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SOME SPECIAL SOLUTIONS FOR DIVERGENCE STRUCTURE QUASILINEAR EQUATION

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Abstract We consider the divergence structure quasilinear equation

$$-\text{div}\vec{a}(\nabla u) = f(x, u, \nabla u),$$

which is not a variational equation. By applying the method of Galerkin approximation, we give some special solutions of the above equation.

Keywords Nonvariational elliptic equations, plane-like solutions, topological method.

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1. Introduction

The classical Aubry-Mather theory was established independently by Aubry [1] and Mather [9] when they respectively studied the plane Hamiltonian system and one dimensional Frenkel-Kontorova model. Moser [10] extended the Aubry-Mather theory for high dimension and set up the relationship between this theory and the variational problem on tori. The variational problem with integrand $F : \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n \ni (x, t, p) \mapsto F(x, t, p) \in \mathbb{R}^1$ is to look for a minimal solution $u : \mathbb{R}^n \to \mathbb{R}^1$ such that

$$\int_{\mathbb{R}^n} (F(x, u(x) + \varphi(x), \nabla(\varphi + u)(x)) - F(x, u(x), \nabla u(x)))dx \geq 0$$

for every C^1-function $\varphi : \mathbb{R}^n \to \mathbb{R}^1$ with compact support. The minimal solution of the variation problem satisfies the Euler-Lagrange equation:

$$\sum_{k=1}^{n} \frac{\partial}{\partial x_k} F_{p_k}(x, u, \nabla u) = F_u(x, u, \nabla u), \tag{1.1}$$

which is an elliptic equation under appropriate conditions on F. Moser [10] proved that for any $\omega \in \mathbb{R}^n$ there exists a classical minimal solution, denoted by u, with the following plane-like property:

$$u(x) - \omega \cdot x \in L^\infty(\mathbb{R}^n). \tag{1.2}$$

Here, ω is called the rotation vector of u. Moreover, Bangert continued studying the lamination of the minimal solutions with the same rotation vector in [2,3]. Related problems about minimal solutions satisfying (1.2) can be found in [5,6,11].

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When $F(x, u, \nabla u) = \frac{1}{2}|\nabla u|^2 + V(x, u)$, the equation (1.1) turns into

$$\triangle u = \partial_u V(x, u). \tag{1.3}$$

In [7], de la Llave and Valdinoci made use of the method of gradient semi-flow to get a solution satisfying (1.2),(1.3), which may not be a minimal solution. This method can be extended to an abstract formulation to study the pseudo-differential equations. See [7].

All the equations mentioned above are in the variational setting. Naturally it is interesting to investigate some non-variational equations. In that direction, Berti, Matzeu and Valdinoci [4] studied the equation:

$$\triangle u = f(x, u, \nabla u) \tag{1.4}$$

and stated that for all $\omega \in \mathbb{Z}^n$ there exists a special solution u such that

$$u(x) - \omega \cdot x \in H^2(\mathbb{T}^n).$$

The proof is based on a Galerkin approximation. There, the authors introduced a N-length cutoff on the Fourier coefficients to construct a vector field, and proved that the vector field has a zero denoted by $X^{(N)}$ when f fulfills some assumptions. Then the approximating sequence is written as follows

$$U_N(x) := \sum_{k \in S_N} X_k^{(N)} \sin(2\pi k \cdot x).$$

It is easy to check that every element of the sequence satisfies the corresponding approximating equation:

$$\triangle U_N = f_N(x), \tag{1.5}$$

where f_N is defined in [4] and we omit the specific form here. Then taking advantage of the elliptic L^2-estimates in equation (1.5), they obtained that $\| U_N \|_{H^2(\mathbb{T}^n)}$ are uniformly bounded. Hence there exist a subsequence, still denoted by U_N, and a function $U \in H^2(\mathbb{T}^n)$ such that U_N converges to U in $H^1(\mathbb{T}^n)$. Thus the left part of (1.5) weakly converges to $\triangle U$ in $L^2(\mathbb{T}^n)$. On the other hand, they also verified that the right part of (1.5) converges to $f(x, \omega \cdot x + U, \omega + \nabla U)$ in distributive sense by Vitali Convergence Theorem. Therefore, $\omega \cdot x + U$ is the weak solution of (1.4).

In this paper, we are concerned with the general divergence structure quasilinear elliptic equation:

$$-(\mathrm{div}\vec{a}(\nabla u))(x) = f(x, u(x), \nabla u(x)), \quad x \in \mathbb{R}^n, \tag{1.6}$$

where $\vec{a} : \mathbb{R}^n \ni x \mapsto (a^1(x), \cdots, a^n(x)) \in \mathbb{R}^n$ belongs to C^1 maps and $f : \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n \ni (x, t, p) \mapsto f(x, t, p) \in \mathbb{R}^1$ is a continuous function. Before giving our main theorem, we first make some assumptions on $\vec{a}$ and f.

H1): Suppose that f is 1-periodic and odd in (x, t). Moreover, suppose that there exist three positive constants K_0, K_1, K_2 such that

$$|f(x, t, p)| \leq K_0 + K_1|p| \tag{1.7}$$

and

$$|f(x, t, p) - f(x, t, q)| \leq K_2|p - q|. \tag{1.8}$$

We can easily find an example satisfying H1 as follows:

$$f(x,t,p) = cos(2\pi x_1)\cdots cos(2\pi x_n) sin(2\pi t)(1+\min\{K_1,K_2\}|p|).$$

H2): Suppose that $\vec{a}$ enjoys the following properties:

$$\text{a)}\quad |\vec{a}(p)| \leq K_3(1+|p|), \tag{1.9}$$

$$\text{b)}\quad \sum_{i,j} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{where } \lambda > 0,\ a_{ij} := \frac{\partial a^i}{\partial x_j}. \tag{1.10}$$

Here (1.10) implies that $\vec{a}$ satisfies the condition of strict monotonicity, i.e., there exists some $r > 0$ such that

$$(\vec{a}(p) - \vec{a}(q), p - q) \geq r|p-q|^2 \text{ for all } p, q \in \mathbb{R}^n.$$

Then we can obtain our main result which is similar to that of Theorem 1.1 in [4].

Theorem 1.1. *If f and $\vec{a}$ in equation (1.6) satisfy H1 and H2, then there exists a constant $c > 0$, depending on n, λ, such that when $K_1 \leq c$, for any $\omega \in \mathbb{Z}^n$, there exists $u \in H^2_{loc}(\mathbb{R}^n)$ which is a weak solution of equation (1.6). Set $U(x) := u(x) - \omega \cdot x$, then U has the following properties:*

$$U(x+l) = U(x) = U(-x) \quad \text{for any } x \in \mathbb{R}^n \text{ and } l \in \mathbb{Z}^n$$

and

$$\| U \|_{H^2(\mathbb{T}^n)} \leq C(K_0 + K_1|\omega|), \tag{1.11}$$

where C depends on n, λ.

When $n = 1$, we can get that $U \in L^\infty(\mathbb{R}^n)$. Therefore, we can get the plane-like solution of equation (1.6).

We remark that, our proof of the main theorem is also based on the method of Galerkin approximation. However, comparing our results with Berti's, we construct a new vector field and claim a strong convergence statement about the sequence f_N in the right side of equation (1.12). Besides, we adopt the monotonicity method to deal with the limit procedure in the divergence item of equation (1.6).

Finally, we give an outline of the proof. Firstly, we construct a new vector field (see (2.1)) and prove that it has a zero by Brouwers Fixed Point Theorem (see Lemma 2.3 and Lemma 2.4). Then an analogous approximating sequence can be constructed and satisfies

$$-\operatorname{div} \vec{a}(\nabla U_N(x) + \omega) = f_N(x). \tag{1.12}$$

Secondly, we prove the strong convergence statement, that is, f_N converges to $f(x, \omega\cdot x + U, \omega + \nabla U)$ in $L^2(\mathbb{T}^n)$, at the cost of an extra Lipschitz condition on the p-component of f. At last, we prove our main theorem using the monotonicity method.

2. Proof of our main theorem

In this section, firstly we will construct a finite dimension vector field by cutting off the Fourier coefficient and prove that there exists a zero of the vector field.

Given an odd function $\phi \in L^2(\mathbb{T}^n)$, then its Fourier expansion has the following form:

$$\phi(x) = \sum_{k \in \mathbb{Z}^n} \phi_k \sin(2\pi k \cdot x)$$

and we define the projection on its kth Fourier coefficient as $\Pi_k(\phi) := \phi_k$. It is clear that $\| \phi \|^2_{L^2(\mathbb{T}^n)} = \frac{1}{2} \sum_{k \in \mathbb{Z}^n} \phi_k^2$.

For an given $N \in \mathbb{N}$, we set

$$S_N := \{k \in \mathbb{Z}^n | 1 \leq |k_1| + \cdots + |k_n| \leq N\}$$

and we denote by $m(N)$ the cardinality of S_N.

Next we will construct the vector field $v^N : \mathbb{R}^{m(N)} \to \mathbb{R}^{m(N)}$.

Given $X = (X_1, \cdots, X_{m(N)}) \in \mathbb{R}^{m(N)}$, we define

$$g_X(x) := f(x, \omega \cdot x + \sum_{l \in S_N} X_l \sin(2\pi l \cdot x), \omega + 2\pi \sum_{l \in S_N} l X_l \cos(2\pi l \cdot x)).$$

We can easily get the conclusion that g_X is odd and 1-periodic by directly check. According to the inequality (1.7), one could also prove that $g_X \in L^2(\mathbb{T}^n)$. Moreover, the following lemma which can be found in [4] will provide an useful and precise estimate on the L^2-norm of g_X.

Lemma 2.1. *For any $X \in \mathbb{R}^{m(N)}$, g_X defined above and set $\omega_k(X) := \Pi_k(g_X)$, we have that*

$$\sum_{k \in \mathbb{Z}^n} |\omega_k(X)|^2 \leq C(K_0^2 + K_1^2 |\omega|^2 + K_1^2 \sum_{k \in S_N} |k|^2 X_k^2),$$

where C depend on only n.

Before giving the proof, we state that the constant C may change from line to line in the rest part of the present paper, but depends only on n, λ.

Proof. Fix $X \in \mathbb{R}^{m(N)}$, let us set $\eta_X(x) := \sum_{k \in S_N} k X_k \cos(2\pi k \cdot x)$, then we have

$$\| \eta_X \|^2_{L^2(\mathbb{T}^n)} \leq C \sum_{k \in S_N} |k|^2 X_k^2.$$

Also by (1.7), it is obvious to see

$$|g_X(x)| \leq C(K_0 + K_1|\omega| + K_1|\eta_X(x)|).$$

Therefore,

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} |\omega_k(X)|^2 &= 2 \| g_X \|^2_{L^2(\mathbb{T}^n)} \\ &\leq C(K_0^2 + K_1^2|\omega|^2 + K_1^2 \| \eta_X \|^2_{L^2(\mathbb{T}^n)}) \\ &\leq C(K_0^2 + K_1^2|\omega|^2 + K_1^2 \sum_{k \in S_N} |k|^2 X_k^2). \end{aligned}$$

□

In view of the above discussion, we know that $\omega_k(X)$ is well-defined for any $X \in \mathbb{R}^{m(N)}$ and $k \in \mathbb{Z}^n$, hence we can define the vector field's kth component as below:

$$v_k^N(X) := 4\pi^2 X_k \sum_{i=1}^n \sum_{j=1}^n a_j^i (2\pi \sum_{k \in S_N} k X_k \cos(2\pi k \cdot x) + \omega) k_i k_j - \omega_k(X). \quad (2.1)$$

Then we will prove that the vector field we constructed above has zeroes based on the lemma in [8, p493]. For the reader's convenience, we formulate the lemma below.

Lemma 2.2. *(Zeroes of a vector field). Assume the continuous function $v:\mathbb{R}^n\to\mathbb{R}^n$ satisfies*

$$v(x)\cdot x\ge 0 \quad \text{if} \quad |x|=r \tag{2.2}$$

for some $r>0$. Then there exists a point $x\in B(0,r)$ such that $v(x)=0$.

According to Lemma 2.2, we have to check that the vector field is continuous and also satisfies (2.2). And the two lemmata below state the corresponding claim respectively.

Lemma 2.3. *Fix any $N\in\mathbb{N}$, the vector field $v^N:\mathbb{R}^{m(N)}\to\mathbb{R}^{m(N)}$ is continuous.*

Proof. It is enough to prove $v_k^N:\mathbb{R}^{m(N)}\to\mathbb{R}$ to be continuous for any fixed $k\in S_N$. By the definition of v_k^N, we only have to prove that $\omega_k(\cdot):\mathbb{R}^{m(N)}\to\mathbb{R}$ is continuous. Assume that $\{X^M\}_{M\ge0}\subseteq\mathbb{R}^{m(N)}$ and $X^M\to X^0$ if $M\to\infty$, we get

$$g_{X^M}(x)=f(x,\omega\cdot x+\sum_{l\in S_N}X_l^M\sin(2\pi l\cdot x),\omega+2\pi\sum_{l\in S_N}lX_l^M\cos(2\pi l\cdot x))$$

and

$$g_{X^0}(x)=f(x,\omega\cdot x+\sum_{l\in S_N}X_l^0\sin(2\pi l\cdot x),\omega+2\pi\sum_{l\in S_N}lX_l^0\cos(2\pi l\cdot x)).$$

We clearly know that g_{X^M} converges to g_{X^0} pointwise when M converges to infinite. As we know that $\omega_k(X^M)$ is the kth Fourier coefficient of g_{X^M}, hence

$$\omega_k(X^M)=2\int_{\mathbb{T}^n}g_{X^M}(x)\sin(2\pi k\cdot x)dx, \qquad M\ge0. \tag{2.3}$$

By (1.7), for M large enough, we get that

$$\begin{aligned}|g_{X^M}(x)|&=|f(x,\omega\cdot x+\sum_{l\in S_N}X_l^M\sin(2\pi l\cdot x),\omega+2\pi\sum_{l\in S_N}lX_l^M\cos(2\pi l\cdot x))|\\&\le K_0+K_1(|\omega|+2\pi\sum_{l\in S_N}|l||X_l^M|)\\&\le K_0+K_1(|\omega|+2\pi\sum_{l\in S_N}|l|(|X_l^0|+1)).\end{aligned}$$

Thanks to the Lebesgue's Dominated Convergence Theorem and (2.3), we obtain

$$\omega_k(X^M)\to\omega_k(X^0) \quad \text{for any } k\in S_N.$$

□

Lemma 2.4. *Fix any $N\in\mathbb{N}$, there exists $R>0$ such that $v^N(X)\cdot X\ge0$ as long as $|X|\ge R$.*

Proof. Fix $X\in\mathbb{R}^{m(N)}$, we set

$$|X|_1:=\sqrt{\sum_{k\in S_N}|k|^2X_k^2}.$$

Combining the fact that

$$|X|_1 \geq \sqrt{\sum_{k\in S_N} X_k^2} = |X|$$

with the Lemma 2.1, we get

$$\sum_{k\in\mathbb{Z}^n} |\omega_k(X)|^2 \leq CK_1^2|X|_1^2,$$

as long as $|X| \geq R := \sqrt{\frac{K_0^2}{K_1^2} + |\omega|^2}$.

As a consequence,

$$\begin{aligned}
v^N(X)\cdot X &= \sum_{k\in S_N} v_k^N(X)X_k \\
&= \sum_{k\in S_N} 4\pi^2 X_k^2 \sum_{i=1}^n\sum_{j=1}^n a_j^i k_i k_j - \sum_{k\in S_N}\omega_k(X)X_k \\
&\geq 4\pi^2\lambda \sum_{k\in S_N} |k|^2 X_k^2 - \sqrt{\sum_{k\in S_N}|\omega_k(X)|^2}\sqrt{\sum_{k\in S_N} X_k^2} \\
&\geq 4\pi^2\lambda|X|_1^2 - CK_1|X|_1|X| \\
&\geq (4\pi^2\lambda - CK_1)|X|_1^2,
\end{aligned}$$

where the third line dues to formula (1.10). As a result, we conclude our claim if K_1 is small enough. □

Up to now, we can easily get a zero of the vector field based on above Lemmata, that is, for any $N\in\mathbb{N}$, there exists $X^{(N)}\in\mathbb{R}^{m(N)}$ in such a way that $v^N(X^{(N)}) = 0$. Then we can define the approximating sequence as follows:

$$U_N(x) := \sum_{k\in S_N} X_k^{(N)} \sin(2\pi k\cdot x).$$

It is obvious that $U_N \in C^\infty(\mathbb{T}^n)$ and U_N is odd. Besides, the fact that $v^N(X^{(N)}) = 0$ yields that

$$\begin{aligned}
&-\mathrm{div}\vec{a}(\nabla U_N(x) + \omega) \\
=&-\sum_{i=1}^n [a^i(2\pi\sum_{k\in S_N} kX_k^{(N)}\cos(2\pi k\cdot x) + \omega)]_{x_i} \\
=&\sum_{k\in S_N} 4\pi^2 X_k \sum_{i=1}^n\sum_{j=1}^n a_j^i(2\pi\sum_{k\in S_N} kX_k\cos(2\pi k\cdot x) + \omega)k_i k_j \sin(2\pi k\cdot x) \\
=&\sum_{k\in S_N}\omega_k(X^{(N)})\sin(2\pi k\cdot x) \\
=&(\Pi_N g_{X^{(N)}})(x),
\end{aligned}$$

where Π_N denotes the projection from $L^2(\mathbb{T}^n)$ to its subspace

$$\{g|g(x) = \sum_{l\in S_N} X_l\sin(2\pi l\cdot x), \quad X_l\in\mathbb{R}\}.$$

When N converges to infinite, $\Pi_N g \to g$ in $L^2(\mathbb{T}^n)$ for any odd function $g \in L^2(\mathbb{T}^n)$.

Now we have two questions: the first one is whether the approximating sequence has a subsequence U_{N_j} converging to some U in a sense. If U exists, the second one is whether we have

$$\mathrm{div}\vec{a}(\nabla U_{N_j} + \omega) \to \mathrm{div}\vec{a}(\nabla U + \omega) \tag{2.4}$$

and

$$\Pi_N g_{X^{(N)}} \to f(x, \omega \cdot x + U(x), \omega + \nabla U(x)), \tag{2.5}$$

in a weak sense.

The following lemma gives a positive answer to the first question.

Lemma 2.5. *Let U_N defined above, then we obtain*

$$\| U_N \|_{H^2(\mathbb{T}^n)} \leq C(K_0 + K_1|\omega|), \tag{2.6}$$

where C depends on n, λ not on N.

Proof. At first, we claim that if $v^N(X) = 0$, then

$$\sqrt{\sum_{k \in S_N} |k|^2 X_k^2} \leq C(K_0 + K_1|\omega|),$$

where C is independent on N. Indeed,

$$\begin{aligned}
4\pi^2 \lambda \sum_{k \in S_N} |k|^2 X_k^2 &\leq \sum_{k \in S_N} 4\pi^2 X_k \sum_{i=1}^{n} \sum_{j=1}^{n} a_j^i k_i k_j X_k \\
&= \sum_{k \in S_N} \omega_k(X) X_k \\
&\leq \sqrt{\sum_{k \in S_N} |\omega_k(X)|^2} \sqrt{\sum_{k \in S_N} X_k^2} \\
&\leq C(K_0 + K_1|\omega| + K_1 \sqrt{\sum_{k \in S_N} |k|^2 X_k^2}) \sqrt{\sum_{k \in S_N} |k|^2 X_k^2}.
\end{aligned}$$

Therefore,

$$\sqrt{\sum_{k \in S_N} |k|^2 X_k^2} \leq C(K_0 + K_1|\omega| + K_1 \sqrt{\sum_{k \in S_N} |k|^2 X_k^2}),$$

which concludes the claim if K_1 is small enough.

Since U_N is periodic with zero average, by Poincaré inequality, we have

$$\| U_N \|_{H^1(\mathbb{T}^n)} \leq C \| \nabla U_N \|_{L^2(\mathbb{T}^n)} \leq C \sqrt{\sum_{k \in S_N} |k|^2 (X_k^{(N)})^2} \leq C(K_0 + K_1|\omega|),$$

where the last inequality is induced by the claim.

Combining Lemma 2.1 and the claim, we also get

$$\begin{aligned}\| \Pi_N g_{X^{(N)}} \|_{L^2(\mathbb{T}^n)} &\leq \| g_{X^{(N)}} \|_{L^2(\mathbb{T}^n)} \\ &= \sqrt{\frac{1}{2}\sum_{k\in\mathbb{Z}^n} |\omega_k(X^{(N)})|^2} \\ &\leq C(K_0 + K_1|\omega| + K_1\sqrt{\sum_{k\in S_N} |k|^2 (X_k^{(N)})^2}) \\ &\leq C(K_0 + K_1|\omega|).\end{aligned}$$

By the knowledge of elliptic estimates (you can refer to the remark in [8, 498]) on the elliptic equations

$$-\operatorname{div}\vec{a}(\nabla U_N(x) + \omega) = (\Pi_N g_{X^{(N)}})(x), \tag{2.7}$$

ones know that

$$\| U_N \|_{H^2(\mathbb{T}^n)} \leq C(\| U_N \|_{L^2(\mathbb{T}^n)} + \| \Pi_N g_{X^{(N)}} \|_{L^2(\mathbb{T}^n)}) \leq C(K_0 + K_1|\omega|).$$

□

Then we can choose a subsequence still marked as U_N and a function $U \in H^2(\mathbb{T}^n)$, satisfying (1.11), such that

$$\nabla U_N \to \nabla U \qquad \text{and} \qquad U_N \to U \quad \text{in } L^2(\mathbb{T}^n).$$

We can also choose a subsequence from the sequence U_N, still denoted by U_N, such that U_N, resp. ∇U_N converges to U, resp. ∇U almost everywhere. So U is odd and 1-periodic while ∇U is even and 1-periodic. Finally, we will prove that U is our weak solution for equation (1.6).

Now we give the proof of (2.5).

We will give a strong L^2 convergence in (2.5). Set

$$g_N(x) := f(x, \omega\cdot x + U_N(x), \omega + \nabla U(x)) \text{ and } f_1(x) := f(x, \omega\cdot x + U(x), \omega + \nabla U(x)),$$

we obtain

$$\begin{aligned}&\int_{\mathbb{T}^n} |(\Pi_N g_{X^{(N)}})(x) - f(x, \omega\cdot x + U(x), \omega + \nabla U(x))|^2 dx \\ \leq\ & 3\int_{\mathbb{T}^n} |(\Pi_N g_{X^{(N)}})(x) - (\Pi_N g_N)(x)|^2 dx + 3\int_{\mathbb{T}^n} |(\Pi_N g_N)(x) - (\Pi_N f_1)(x)|^2 dx \\ &+ 3\int_{\mathbb{T}^n} |(\Pi_N f_1)(x) - f(x, \omega\cdot x + U(x), \omega + \nabla U(x))|^2 dx \\ \leq\ & 3\int_{\mathbb{T}^n} |g_{X^{(N)}}(x) - g_N(x)|^2 dx + 3\int_{\mathbb{T}^n} |g_N(x) - f_1(x)|^2 dx \\ &+ 3\int_{\mathbb{T}^n} |(\Pi_N f_1)(x) - f_1(x)|^2 dx \\ :=\ & I + II + III,\end{aligned}$$

where I, II, III respectively stand for the three parts of the right side of the second inequality. Then we will give some estimates of I, II, III.

By (1.8),

$$\begin{aligned} I &= 3\int_{\mathbb{T}^n} |g_{X^{(N)}}(x) - g_N(x)|^2 dx \\ &= 3\int_{\mathbb{T}^n} |f(x,\omega\cdot x + U_N(x), \omega + \nabla U_N(x)) - f(x,\omega\cdot x + U_N(x),\omega + \nabla U(x))|^2 dx \\ &\le 3\int_{\mathbb{T}^n} K_2^2 |\nabla U_N - \nabla U|^2 dx, \end{aligned}$$

which implies that I converges to zero when $N \to +\infty$.

As to II, combining the fact that g_N converges to f_1 almost everywhere and

$$|g_N(x) - f_1(x)|^2 \le 2(K_0 + K_1|\omega + \nabla U(x)|)^2,$$

we get that II converges to zero thanks to the Lebesgue's Dominated Convergence Theorem.

To estimate the third item, according to the fact that $\sin 2\pi k\cdot x,\ k \in \mathbb{Z}^n$ are an orthonormal and complete family within the space of all odd L^2-functions on $\mathbb{T}^n$, it is sufficient to show that $f_1 \in L^2(\mathbb{T}^n)$ and it is also odd. In fact, due to the condition (1.7), it is obvious to know that $f_1 \in L^2(\mathbb{T}^n)$. And recall that f is 1-periodic and odd in (x,t) in the assumption H1, we could check that f_1 is odd. Then we give the proof of (2.4). The proof will take advantage of the monotonicity methods introduced in the ninth chapter of [8].

By (1.9), $\{\vec{a}(\nabla U_N + \omega)\}_{N\ge 1}$ is bounded in $L^2(\mathbb{T}^n, \mathbb{R}^n)$. Hence we can suppose

$$\vec{a}(\nabla U_N + \omega) \rightharpoonup \xi$$

for some $\xi \in L^2(\mathbb{R}^n, \mathbb{R}^n)$.

Due to (2.7), for any $\phi \in H^1(\mathbb{T}^n)$, it is clear that

$$\int_{\mathbb{T}^n} \vec{a}(\nabla U_N + \omega)\cdot \nabla\phi dx = \int_{\mathbb{T}^n} (\Pi_N g_{X^{(N)}})\phi dx. \tag{2.8}$$

Let $N \to \infty$, combining with the convergence (2.5), we get

$$\int_{\mathbb{T}^n} \xi\cdot\nabla\phi dx = \int_{\mathbb{T}^n} f_1\phi dx. \tag{2.9}$$

In the following part, in view of (2.9), it is enough to show that $\xi = \vec{a}(\nabla U + \omega)$. Using the monotone condition on $\vec{a}$, it is easy to know that

$$\int_{\mathbb{T}^n} [\vec{a}(\nabla U_N + \omega) - \vec{a}(\nabla V + \omega)]\cdot(\nabla U_N - \nabla V)dx \ge 0 \tag{2.10}$$

for any $V \in H^1(\mathbb{T}^n)$ and $N \ge 1$.

Replace ϕ in (2.8) by U_N and substitute into (2.10), one gets

$$\int_{\mathbb{T}^n} [\Pi_N g_{X^{(N)}} U_N - \vec{a}(\nabla U_N + \omega)\cdot\nabla V - \vec{a}(\nabla V + \omega)\cdot(\nabla U_N - \nabla V)]dx \ge 0.$$

Let $N \to \infty$, we are aiming to obtain

$$\int_{\mathbb{T}^n} [f_1 U - \xi\cdot\nabla V - \vec{a}(\nabla V + \omega)\cdot(\nabla U - \nabla V)]dx \ge 0. \tag{2.11}$$

When taking limits above ($N \to +\infty$), the most difficult part is to show

$$\lim_{N\to+\infty}\int_{\mathbb{T}^n}\Pi_N g_{X^{(N)}}U_N dx = \int_{\mathbb{T}^n} f_1 U dx.$$

Indeed, using Lemma 2.5, we can get

$$\begin{aligned}
&\int_{\mathbb{T}^n}|(\Pi_N g_{X^{(N)}})(x)U_N(x) - f_1(x)U(x)|dx\\
\le&\int_{\mathbb{T}^n}|(\Pi_N g_{X^{(N)}})(x) - f_1(x)||U_N(x)|dx + \int_{\mathbb{T}^n}|f_1(x)||U_N(x) - U(x)|dx\\
\le&(\int_{\mathbb{T}^n}|(\Pi_N g_{X^{(N)}})(x) - f_1(x)|^2dx)^{\frac{1}{2}}(\int_{\mathbb{T}^n}|U_N(x)|^2dx)^{\frac{1}{2}}\\
&+(\int_{\mathbb{T}^n}|f_1(x)|^2dx)^{\frac{1}{2}}(\int_{\mathbb{T}^n}|U_N(x) - U(x)|^2dx)^{\frac{1}{2}} \quad \to 0.
\end{aligned}$$

By (2.9) (replace ϕ by U) and (2.11), we deduce

$$\int_{\mathbb{T}^n}(\xi - \vec{a}(\nabla V + \omega))\cdot(\nabla U - \nabla V)dx \ge 0 \qquad \text{for any } V \in H^1(\mathbb{T}^n). \tag{2.12}$$

Choose $V := U - rW(r > 0)$ in (2.12), we get

$$\int_{\mathbb{T}^n}(\xi - \vec{a}(\nabla U - r\nabla W + \omega))\cdot \nabla W dx \ge 0.$$

Let $r \to 0$, to find

$$\int_{\mathbb{T}^n}(\xi - \vec{a}(\nabla U + \omega))\cdot \nabla W dx \ge 0 \qquad \text{for any } W \in H^1(\mathbb{T}^n). \tag{2.13}$$

Replacing W by $-W$, in fact, the equality holds above. At last, due to (2.9) and (2.13), we conclude

$$\int_{\mathbb{T}^n}\vec{a}(\nabla U + \omega)\cdot\nabla W dx = \int_{\mathbb{T}^n} f_1 W dx \qquad \text{for any } W \in H^1(\mathbb{T}^n).$$

Finally, we have proved not only the statement (2.4) but also the main theorem.

Corollary 2.1. *When $n = 1$, $U \in L^\infty$.*

Proof. Since

$$\begin{aligned}
|U_N(x)| &= |\sum_{k\in S_N} X_k^{(N)}\sin(2\pi k\cdot x)|\\
&\le \sum_{k\in S_N}|X_k^{(N)}|\\
&\le \sum_{k\in S_N}|X_k^{(N)}||k|\frac{1}{|k|}\\
&\le (\sum_{k\in S_N}|k|^2|X_k^{(N)}|^2)^{\frac{1}{2}}(\sum_{k\in S_N}\frac{1}{|k|^2})^{\frac{1}{2}},
\end{aligned}$$

we obtain that U_N is uniformly bounded. Indeed, when $n = 1$, $\sum_{k\in S_N}\frac{1}{|k|^2}$ is convergent as $N \to +\infty$, while $\sum_{k\in S_N}|k|^2|X_k^{(N)}|^2$ is uniformly bounded (see Lemma 2.5). Therefore, U is bounded because of U_N's converging to U almost everywhere. □

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