(0)\|, \\ \|x(t)\| < \theta(u)^{-m}c(s)\|x(0)\|, \quad 0 \leq t < t_m, \end{cases}$$

then $h < t_1 < t_2 < ... < t_m < ..., t_m \to +\infty$.

Now, we proof $t_{m+1} \leq t_m + h$.

If not, there exists m_0, such that $t_{m_0+1} > t_{m_0} + h$. However,

$$\begin{aligned}
\|x(t_{m_0})\| &\leq \theta(u) \sup_{|u-t_{m_0}|\leq h} \|x(u)\| \\
&\leq \theta(u) \sup_{0\leq u\leq t_{m_0}+h} \|x(u)\| \\
&< \theta(u)\|x(t_{m_0+1})\|,
\end{aligned}$$

which is in contradiction with $\|x(t_{m_0+1})\| = \theta^{-1}(u)\|x(t_{m_0})\|$. So we have $t_{m+1} \leq t_m + h$. For $0 \leq t \leq s$, assume $0 < t_m \leq t < t_{m+1}, t_k \leq s < t_{k+1}$, then

$$\begin{aligned}
\|x(t)\| &< \|x(t_{m+1})\| \\
&= \theta^{k-m}(u)\|x(t_{k+1})\| \\
&\leq c(s)\theta^{-1}(u)\theta^{k-m+1}(u)\|x(s)\| \\
&= \mu\theta^{-1}(u) \exp\{\int_s^{s+h} [\varrho(\tau) - (k-m+1)\kappa(\tau)]d\tau\}\|x(s)\|.
\end{aligned}$$

Then there exists a nonnegative continuous function $\varrho(t)$, such that

$$\int_s^{s+h}[\varrho(\tau)-(k-m+1)\kappa(\tau)]d\tau \le \int_s^t \varrho(\tau)d\tau, \quad 0\le t\le s,$$

hold. Take $K=\mu\max\{\theta^{-1}(u)\}\ge 1$, then

$$\|x(t)\| \le K\exp\big(\int_s^t \varrho(\tau)d\tau\big)\|x(s)\|, \quad 0\le t\le s.$$

This completes the proof of Lemma 3.1. □

Theorem 3.4. *If linear system (2.1) is generalized bounded growth, and there exists a nonnegative continuous functions $\kappa(t)$, a constant $h>0$, n-linearly independent solutions of linear system(2.1) $x_1(t),x_2(t),...,x_n(t)$, and any solution of system (2.1) $x(t)$ satisfies*

$$\begin{cases} \|x(t)\|\le\theta(u)\sup\limits_{|u-t|\le h}\|x(u)\|, \\ \liminf\limits_{t\to+\infty}\|x_i(t)\|<\infty, (i=1,2,...,r), \\ \liminf\limits_{t\to-\infty}\|x_i(t)\|<\infty, (i=r+1,r+2,...,n), \end{cases}$$

where $\theta(u)=\exp\{-\int_u^{u+h}\kappa(\tau)d\tau\}$ and $\kappa(\tau+u)$ is non-increasing for u. Then linear system (2.1) has a GED, and the rank of projection P is r.

Proof. Since linear system (2.1) is of generalized bounded growth, for $h>0$, there exists a continuous function $\varrho(t)\ge 0$, such that any solution of linear system (2.1) $x(t)$ satisfies

$$\|x(t)\|\le c(s)\|x(s)\|, \quad s\le t\le s+h,$$

where $c(s)$ and μ have been defined in Definition 3.1. Therefore, for arbitrary constants $a_1,a_2,...,a_n$, we have

$$\|\sum_{i=1}^n a_ix_i(t)\|\le c(s)\|\sum_{i=1}^n a_ix_i(s)\|, \quad s\le t\le s+h,$$

as $\|x(t)\|\le\theta(u)\sup\limits_{|u-t|\le h}\|x(u)\|$, now, we consider the case on $\mathbb{R}^+$.
From Lemma 3.1, for arbitrary constants $a_1,a_2,...,a_r$, we have

$$\liminf_{t\to+\infty}\|\sum_{i=1}^r a_ix_i(t)\|<+\infty,$$

then, for $t\ge s\ge 0$, there exists a continuous function $\alpha_1(t)\ge 0$, and a constant $K_1\ge 1$, satisfying

$$\|\sum_{i=1}^r a_ix_i(t)\|\le K_1\|\sum_{i=1}^r a_ix_i(s)\|\exp\{-\int_s^t\alpha_1(\tau)d\tau\}. \tag{3.6}$$

Then we consider the case on $\mathbb{R}^-$.
From Lemma 3.1, we derive for $0\ge t\ge s$, there exists a nonnegative continuous function $\alpha_2(t)$, and a constant $K_2\ge 1$ satisfying

$$\|\sum_{i=1}^r a_ix_i(t)\|\le K_2\|\sum_{i=1}^r a_ix_i(s)\|\exp\{-\int_s^t\alpha_2(\tau)d\tau\}, \tag{3.7}$$

or for $0 \geq s \geq t$,

$$\|\sum_{i=1}^{r} a_i x_i(t)\| \leq K_2 \|\sum_{i=1}^{r} a_i x_i(s)\| \exp\{\int_s^t \alpha_2(\tau) d\tau\}. \tag{3.8}$$

If (3.8) hold, then from (3.6) and (3.8), we can derive

$$\lim_{t\to+\infty} \|\sum_{i=1}^{r} a_i x_i(t)\| = 0,$$

and

$$\lim_{t\to-\infty} \|\sum_{i=1}^{r} a_i x_i(t)\| = 0.$$

Therefore, there exists $t_0 \in \mathbb{R}$, such that

$$\|\sum_{i=1}^{r} a_i x_i(t_0)\| = \sup_{t\in R} \|\sum_{i=1}^{r} a_i x_i(t)\|,$$

hold. However, we can get from our conditions, for arbitrary $t \in \mathbb{R}$, we have

$$\|\sum_{i=1}^{r} a_i x_i(t_0)\| \leq \theta(u) \sup_{t\in R} \|\sum_{i=1}^{r} a_i x_i(t)\|.$$

Hence $\theta(u) \geq 1$, which is a contradiction to $\theta(u) < 1$ due to the definition

$$\theta(u) = \exp\{-\int_u^{u+h} \kappa(\tau) d\tau\}.$$

Then (3.7) hold. take $K = \max\{K_1, K_2\}$, $\alpha(t) = \min\{\alpha_1(t), \alpha_2(t)\}$, then from (3.6) and (3.7), we can get

$$\|\sum_{i=1}^{k} a_i x_i(t)\| \leq K \|\sum_{i=1}^{k} a_i x_i(s)\| \exp\{-\int_s^t \alpha(\tau) d\tau\}, \quad t \geq s.$$

Similarly,we also have

$$\|\sum_{i=k+1}^{n} a_i x_i(t)\| \leq K \|\sum_{i=k+1}^{n} a_i x_i(s)\| \exp\{\int_s^t \alpha(\tau) d\tau\}, \quad t \leq s.$$

From Theorem 3.3, we know linear system (2.1) has a GED. This completes the proof of Theorem 3.4.

□

Now we need an interesting lemma.

Lemma 3.2. *(i) If $x(t)$ is a continuous real vector function defined on $\mathbb{R}^+(\mathbb{R}^+ = [0, +\infty))$, $x(t) \neq 0$, and there exist nonnegative continuous functions $\varrho(t)$ and $\widetilde{\kappa}(t)$, a constant $h > 0$, such that the following inequalities*

$$\|x(t)\| \leq c(s)\|x(s)\|, \quad 0 \leq s \leq t \leq s+h,$$

$$\|x(t)\| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h} \|x(u)\|, \quad t \geq h,$$

hold, where $c(s)$ *and* μ *have been defined in Definition 3.1,* $\widetilde{\theta}(u) = \exp\{\int_u^{u+h} \widetilde{\kappa}(\tau)d\tau\}$ *and* $\widetilde{\kappa}(\tau + u)$ *is non-increasing for* u*. Then there exists a nonnegative continuous function* $\alpha(t)$*, and a constant* $K \geq 1$*, such that one of the following inequalities*

$$\|x(t)\| \leq K \exp\{-\int_s^t \alpha(\tau)d\tau\}, \quad t \geq s \geq 0,$$

$$\|x(t)\| \leq K \exp\{\int_s^t \alpha(\tau)d\tau\}, \quad s \geq t \geq 0,$$

hold.

(ii) If $x(t)$ *is a continuous real vector function defined on* $\mathbb{R}^-(\mathbb{R}^- = (-\infty, 0))$*,* $x(t) \neq 0$*, and there exist nonnegative continuous functions* $\varrho(t)$ *and* $\widetilde{\kappa}(t)$*, a constant* $h > 0$*, such that the following inequalities*

$$\|x(t)\| \leq c(s)\|x(s)\|, \quad s \leq t \leq s + h \leq 0,$$

$$\|x(t)\| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h} \|x(u)\|, \quad t \leq -h,$$

hold, where $c(s)$ *and* μ *have been defined in Definition 3.1,* $\widetilde{\theta}(u) = \exp\{\int_u^{u+h} \widetilde{\kappa}(\tau)d\tau\}$ *and* $\widetilde{\kappa}(\tau + u)$ *is non-increasing for* u*. Then there exists a continuous function* $\alpha(t) \geq 0$*, a constant* $K \geq 1$*, such that one of the following inequalities*

$$\|x(t)\| \leq K \exp\{-\int_s^t \alpha(\tau)d\tau\}, \quad 0 \geq t \geq s,$$

$$\|x(t)\| \leq K \exp\{\int_s^t \alpha(\tau)d\tau\}, \quad 0 \geq s \geq t,$$

hold.

Proof. We prove conclusion (i) only the proof of (ii) is same as that of (i). If $\inf_{t\geq 0} \|x(t)\| > 0$, we take $\lambda(s) = \inf_{u\geq s} \|x(u)\|$. Then for $s \geq t \geq h$,

$$\|x(s)\| \geq \widetilde{\theta}(u) \inf_{|u-s|\leq h} \|x(u)\| \geq \widetilde{\theta}(u)\lambda(s-h).$$

By the definition of $\lambda(s)$, there are $u_m \geq s$ with

$$\lim_{m\to+\infty} \|x(u_m)\| = \lambda(s). \tag{3.9}$$

Now we proof that there exists a N such that $u_m \leq s + h$ for $m \geq N$. Or else, there exists a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ with $u_{m_k} > s + h$. But

$$\begin{aligned}\|x(u_{m_k})\| &\geq \widetilde{\theta}(u) \inf_{|u-u_{m_k}|\leq h} \|x(u)\| \\ &\geq \widetilde{\theta}(u)\lambda(u_{m_k} - h) \\ &\geq \widetilde{\theta}(u)\lambda(s).\end{aligned}$$

This is contrary to (3.9). So there exists N such that $u_m \leq s+h$ for $m \geq N$. Then $\lambda(s) = \inf_{s\leq u\leq s+h} \|x(u)\|$ and for $0 \leq s - 2h < s + h \leq t \leq s$,

$$\begin{aligned}\|x(s)\| &\geq \widetilde{\theta}^2(u) \inf_{|u-s|\leq 2h} \|x(u)\| \\ &\geq \widetilde{\theta}^2(u)\lambda(s-2h) \\ &= \widetilde{\theta}^2(u) \inf_{s-2h\leq u\leq s-h} \|x(u)\| \\ &\geq \widetilde{\theta}^2(u)c^{-2}(t)\|x(t)\|,\end{aligned}$$

for $t+(k-1)h \le s \le t+kh$,

$$\begin{aligned}
\|x(s)\| &\ge \widetilde{\theta}(u) \inf_{|u-s|\le h} \|x(u)\| \\
&\ge \widetilde{\theta}^2(u) \inf_{|u-s|\le 2h} \|x(u)\| \\
&\ge ... \\
&\ge \widetilde{\theta}^{k+1}(u) \inf_{|u-s|\le (k+1)h} \|x(u)\| \\
&\ge \widetilde{\theta}^{k+1}(u)\lambda(s-(k+1)h) \\
&= \widetilde{\theta}^{k+1}(u) \inf_{s-(k+1)h\le u\le s-kh} \|x(u)\| \\
&\ge \widetilde{\theta}^{k+1}(u)c^{-2}(t)\|x(t)\| \\
&= \mu^{-2}\exp\{(k+1)\int_u^{u+h} \widetilde{\kappa}(\tau)d\tau - 2\int_t^{t+h} \varrho(\tau)d\tau\}\|x(t)\|.
\end{aligned}$$

Then there exists a continuous function $\widetilde{\alpha}(t)\ge 0$ such that

$$-(k+1)\int_u^{u+h} \widetilde{\kappa}(\tau)d\tau + 2\int_t^{t+h} \varrho(\tau)d\tau \le \int_s^t \widetilde{\alpha}(\tau)d\tau,$$

hold. Take $K=\mu^2\ge 1$, then

$$\|x(t)\| \le K\exp\{\int_s^t \widetilde{\alpha}(\tau)d\tau\}\|x(s)\|, \quad s\ge t\ge 0.$$

If

$$\inf_{t\ge 0} \|x(u)\| = 0.$$

If $\inf\limits_{t\ge 0} \|x(t)\| = 0$, we take $t_m \ge 0$ such that

$$\begin{cases} \|x(t_m)\| = \widetilde{\theta}^{-m}(u)c(s)\|x(0)\|, \\ \|x(t)\| > \widetilde{\theta}^{-m}(u)c(s)\|x(0)\|, \quad 0\le t< t_m. \end{cases}$$

So $h\le t_1 < t_2 < ... < t_m < t_{m+1} < ...$, now we prove that $t_{m+1} \le t_m + h$. Or else, there exists m_0, such that $t_{m_0+1} > t_{m_0}+h$. So

$$\begin{aligned}
\|x(t_{m_0+1})\| &< \inf_{0\le u\le t_{m_0+h}} \|x(u)\| \\
&\le \inf_{|u-t_{m_0}|\le h} \|x(u)\| \\
&\le \widetilde{\theta}^{-1}(u)\|x(t_{m_0})\|.
\end{aligned}$$

This is in contrary to $\|x(t_{m_0+1})\| = \widetilde{\theta}^{-1}(u)\|x(t_{m_0})\|$. So $t_{m+1}\le t_m+h$.

For $t\ge s\ge 0$, suppose that $t_m \le t < t_{m+1}$, $t_k \le s < t_{k+1}$, then

$$\begin{aligned}
\|x(s)\| &> \|x(t_{k+1})\| \\
&= \widetilde{\theta}^{m-k-1}(u)\|x(t_m)\| \\
&\ge c^{-1}(s)\widetilde{\theta}^{m-k-1}(u)\|x(t)\| \\
&= \mu^{-1}\exp\{(m-k-1)\int_u^{u+h} \widetilde{\kappa}(\tau)d\tau - \int_s^{s+h} \varrho(\tau)d\tau\}\|x(t)\|.
\end{aligned}$$

Then there exists a continuous function $\overline{\alpha}(t) \geq 0$ such that

$$-(m-k-1)\int_{u}^{u+h}\widetilde{\kappa}(\tau)d\tau+\int_{s}^{s+h}\varrho(\tau)d\tau \leq -\int_{s}^{t}\overline{\alpha}(\tau)d\tau,$$

hold. Take $K=\mu \geq 1$, then

$$\|x(t)\| \leq K\exp\{-\int_{s}^{t}\overline{\alpha}(\tau)d\tau\}\|x(s)\|, \quad t\geq s\geq 0.$$

This completes the proof of Lemma 3.2. □

Theorem 3.5. *If linear system (2.1) is of generalized bounded growth, and there exists a nonnegative continuous function $\widetilde{\kappa}(t)$, a constant $h>0$ such that n linearly independent solutions of linear system(2.1) $x_1(t), x_2(t), ..., x_n(t)$ satisfy*

$$\begin{cases} \|\sum\limits_{i=1}^{r} a_i x_i(t)\| \geq \widetilde{\theta}(u)\inf\limits_{|u-t|\leq h}\|\sum\limits_{i=1}^{r} a_i x_i(u)\|, \\ \|\sum\limits_{i=r+1}^{n} a_i x_i(t)\| \geq \widetilde{\theta}(u)\inf\limits_{|u-t|\leq h}\|\sum\limits_{i=r+1}^{n} a_i x_i(u)\|, \\ \limsup\limits_{t\to-\infty}\|\sum\limits_{i=1}^{r} a_i x_i(t)\| > 0, \\ \limsup\limits_{t\to+\infty}\|\sum\limits_{i=r+1}^{n} a_i x_i(t)\| > 0, \end{cases}$$

where $\widetilde{\theta}(u)=\exp\{\int_u^{u+h}\widetilde{\kappa}(\tau)d\tau\}$ and $\widetilde{\kappa}(\tau+u)$ is non-increasing for u. Then linear system (2.1) has a GED, and the rank of projection P is r.

Proof. Since linear system (2.1) has generalized bounded growth, for $h>0$, there exists a continuous function $\varrho(t)\geq 0$, such that any solution of linear system (2.1) $x(t)$ satisfies

$$\|x(t)\| \leq c(s)\|x(s)\|, \quad s\leq t\leq s+h,$$

where $c(s)$ and μ have been defined in Definition 3.1. That is

$$\|\sum_{i=1}^{r} a_i x_i(t)\| \leq c(s)\|\sum_{i=1}^{r} a_i x_i(s)\|, \quad s\leq t\leq s+h.$$

For $t\in\mathbb{R}^-$, because $\limsup\limits_{t\to-\infty}\|\sum\limits_{i=1}^{r} a_i x_i(t)\|>0$ and Lemma 3.2, then for $0\geq t\geq s$, there exists a continuous function $\alpha_1(t)\geq 0$, and a constant $K_1\geq 1$, satisfying

$$\|\sum_{i=1}^{r} a_i x_i(t)\| \leq K_1\|\sum_{i=1}^{r} a_i x_i(s)\|\exp\{-\int_{s}^{t}\alpha_1(\tau)d\tau\}. \tag{3.10}$$

For $t\in\mathbb{R}^+$, from Lemma 3.2, we derive for $t\geq s\geq 0$, there exists a nonnegative continuous function $\alpha_2(t)$, and a constant $K_2\geq 1$ satisfying

$$\|\sum_{i=1}^{r} a_i x_i(t)\| \leq K_2\|\sum_{i=1}^{r} a_i x_i(s)\|\exp\{-\int_{s}^{t}\alpha_2(\tau)d\tau\}, \tag{3.11}$$

or for $s\geq t\geq 0$,

$$\|\sum_{i=1}^{r} a_i x_i(t)\| \leq K_2\|\sum_{i=1}^{r} a_i x_i(s)\|\exp\{\int_{s}^{t}\alpha_2(\tau)d\tau\}. \tag{3.12}$$

If (3.11) hold, then from (3.9) and (3.11), we can derive

$$\lim_{t\to-\infty}\|\sum_{i=1}^{r}a_ix_i(t)\|=+\infty,$$

and

$$\lim_{t\to+\infty}\|\sum_{i=1}^{r}a_ix_i(t)\|=+\infty.$$

Therefore, there exists $t_0\in\mathbb{R}$, such that

$$\|\sum_{i=1}^{r}a_ix_i(t_0)\|=\inf_{u\in R}\|\sum_{i=1}^{r}a_ix_i(u)\|,$$

hold. However, we can get from our conditions, for arbitrary $t\in\mathbb{R}$, we have

$$\|\sum_{i=1}^{r}a_ix_i(t)\|\geq\widetilde{\theta}(u)\inf_{|u-t|\leq h}\|\sum_{i=1}^{r}a_ix_i(u)\|.$$

Hence, $\widetilde{\theta}(u)\leq 1$, which is a contradiction to $\widetilde{\theta}(u)>1$ due to the definition

$$\widetilde{\theta}(u)=\exp\{\int_u^{u+h}\widetilde{\kappa}(\tau)d\tau\}.$$

Hence, (3.10) hold. take $K=\max\{K_1,K_2\}$, $\alpha(t)=\min\{\alpha_1(t),\alpha_2(t)\}$, then from (3.9) and (3.10), we can get

$$\|\sum_{i=1}^{k}a_ix_i(t)\|\leq K\|\sum_{i=1}^{k}a_ix_i(s)\|\exp\{-\int_s^t\alpha(\tau)d\tau\},\quad t\geq s.$$

Similarly, we also have

$$\|\sum_{i=k+1}^{n}a_ix_i(t)\|\leq K\|\sum_{i=k+1}^{n}a_ix_i(s)\|\exp\{\int_s^t\alpha(\tau)d\tau\},\quad t\leq s.$$

From Theorem 3.3, we know that linear system (2.1) has a GED. This completes the proof of Theorem 3.5. □

Theorem 3.6. *If linear system (2.1) is of generalized bounded growth, and there exist nonnegative continuous functions $\varrho(t)$, $\kappa(t)$, $\widetilde{\kappa}(t)$, a constant $h>0$ such that n linearly independent solutions of system (2.1) $x_1(t),x_2(t),...,x_n(t)$ satisfy*

$$\begin{cases}\|\sum_{i=1}^{r}a_ix_i(t)\|\leq\theta(u)\sup_{|u-t|\leq h}\|\sum_{i=1}^{r}a_ix_i(u)\|,\\ \|\sum_{i=r+1}^{n}a_ix_i(t)\|\leq\theta(u)\sup_{|u-t|\leq h}\|\sum_{i=r+1}^{n}a_ix_i(u)\|,\\ \|\sum_{i=1}^{r}a_ix_i(t)\|\geq\widetilde{\theta}(u)\inf_{|u-t|\leq h}\|\sum_{i=1}^{r}a_ix_i(u)\|,\\ \|\sum_{i=r+1}^{n}a_ix_i(t)\|\geq\widetilde{\theta}(u)\inf_{|u-t|\leq h}\|\sum_{i=r+1}^{n}a_ix_i(u)\|,\end{cases}$$

where $c(s)$ and μ have been defined in Definition 3.1, $\widetilde{\theta}(u)=\exp\{\int_u^{u+h}\widetilde{\kappa}(\tau)d\tau\}$ $\theta(u)=\exp\{-\int_u^{u+h}\kappa(\tau)d\tau\}$. Moreover, $\widetilde{\kappa}(\tau+u)$ and $\kappa(\tau+u)$ are non-increasing for u. Then linear system (2.1) has a GED, and the rank of projection P is r.

Proof. As linear system (2.1) is of generalized bounded growth, for $h>0$, there exists a continuous function $\varrho(t)\geq 0$, such that any solution of linear system (2.1) $x(t)$ satisfies

$$\|x(t)\|\leq c(s)\|x(s)\|,\quad s\leq t\leq s+h,$$

where $c(s)$ and μ have been defined in Definition 3.1. Then for any solution $x(t)$ of system (2.1), we have

$$\|\sum_{i=1}^{r}a_ix_i(t)\|\leq c(s)\|\sum_{i=1}^{r}a_ix_i(s)\|,\quad s\leq t\leq s+h.$$

For $t\in\mathbb{R}^+$, by Lemma 3.1, there exists a nonnegative continuous function $\alpha_1(t)$ and a constant $K_1\geq 1$, such that

$$\|\sum_{i=1}^{r}a_ix_i(t)\|\leq K_1\|\sum_{i=1}^{r}a_ix_i(s)\exp\{-\int_s^t\alpha_1(\tau)d\tau\},\quad t\geq s\geq 0, \tag{3.13}$$

or

$$\|\sum_{i=1}^{r}a_ix_i(t)\|\leq K_1\|\sum_{i=1}^{r}a_ix_i(s)\exp\{\int_s^t\alpha_1(\tau)d\tau\},\quad s\geq t\geq 0. \tag{3.14}$$

For $t\in\mathbb{R}^-$, by Lemma 3.1, there exists a nonnegative continuous function $\alpha_2(t)$ and a constant $K_2\geq 1$, such that

$$\|\sum_{i=1}^{r}a_ix_i(t)\|\leq K_2\|\sum_{i=1}^{r}a_ix_i(s)\exp\{\int_s^t-\alpha_2(\tau)d\tau\},\quad 0\geq t\geq s, \tag{3.15}$$

or

$$\|\sum_{i=1}^{r}a_ix_i(t)\|\leq K_2\|\sum_{i=1}^{r}a_ix_i(s)\exp\{\int_s^t\alpha_2(\tau)d\tau\},\quad 0\geq s\geq t. \tag{3.16}$$

(I) Suppose that (3.13) is true. Then (3.15) holds. Or else, (3.16) is true. By (3.13) and (3.16),

$$\lim_{t\to+\infty}\|\sum_{i=1}^{r}a_ix_i(t)\|=0,$$

$$\lim_{t\to-\infty}\|\sum_{i=1}^{r}a_ix_i(t)\|=0.$$

So there exists $t_0\in\mathbb{R}$, such that

$$\|\sum_{i=1}^{r}a_ix_i(t_0)\|=\sup_{t\in\mathbb{R}}\|\sum_{i=1}^{r}a_ix_i(t)\|.$$

This is contrary to condition to

$$\|\sum_{i=1}^{r}a_ix_i(t)\|\leq\theta(u)\sup_{|t-u|\leq h}\|\sum_{i=1}^{r}a_ix_i(t)\|.$$

So (3.15) is true. Let $\alpha(t)=\min\{\alpha_1(t),\alpha_2(t)\}$, $K=\max\{K_1,K_2\}$. By (3.13) and (3.15), we have

$$\|\sum_{i=1}^{r}a_ix_i(t)\|\leq K\|\sum_{i=1}^{r}a_ix_i(s)\exp\{\int_s^t-\alpha(\tau)d\tau\},\quad t\geq s.$$

(II) Suppose that (3.14) is true. Then (3.16) holds. Or else, (3.15) is true. By (3.14) and (3.15),

$$\lim_{t\to+\infty} \| \sum_{i=1}^{r} a_i x_i(t) \| = +\infty,$$

$$\lim_{t\to-\infty} \| \sum_{i=1}^{r} a_i x_i(t) \| = +\infty.$$

So there exists $\bar{t} \in \mathbb{R}$, such that

$$\| \sum_{i=1}^{r} a_i x_i(\bar{t}) \| = \inf_{t\in\mathbb{R}} \| \sum_{i=1}^{r} a_i x_i(t) \|.$$

This is contrary to the condition

$$\| \sum_{i=1}^{r} a_i x_i(t) \| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h} \| \sum_{i=1}^{r} a_i x_i(u) \|.$$

So (3.15) is true. Let $\widetilde{\alpha}(t) = \min\{\alpha_1(t), \alpha_2(t)\}$, $\widetilde{K} = \max\{K_1, K_2\}$. By (3.13) and (3.15), we have

$$\| \sum_{i=1}^{r} a_i x_i(t) \| \leq \widetilde{K} \| \sum_{i=1}^{r} a_i x_i(s) \exp\{\int_s^t \widetilde{\alpha}(\tau) d\tau\}, \quad t \leq s.$$

Similarly, we can prove

$$\| \sum_{i=r+1}^{n} a_i x_i(t) \| \leq K \| \sum_{i=r+1}^{n} a_i x_i(s) \exp\{\int_s^t -\alpha(\tau) d\tau\}, \quad t \geq s.$$

or

$$\| \sum_{i=r+1}^{n} a_i x_i(t) \| \leq K \| \sum_{i=r+1}^{n} a_i x_i(s) \exp\{\int_s^t \alpha(\tau) d\tau\}, \quad t \leq s.$$

From Theorem 3.3, we can deduce that system (2.1) has a GED.
This completes the proof of Theorem 3.6.

□

4. Properties of GED

Now, we prove some important properties of GED.

Theorem 4.1. *If linear system (2.1) has a GED, then system (2.1) has non trivial bounded solutions.*

Proof. By way of contradiction, suppose that $x(t)$ is any nontrivial bounded solution of linear system (2.1), then

$$x(t) = X(t)X^{-1}(0)x(0) = X(t)PX^{-1}(0)x(0) + X(t)(I-P)X^{-1}(0)x(0).$$

In view of $x(0) \neq 0$, we know that at least one of $PX^{-1}(0)x(0)$ and $(I-P)X^{-1}(0)x(0)$ is not equal to zero. So we proceed with two cases:

Case 1: If $PX^{-1}(0)x(0) \neq 0$, take $\xi = X^{-1}(0)x(0)$, from the first inequality of (2.5), we have

$$\|X(t)PX^{-1}(0)x(0)\| \geq K^{-1} \|X(0)PX^{-1}(0)x(0)\| \exp\{\int_t^0 \alpha(\tau) d\tau\}.$$

From the second inequality of (2.5), we have

$$\begin{aligned}\|X(t)(I-P)X^{-1}(0)x(0)\| &\le K\|X(0)(I-P)X^{-1}(0)x(0)\|\exp\{\int_0^t \alpha(\tau)d\tau\} \\ &= K\|X(0)(I-P)X^{-1}(0)x(0)\|\exp\{-\int_t^0 \alpha(\tau)d\tau\}.\end{aligned}$$

Letting $t\to-\infty$, we have

$$\|x(t)\| \ge \|X(t)PX^{-1}(0)x(0)\| - \|X(t)(I-P)X^{-1}(0)x(0)\| \ge \to +\infty - 0 = +\infty,$$

which contradicts to the boundedness of $X(t)$.

Case 2: If $(I-P)X^{-1}(0)x(0) \neq 0$, similar arguments show that as $t\to+\infty$, we have $\|x(t)\|\to+\infty$, which also contradicts to the boundedness of $X(t)$. So linear system has nontrivial bounded solution. This completes the proof of Theorem 4.1. □

Remark 4.1. Theorem 4.1 doesn't hold, if linear system (2.1) has a GED only on $\mathbb{R}^+$ or $\mathbb{R}^-$. In fact, if linear system (2.1) has a GED on $\mathbb{R}^+$, then linear system (2.1) has a r-dimension bounded solution, where r is the rank of the projection P.

Theorem 4.2. *Suppose that system (2.1) has a GED and $P \neq 0$ or I, then system (2.1) is a diagonal block.*

Proof. To prove system (2.1) is a diagonal block, our main task is to apply Definition 2.3. To this ends, we proceed two steps.
Step1: We need prove system (2.3) is kinematically similar to system (2.2). From (2.3), we know $\|S(t)\|$ is bounded. From (2.4) and (1.3), we have

$$\begin{aligned}\|S^{-1}(t)\| &\le [\|X(t)PX^{-1}(t)\|^2 + \|X(t)(I-P)X^{-1}(t)\|^2]^{\frac{1}{2}} \\ &\le (K^2+K^2)^{\frac{1}{2}} = \sqrt{2}K,\end{aligned}$$

that is, $\|S^{-1}(t)\|$ is bounded. From Lemma 2.2, we know $S(t)$ is continuous and differentiable. Then $S(t)$ is a Lyapunov matrix. Set

$$R'(t)R^{-1}(t) = B(t), \tag{4.1}$$

then $R(t)$ is a fundamental square matrix of the linear system (2.2). Moreover,

$$\begin{aligned}S'(t) &= (X(t)R^{-1}(t))' \\ &= X'(t)R^{-1}(t) + X(t)(R^{-1}(t))' \\ &= A(t)X(t)R^{-1}(t) - X(t)R^{-1}(t)R'(t)R^{-1}(t).\end{aligned}$$

From (4.1), we have

$$S'(t) = A(t)S(t) - S(t)B(t).$$

That is, system (2.1) is kinematically similar to system (2.2).

Step2: We show that $B(t)$ has a diagonal block of the form $\begin{pmatrix} B_1(t) & \\ & B_2(t)\end{pmatrix}$, where the ranks of $B_1(t), B_2(t)$ are lower than $B(t)$. Since $R(t)$ is a diagonal block, then $R'(t), R^{-1}(t)$ are also diagonal blocks, so $B(t) = R'(t)R^{-1}(t)$ is also a diagonal block. Suppose $B(t) = \begin{pmatrix} B_1(t) & \\ & B_2(t)\end{pmatrix}$, obviously, the rank of $B_1(t)$ is the rank of the projection P, the rank of $B_2(t)$ is the rank of the projection $(I-P)$. This completes the proof of Theorem 4.2. □

Theorem 4.3. *If linear system (2.1) has a GED, then there exists a nonnegative continuous function $\kappa(t)$, a constant $h > 0$, such that any solution of linear system (2.1) $x(t)$ satisfies*

$$\|x(t)\| \le \theta(u) \sup_{|u-t|\le h} \|x(u)\|,$$

and linear system(2.1) has n linearly independent solutions $x_1(t), x_2(t), ..., x_n(t)$, satisfying

$$\liminf_{t\to+\infty} \|x_i(t)\| < \infty, (i = 1, 2, ..., r),$$

and

$$\liminf_{t\to-\infty} \|x_i(t)\| < \infty, (i = r+1, r+2, ..., n),$$

where $\theta(u) = \exp\{-\int_u^{u+h} \kappa(\tau)d\tau\}$ and r is the rank of projection P.

Proof. As there exists a nonnegative continuous function $\alpha(t)$, a constant $K \ge 1$, and a projection P, such that (1.3) hold, then for a nonnegative continuous function

$$\theta(u) = \exp\{-\int_u^{u+h} \kappa(\tau)d\tau\}.$$

Take $h > 0$, such that the following inequality

$$K^{-1}\exp\{\int_s^{s+h} \alpha(\tau)d\tau\} - K\exp\{-\int_s^{s+h} \alpha(\tau)d\tau\} \ge 2\theta^{-1}(u),$$

hold. For $\xi\in\mathbb{R}^n$, take any $s\in\mathbb{R}$, assume that $\widetilde{X}(t)$ is a standard matrix of linear system (2.1), then $x(t) = \widetilde{X}(t)\xi$.
Case 1: If $\|\widetilde{X}(s)(I-P)\xi\| \ge \|\widetilde{X}(s)P\xi\|$, then

$$\begin{aligned}\|\widetilde{X}(t)P\xi\| &= \|\widetilde{X}(t)P\widetilde{X}^{-1}(s)\widetilde{X}(s)P\xi\| \\ &\le \|\widetilde{X}(t)P\widetilde{X}^{-1}(s)\|\cdot\|\widetilde{X}(s)P\xi\| \\ &\le K\|\widetilde{X}(s)P\xi\|\exp\{-\int_s^t \alpha(\tau)d\tau\}, \quad t \ge s,\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\|\widetilde{X}(t)(I-P)\xi\| &= \|\widetilde{X}(t)(I-P)\widetilde{X}^{-1}(s)\widetilde{X}(s)(I-P)\xi\| \\ &\le \|\widetilde{X}(t)(I-P)\widetilde{X}^{-1}(s)\|\cdot\|\widetilde{X}(s)(I-P)\xi\| \\ &\le K\|\widetilde{X}(s)(I-P)\xi\|\exp\{\int_s^t \alpha(\tau)d\tau\}, \quad t \le s.\end{aligned} \tag{4.3}$$

From (4.3), we have

$$\|\widetilde{X}(s)(I-P)\xi\| \ge K^{-1}\|\widetilde{X}(t)(I-P)\xi\|\exp\{\int_t^s \alpha(\tau)d\tau\}, \quad t \le s,$$

then exchange s and t, for $t \ge s$, we have

$$\|\widetilde{X}(t)(I-P)\xi\| \ge K^{-1}\|\widetilde{X}(s)(I-P)\xi\|\exp\{\int_s^t \alpha(\tau)d\tau\}. \tag{4.4}$$

From (4.2) and (4.4), for $t \geq s$, we have

$$\begin{aligned}
\|x(t)\| &= \|\widetilde{X}(t)(I-P)\xi + \widetilde{X}(t)P\xi\| \\
&\geq \|\widetilde{X}(t)(I-P)\xi\| - \|\widetilde{X}(t)P\xi\| \\
&\geq K^{-1}\|\widetilde{X}(s)(I-P)\xi\|\exp\{\int_s^t \alpha(\tau)d\tau\} - K\|\widetilde{X}(s)P\xi\|\exp\{-\int_s^t \alpha(\tau)d\tau\} \\
&\geq K^{-1}\|\widetilde{X}(s)(I-P)\xi\|\exp\{\int_s^t \alpha(\tau)d\tau\} \\
&\quad - K\|\widetilde{X}(s)(I-P)\xi\|\exp\{-\int_s^t \alpha(\tau)d\tau\} \\
&= [K^{-1}\exp\{\int_s^t \alpha(\tau)d\tau\} - K\exp\{-\int_s^t \alpha(\tau)d\tau\}]\cdot\|\widetilde{X}(s)(I-P)\xi\|,
\end{aligned}$$

take $t = s + h$, then

$$\|x(s+h)\| \geq [K^{-1}\exp\{\int_s^{s+h} \alpha(\tau)d\tau\} - K\exp\{-\int_s^{s+h} \alpha(\tau)d\tau\}]\|\widetilde{X}(s)(I-P)\xi\|. \tag{4.5}$$

Moreover,

$$\begin{aligned}
\|x(s)\| &= \|\widetilde{X}(s)(I-P)\xi + \widetilde{X}(s)P\xi\| \\
&\leq \|\widetilde{X}(s)(I-P)\xi\| + \|\widetilde{X}(s)P\xi\| \\
&\leq 2\|\widetilde{X}(s)(I-P)\xi\|,
\end{aligned}$$

that is $\|\widetilde{X}(s)(I-P)\xi\| \geq \frac{1}{2}\|x(s)\|$.
From (4.5), we have

$$\|x(s+h)\| \geq \frac{1}{2}\|x(s)\|\cdot 2\theta^{-1}(u) = \theta^{-1}(u)\|x(s)\|,$$

that is, $\|x(s)\| \leq \theta(u)\|x(s+h)\|$, then we have

$$\|x(s)\| \leq \theta(u) \sup_{|s-u|\leq h} \|x(u)\|.$$

Case 2: If $\|\widetilde{X}(s)(I-P)\xi\| < \|\widetilde{X}(s)P\xi\|$, Similarly, we can derive

$$\|x(s)\| \leq \theta(u) \sup_{|u-s|\leq h} \|x(u)\|.$$

In conclusion, for any solution of linear system (2.1) $x(t)$, we have

$$\|x(s)\| \leq \theta(u) \sup_{|s-u|\leq h} \|x(u)\|.$$

From Theorem 3.3, we have n linearly independent solutions of linear system (2.1) $x_1(t), x_2(t), ..., x_n(t)$ satisfy

$$\begin{cases}
\|\sum\limits_{i=1}^{r} a_i x_i(t) \leq K\|\sum\limits_{i=1}^{r} a_i x_i(s)\|\exp\{-\int_s^t \alpha(\tau)d\tau\}, & t \geq s, \\
\|\sum\limits_{i=r+1}^{n} a_i x_i(t)\| \leq K\|\sum\limits_{i=r+1}^{n} a_i x_i(s)\|\exp\big(\int_s^t \alpha(\tau)d\tau\big), & t \leq s,
\end{cases}$$

where r is the rank of projection P, $a_1, a_2, ..., a_n$ are arbitrary constants. Obviously, we have

$$\begin{cases} \liminf_{t\to+\infty} \|x_i(t)\| < \infty, (i = 1, 2, ..., r), \\ \liminf_{t\to-\infty} \|x_i(t)\| < \infty, (i = r+1, r+2, ..., n). \end{cases}$$

This completes the proof of Theorem 4.3. □

Theorem 4.4. *If linear system (2.1) has a GED, then there exists a nonnegative continuous function $\widetilde{\kappa}(t)$, a constant $h > 0$ such that n linearly independent solutions of linear system(2.1) $x_1(t), x_2(t), ..., x_n(t)$ satisfy*

$$\begin{cases} \|\sum_{i=1}^{r} a_i x_i(t)\| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h} \|\sum_{i=1}^{r} a_i x_i(u)\|, \\ \|\sum_{i=r+1}^{n} a_i x_i(t)\| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h} \|\sum_{i=r+1}^{n} a_i x_i(u)\|, \\ \limsup_{t\to-\infty} \|\sum_{i=1}^{r} a_i x_i(t)\| > 0, \\ \limsup_{t\to+\infty} \|\sum_{i=r+1}^{n} a_i x_i(t)\| > 0, \end{cases}$$

where $\widetilde{\theta}(u) = \exp\{\int_u^{u+h} \widetilde{\kappa}(\tau)d\tau\}$ and $\widetilde{\kappa}(\tau + u)$ is non-increasing for u, the rank of projection P is r, $a_1, a_2, ..., a_n$ are arbitrary constants.

Proof. Since system (2.1) has a GED, from Theorem 3.3, we know there exists a nonnegative continuous function $\alpha(t)$, a constant $K\geq 1$ and n linearly independent solutions $x_1(t), x_2(t), ..., x_n(t)$ such that

$$\begin{cases} \|\sum_{i=1}^{r} a_i x_i(t)\| \leq K \|\sum_{i=1}^{r} a_i x_i(s)\| \exp\{-\int_s^t \alpha(\tau)d\tau\}, \quad t \geq s, \\ \|\sum_{i=r+1}^{n} a_i x_i(t)\| \leq K \|\sum_{i=r+1}^{n} a_i x_i(s)\| \exp\{\int_s^t \alpha(\tau)d\tau\}, \quad t \leq s, \end{cases}$$

where r is the rank of projection P, $a_1, a_2, ..., a_n$ are arbitrary constants, then

$$\limsup_{t\to-\infty} \|\sum_{i=1}^{r} a_i x_i(t)\| = \lim_{t\to-\infty} \|\sum_{i=1}^{r} a_i x_i(t)\| = +\infty,$$

$$\limsup_{t\to+\infty} \|\sum_{i=r+1}^{n} a_i x_i(t)\| = \lim_{t\to+\infty} \|\sum_{i=r+1}^{n} a_i x_i(t)\| = +\infty.$$

Now it is suffice to prove that there exists a nonnegative continuous function

$$\widetilde{\theta}(u) = \exp\{\int_u^{u+h} \widetilde{\kappa}(\tau)d\tau\},$$

and a constant $h > 0$, such that

$$\|\sum_{i=1}^{r} a_i x_i(t)\| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h} \|\sum_{i=1}^{r} a_i x_i(u)\|,$$

$$\|\sum_{i=r+1}^{n} a_i x_i(t)\| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h} \|\sum_{i=r+1}^{n} a_i x_i(u)\|.$$

For arbitrary $s \in \mathbb{R}$, take $t = s + h$, then

$$\|\sum_{i=1}^{r} a_i x_i(s+h)\| \leq K \|\sum_{i=1}^{r} a_i x_i(s)\| \exp\{-\int_s^{s+h} \alpha(\tau) d\tau\},$$

that is

$$\begin{aligned}\|\sum_{i=1}^{r} a_i x_i(s)\| &\geq K^{-1} \|\sum_{i=1}^{r} a_i x_i(s+h)\| \exp\{\int_s^{s+h} \alpha(\tau) d\tau\} \\ &\geq \widetilde{\theta}(u) \|\sum_{i=1}^{r} a_i x_i(s+h)\| \\ &\geq \widetilde{\theta}(u) \inf_{|u-s|\leq h} \|\sum_{i=1}^{r} a_i x_i(u)\|.\end{aligned}$$

Similarly, we can prove

$$\|\sum_{i=r+1}^{n} a_i x_i(s)\| \geq \widetilde{\theta}(u) \inf_{|u-s|\leq h} \|\sum_{i=r+1}^{n} a_i x_i(u)\|.$$

This completes the proof of Theorem 4.4. □

Theorem 4.5. *Suppose that system (2.1) has a GED, then there exist nonnegative continuous functions $\varrho(t)$, $\kappa(t)$, $\widetilde{\kappa}(t)$, a constant $h > 0$ such that n linearly independent solutions of system (2.1) $x_1(t), x_2(t), ..., x_n(t)$ satisfy*

$$\begin{cases} \|\sum_{i=1}^{r} a_i x_i(t)\| \leq \theta(u) \sup_{|u-t|\leq h} \|\sum_{i=1}^{r} a_i x_i(u)\|, \\ \|\sum_{i=r+1}^{n} a_i x_i(t)\| \leq \theta(u) \sup_{|u-t|\leq h} \|\sum_{i=r+1}^{n} a_i x_i(u)\|, \\ \|\sum_{i=1}^{r} a_i x_i(t)\| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h} \|\sum_{i=1}^{r} a_i x_i(u)\|, \\ \|\sum_{i=r+1}^{n} a_i x_i(t)\| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h} \|\sum_{i=r+1}^{n} a_i x_i(u)\|, \end{cases}$$

where $c(s)$ and μ have been defined in Definition 3.1, $\widetilde{\theta}(u) = \exp\{\int_u^{u+h} \widetilde{\kappa}(\tau) d\tau\}$ $\theta(u) = \exp\{-\int_u^{u+h} \kappa(\tau) d\tau\}$. Moreover, $\widetilde{\kappa}(\tau + u)$ and $\kappa(\tau + u)$ are non-increasing for u. Then linear system (2.1) has a GED, and the rank of projection P is r, $a_1, a_2, ..., a_n$ are arbitrary constants.

Proof. Because system (2.1) has a GED, from Theorem 4.4, there exist nonnegative continuous functions $\alpha(t)$, $\widetilde{\theta}(u) = \exp\{\int_u^{u+h} \widetilde{\kappa}(\tau) d\tau\}$, constants $h_1 > 0$, $K \geq 1$ and n linear independent solutions of (2.1) $x_1(t), x_2(t), ..., x_n(t)$ such that

$$\|\sum_{i=1}^{r} a_i x_i(t)\| \geq \widetilde{\theta}(u) \inf_{|u-t|\leq h_1} \|\sum_{i=1}^{r} a_i x_i(u)\|.$$

By Coppel [4], there is $h_2 > 0$, for any solution $x(t)$ of system (2.1) such that

$$\|x(t)\| \leq \theta(u) \sup_{|u-t|\leq h_2} \|x(u)\|.$$

So

$$\|\sum_{i=1}^{r} a_i x_i(t)\| \leq \theta(u) \sup_{|u-t|\leq h_2} \|\sum_{i=1}^{r} a_i x_i(u)\|.$$

Let $h = \max\{h_1, h_2\}$. Hence,

$$\begin{cases} \|\sum_{i=r+1}^{n} a_i x_i(t)\| \le \theta(u) \sup_{|u-t|\le h_2} \|\sum_{i=r+1}^{n} a_i x_i(u)\|, \\ \|\sum_{i=1}^{r} a_i x_i(t)\| \ge \widetilde{\theta}(u) \inf_{|u-t|\le h_1} \|\sum_{i=1}^{r} a_i x_i(u)\|. \end{cases}$$

Similarly, we can prove that

$$\begin{cases} \|\sum_{i=1}^{r} a_i x_i(t)\| \le \theta(u) \sup_{|u-t|\le h_2} \|\sum_{i=1}^{r} a_i x_i(u)\|, \\ \|\sum_{i=r+1}^{n} a_i x_i(t)\| \ge \widetilde{\theta}(u) \inf_{|u-t|\le h_1} \|\sum_{i=r+1}^{n} a_i x_i(u)\|. \end{cases}$$

This completes the proof of Theorem 4.5. □

Theorem 4.6. *Suppose that system (2.1) has a GED and system (2.1) is kinematically similar to system (2.2), then system (2.2) also has a GED.*

Proof. Let $Y(t)$ be a fundamental matrix of system (2.2). As system (2.1) has a GED, then there exists a projection P and $K > 0$ such that

$$\begin{cases} \|X(t)PX^{-1}(s)\| \le K \exp\{-\int_s^t \alpha(\tau)d\tau\}, & \text{for } t \ge s,\ s,t \in \mathbb{R}, \\ \|X(t)(I-P)X^{-1}(s)\| \le K \exp\{\int_s^t \alpha(\tau)d\tau\}, & \text{for } t \le s,\ s,t \in \mathbb{R}, \end{cases}$$

hold. Since system (2.1) is kinematically similar to system (2.2), from Lemma 2.1, we know there exists a Lyapunov transformation $y = S(t)x$ which can send system (2.1) into system (2.2), then for $t \ge s$, we have

$$\begin{aligned} \|Y(t)PY^{-1}(s)\| &= \|S(t)X(t)PX^{-1}(s)S^{-1}(t)\| \\ &\le \|S(t)\| \cdot \|X(t)PX^{-1}(s)\| \cdot \|S^{-1}(t)\| \\ &\le \|X(t)PX^{-1}(s)\| \\ &\le K \exp\{-\int_s^t \alpha(\tau)d\tau\}. \end{aligned}$$

Similarly, for $t \le s$, we have

$$\|Y(t)(I-P)Y^{-1}(s)\| \le K \exp\{\int_s^t \alpha(\tau)d\tau\}.$$

That is to say, system (2.2) also has a GED. This completes the proof of Theorem 4.6.

□

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