

A LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION INVOLVING THE RATIO OF GAMMA FUNCTIONS*

Feng Qi^{1,2,3,†} and Wen-Hui Li³

Abstract In the paper, the authors concisely survey and review some functions involving the gamma function and its various ratios, simply state their logarithmically complete monotonicity and related results, and find necessary and sufficient conditions for a new function involving the ratio of two gamma functions and originating from the coding gain to be logarithmically completely monotonic.

Keywords Logarithmically completely monotonic function, ratio, gamma function, coding gain, necessary and sufficient condition.

MSC(2010) Primary 33B15; Secondary 26A48, 65R10.

1. Introduction

In [22, Appendix B], the function

$$h(x) = \frac{(2\sqrt{\pi})^{1/x} [\Gamma(x+1)]^{1/x}}{[\Gamma(x+1/2)]^{1/x}}, \quad (1.1)$$

which originated from a coding gain in [22, p. 3171, eq. (15)] and where the classical Euler gamma function Γ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

was proved to be logarithmically completely monotonic on $(0, \infty)$. An infinitely differentiable function f is said to be logarithmically completely monotonic on an interval I if inequalities

$$(-1)^k [\ln f(x)]^{(k)} \geq 0$$

hold on I for all $k \in \mathbb{N} = \{1, 2, \dots\}$. For more information about the notion “the logarithmically completely monotonic function”, please refer to [2, 3, 9, 37, 40, 44, 47, 53, 57] and closely related references therein.

[†]Corresponding author. Email address: qifeng618@gmail.com (F. Qi)

¹Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

²College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China

³Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

*The first author was supported in part by the National Natural Science Foundation of China (11361038).

In [42, 43], the function

$$\frac{[\Gamma(x + \alpha + 1)]^{1/(x+\alpha)}}{[\Gamma(x + 1)]^{1/x}} \tag{1.2}$$

was proved to be logarithmically completely monotonic on $(-1, \infty)$ if and only if $\alpha > 0$.

In the preprint [44] and its formally published version [45], the function

$$\frac{[\Gamma(x + 1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x \tag{1.3}$$

was verified to be logarithmically completely monotonic on $(0, \infty)$. This result was promptly strengthened in [3] to the function (1.3) being a Stieltjes transform. A Stieltjes transform is a function $f : (0, \infty) \rightarrow [0, \infty)$ which can be written in the form

$$f(x) = \frac{a}{x} + b + \int_0^\infty \frac{1}{s+x} d\mu(s),$$

where a, b are nonnegative constants and μ is a nonnegative measure on $(0, \infty)$ such that

$$\int_0^\infty \frac{1}{1+s} d\mu(s) < \infty.$$

More generally, the inclusions

$$\mathcal{L}[I] \subset \mathcal{C}[I] \quad \text{and} \quad \mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0, \infty)] \tag{1.4}$$

were discovered in [3, 9, 37, 40], where \mathcal{S} , $\mathcal{L}[I]$, and $\mathcal{C}[I]$ denote respectively the set of all Stieltjes transforms, the set of all logarithmically completely monotonic functions on an interval I , and the set of all completely monotonic functions on I . An infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies

$$(-1)^i f^{(i)}(x) \geq 0$$

on I for all $i \in \{0\} \cup \mathbb{N}$. In the literature, we call (1.4) Qi-Berg’s inclusions.

Some properties of the function $[\Gamma(x + 1)]^{1/x}$ and its logarithm can be found in [5, 40, 43] and tightly related references therein.

In [46, 48], the monotonicity of the function

$$G_{s,t}(x) = \frac{[\Gamma(1 + tx)]^s}{[\Gamma(1 + sx)]^t}, \tag{1.5}$$

for $x, s, t \in \mathbb{R}$ such that $1 + sx > 0$ and $1 + tx > 0$ with $s \neq t$, and its general form

$$g_{a,b}(x) = \frac{[f(bx)]^a}{[f(ax)]^b},$$

for $ax \in I$ and $bx \in I$, where a and b are two real numbers and $f(x)$ is a positive function on an interval I , are investigated. For a much complete survey of this topic, please read [27, pp. 73–76, Section 7.6]. Let s and t be two real numbers with $s \neq t$, $\alpha = \min\{s, t\}$ and $\beta > -\alpha$. For $x \in (-\alpha, \infty)$, define

$$h_\beta(x) = \begin{cases} \left[\frac{\Gamma(\beta + t)}{\Gamma(\beta + s)} \cdot \frac{\Gamma(x + s)}{\Gamma(x + t)} \right]^{1/(x-\beta)}, & x \neq \beta, \\ \exp[\psi(\beta + s) - \psi(\beta + t)], & x = \beta. \end{cases} \tag{1.6}$$

For $x \in (0, \infty)$ and $\alpha > 0$, let

$$p_\alpha(x) = \begin{cases} \left[\frac{\Gamma(\alpha+1)}{\alpha^\alpha} \cdot \frac{x^x}{\Gamma(x+1)} \right]^{1/(\alpha-x)}, & x \neq \alpha, \\ \frac{\exp[\psi(\alpha+1) - 1]}{\alpha}, & x = \alpha. \end{cases}$$

The logarithmically complete monotonicity of the functions $h_\beta(x)$ and $p_\alpha(x)$ has been studied in [34, 55]. A special case of the function (1.6) came from problems of traffic flow. For more details, please see [27, pp. 63–66, Section 6.5].

For given $y \in (-1, \infty)$ and $\alpha \in (-\infty, \infty)$, let

$$h_{\alpha,y}(x) = \begin{cases} \frac{1}{(x+y+1)^\alpha} \left[\frac{\Gamma(x+y+1)}{\Gamma(y+1)} \right]^{1/x}, & x \in (-y-1, \infty) \setminus \{0\}, \\ \frac{1}{(y+1)^\alpha} \exp[\psi(y+1)], & x = 0. \end{cases}$$

Some inequalities and some necessary and sufficient conditions for the function $h_{\alpha,y}(x)$ and its special cases to be logarithmically completely monotonic were provided in [10, 11, 38, 57, 62, 63] and many other references listed therein.

Let s and t be two real numbers and $\alpha = \min\{s, t\}$. For $x \in (-\alpha, \infty)$, define

$$\Psi_{s,t}(x) = \begin{cases} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)}, & s \neq t, \\ e^{\psi(x+s)}, & s = t. \end{cases} \quad (1.7)$$

There have been a large amount of literature devoted to inequalities, logarithmically complete monotonicity, asymptotic expansions, applications, and the like, of functions related to the function (1.7). The work in this field, dating back to 1948, has been surveyed and reviewed in [27, 28, 50, 51]. Recently, some new results on inequalities and logarithmically complete monotonicity of functions related to (1.7) were obtained in [6, 36, 39, 41, 53].

We may classify newly published papers relating to the gamma and polygamma functions and to the logarithmically complete monotonicity into several groups below.

1. Some papers having something to do with the unit ball in \mathbb{R}^n are [6, 12, 13, 41, 57, 64].
2. Some papers on asymptotic expansions and complete monotonicity for the gamma function Γ or for its ratio such as (1.7) are [4, 7, 16, 18–21, 25, 31, 52, 54].
3. A series of papers on the complete monotonicity of the function $e^{1/t} - \psi'(t)$ and its variants are [15, 23, 33, 35, 56].
4. Some papers on the notion “completely monotonic degree” and its computation are [8, 14, 33, 56, 58].
5. Some papers related to the function $[\psi'(x)]^2 + \psi''(x)$, its divided difference forms, and their q -analogues are [6, 17, 24, 29, 30, 36, 41, 53, 65].
6. Some applications of the (logarithmically) complete monotonicity to number theory and mean values are published in [14, 26, 32, 59–61].

After concisely surveying and reviewing, now let us return to the function $h(x)$ in (1.1). It is easy to see that we may rearrange the function $h(x)$ as the form

$$h(x) = \frac{(2\sqrt{\pi})^{1/x} [\Gamma(x+1)]^{1/x}}{[\Gamma(x+1/2)]^{1/x}} = \left[\frac{2\sqrt{\pi} \Gamma(x+1)}{\Gamma(x+1/2)} \right]^{1/x}.$$

It is not difficult to see that the function $h(x)$ is not a special case of any functions in (1.2) and (1.5) to (1.7). On the other hand, the function $h(x)$ may be written in a general form

$$h_{a,b;c}(x) = \left[c \frac{\Gamma(x+a)}{\Gamma(x+b)} \right]^{1/x}, \tag{1.8}$$

where $a, b, c > 0$ and $x \in (0, \infty)$.

The aim of this paper is to find necessary and sufficient conditions on a, b, c such that the function $h_{a,b;c}(x)$ or its reciprocal is logarithmically completely monotonic on $(0, \infty)$.

Our main results may be stated as the following theorem.

Theorem 1.1. Let $a, b, c > 0$.

1. When $a > b$, the function $h_{a,b;c}(x)$ defined by (1.8) is logarithmically completely monotonic on $(0, \infty)$ if and only if $c \geq \frac{\Gamma(b)}{\Gamma(a)}$.
2. When $a < b$, the reciprocal of $h_{a,b;c}(x)$ is logarithmically completely monotonic on $(0, \infty)$ if and only if $c \leq \frac{\Gamma(b)}{\Gamma(a)}$.

Remark 1.1. It is clear that the main result in [22, Appendix B] is a special case of Theorem 1.1 for $a = 1, b = \frac{1}{2}$, and $c = 2\sqrt{\pi} > \frac{\Gamma(1/2)}{\Gamma(1)} = \sqrt{\pi}$.

2. Proof of Theorem 1.1

Taking the logarithm of $h_{a,b;c}(x)$ and differentiating give

$$\ln h_{a,b;c}(x) = \frac{\ln c + \ln \Gamma(x+a) - \ln \Gamma(x+b)}{x}$$

and

$$\begin{aligned} [\ln h_{a,b;c}(x)]^{(k)} &= \sum_{i=0}^k \binom{k}{i} \left(\frac{1}{x}\right)^{(k-i)} [\ln c + \ln \Gamma(x+a) - \ln \Gamma(x+b)]^{(i)} \\ &= \frac{(-1)^k k!}{x^{k+1}} [\ln c + \ln \Gamma(x+a) - \ln \Gamma(x+b)] \\ &\quad + \sum_{i=1}^k \binom{k}{i} \frac{(-1)^{k-i} (k-i)!}{x^{k-i+1}} [\psi^{(i-1)}(x+a) - \psi^{(i-1)}(x+b)] \\ &= \frac{(-1)^k k!}{x^{k+1}} \left\{ \ln c + \ln \Gamma(x+a) - \ln \Gamma(x+b) \right. \\ &\quad \left. + \sum_{i=1}^k \frac{(-1)^i x^i}{i!} [\psi^{(i-1)}(x+a) - \psi^{(i-1)}(x+b)] \right\} \\ &\triangleq \frac{(-1)^k k!}{x^{k+1}} H_{a,b;c;k}(x), \end{aligned}$$

for $k \in \mathbb{N}$. It is easy to see that

$$H_{a,b;c;k}(0) = \ln c + \ln \Gamma(a) - \ln \Gamma(b) = \ln c - \ln \frac{\Gamma(b)}{\Gamma(a)}$$

and

$$\begin{aligned} H'_{a,b;c;k}(x) &= \frac{(-1)^k x^k}{k!} [\psi^{(k)}(x+a) - \psi^{(k)}(x+b)] \\ &= \frac{x^k}{k!} [(-1)^k \psi^{(k)}(x+a) - (-1)^k \psi^{(k)}(x+b)]. \end{aligned}$$

Since

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1-e^{-t}} e^{-zt} dt,$$

for $\Re(z) > 0$ and $n \in \mathbb{N}$, see [1, p. 260, 6.4.1], the function $(-1)^k \psi^{(k)}(x)$ is increasing on $(0, \infty)$ for $k \in \mathbb{N}$. Hence, when $a > b$, the derivative $H'_{a,b;c;k}(x)$ is positive on $(0, \infty)$ for $k \in \mathbb{N}$. This means that, when $a > b$, the function $H_{a,b;c;k}(x)$ is increasing on $(0, \infty)$ for $k \in \mathbb{N}$. Therefore, when $a > b$ and $c \geq \frac{\Gamma(b)}{\Gamma(a)}$, the function $H_{a,b;c;k}(x)$ is positive on $(0, \infty)$ for $k \in \mathbb{N}$. Consequently, when $a > b$ and $c \geq \frac{\Gamma(b)}{\Gamma(a)}$, it follows that

$$(-1)^k [\ln h_{a,b;c}(x)]^{(k)} = \frac{k!}{x^{k+1}} H_{a,b;c;k}(x) \geq 0, \quad (2.1)$$

for $k \in \mathbb{N}$ on $(0, \infty)$, that is, the function $h_{a,b;c}(x)$ defined by (1.8) is logarithmically completely monotonic.

Similarly, when $a < b$, the derivative $H'_{a,b;c;k}(x)$ is negative and the function $H_{a,b;c;k}(x)$ is decreasing on $(0, \infty)$ for $k \in \mathbb{N}$. Therefore, when $a < b$ and $c \leq \frac{\Gamma(b)}{\Gamma(a)}$, the function $H_{a,b;c;k}(x)$ is negative and the inequality (2.1) is reversed on $(0, \infty)$ for $k \in \mathbb{N}$. This implies that the reciprocal of the function $h_{a,b;c}(x)$ defined by (1.8) is logarithmically completely monotonic.

Conversely, if the function $h_{a,b;c}(x)$ is logarithmically completely monotonic, then, by definition, its first logarithmic derivative is negative, that is,

$$\ln \left[\frac{c\Gamma(a+x)}{\Gamma(b+x)} \right] - x[\psi(a+x) - \psi(b+x)] \geq 0.$$

Taking $x \rightarrow 0^+$ in the above inequality yields

$$\ln \left[\frac{c\Gamma(a)}{\Gamma(b)} \right] \geq 0,$$

which implies the required necessary and sufficient conditions. The proof of Theorem 1.1 is complete.

Remark 2.1. It is apparent that the proof of Theorem 1.1 is slightly simpler than the proof in [22, Appendix B] and that Theorem 1.1 generalizes the result obtained in [22, Appendix B].

Remark 2.2. The techniques in the proof of Theorem 1.1 and [22, Appendix B] were ever appeared in [43].

Remark 2.3. This paper is a slightly modified version of the preprint [49].

References

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, 10th printing, Washington, 1972.
- [2] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci., 41(2)(1988), 21–23.
- [3] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math., 1(4)(2004), 433–439.
- [4] T. Burić and N. Elezović, *New asymptotic expansions of the quotient of gamma functions*, Integral Transforms Spec. Funct., 23(5)(2012), 355–368.
- [5] M. W. Coffey, *Fractional part integral representation for derivatives of a function related to $\ln \Gamma(x + 1)$* , Publ. Math. Debrecen, 80(3-4)(2012), 347–358.
- [6] B.-N. Guo and F. Qi, *A class of completely monotonic functions involving divided differences of the psi and tri-gamma functions and some applications*, J. Korean Math. Soc., 48(3)(2011), 655–667.
- [7] B.-N. Guo and F. Qi, *A class of completely monotonic functions involving the gamma and polygamma functions*, Cogent Math., 1(2014), 1:982896, 8 pages.
- [8] B.-N. Guo and F. Qi, *A completely monotonic function involving the tri-gamma function and with degree one*, Appl. Math. Comput., 218(19)(2012), 9890–9897.
- [9] B.-N. Guo and F. Qi, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 72(2)(2010), 21–30.
- [10] B.-N. Guo and F. Qi, *An extension of an inequality for ratios of gamma functions*, J. Approx. Theory, 163(9)(2011), 1208–1216.
- [11] B.-N. Guo and F. Qi, *Inequalities and monotonicity for the ratio of gamma functions*, Taiwanese J. Math., 7(2)(2003), 239–247.
- [12] B.-N. Guo and F. Qi, *Monotonicity and logarithmic convexity relating to the volume of the unit ball*, Optim. Lett., 7(6)(2013), 1139–1153.
- [13] B.-N. Guo and F. Qi, *Monotonicity of functions connected with the gamma function and the volume of the unit ball*, Integral Transforms Spec. Funct., 23(9)(2012), 701–708.
- [14] B.-N. Guo and F. Qi, *On the degree of the weighted geometric mean as a complete Bernstein function*, Afr. Mat., 26(2015), <http://dx.doi.org/10.1007/s13370-014-0279-2>.
- [15] B.-N. Guo and F. Qi, *Refinements of lower bounds for polygamma functions*, Proc. Amer. Math. Soc., 141(3)(2013), 1007–1015.
- [16] B.-N. Guo and F. Qi, *Sharp inequalities for the psi function and harmonic numbers*, Analysis (Berlin), 34(2)(2014), 201–208.
- [17] B.-N. Guo, J.-L. Zhao, and F. Qi, *A completely monotonic function involving the tri- and tetra-gamma functions*, Math. Slovaca, 63(3)(2013), 469–478.

- [18] S. Koumandos, *On Ruijsenaars' asymptotic expansion of the logarithm of the double gamma function*, J. Math. Anal. Appl., 341(2008), 1125–1132.
- [19] S. Koumandos and M. Lamprecht, *Complete monotonicity and related properties of some special functions*, Math. Comp., 82(282)(2013), 1097–1120.
- [20] S. Koumandos and H. L. Pedersen, *Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function*, J. Math. Anal. Appl., 355(1)(2009), 33–40.
- [21] V. Krasniqi and F. Qi, *Complete monotonicity of a function involving the p -psi function and alternative proofs*, Glob. J. Math. Anal., 2(3)(2014), 204–208.
- [22] J. Lee and C. Tepedelenlioglu, *Space-time coding over fading channels with stable noise*, IEEE Transactions on Vehicular Technology, 60(7)(2011), 3169–3177.
- [23] W.-H. Li, F. Qi, and B.-N. Guo, *On proofs for monotonicity of a function involving the psi and exponential functions*, Analysis (Munich), 33(1)(2013), 45–50.
- [24] F. Qi, *A completely monotonic function involving the divided difference of the psi function and an equivalent inequality involving sums*, ANZIAM J., 48(4)(2007), 523–532.
- [25] F. Qi, *A completely monotonic function related to the q -trigamma function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 76(1)(2014), 107–114.
- [26] F. Qi, *An integral representation, complete monotonicity, and inequalities of Cauchy numbers of the second kind*, J. Number Theory, 144(2014), 244–255.
- [27] F. Qi, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl., 2010(2010), Article ID 493058, 84 pages.
- [28] F. Qi, *Bounds for the ratio of two gamma functions: from Gautschi's and Kershaw's inequalities to complete monotonicity*, Turkish J. Anal. Number Theory, 2(5)(2014), 152–164.
- [29] F. Qi, *Complete monotonicity of functions involving the q -trigamma and q -tetragamma functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM., 109(2015), <http://dx.doi.org/10.1007/s13398-014-0193-3>.
- [30] F. Qi, *Complete monotonicity of a function involving the tri- and tetra-gamma functions*, Proc. Jangjeon Math. Soc., 18(2)(2015), 253–264.
- [31] F. Qi, *Integral representations and complete monotonicity related to the remainder of Burnside's formula for the gamma function*, J. Comput. Appl. Math., 268(2014), 155–167.
- [32] F. Qi, *Integral representations and properties of Stirling numbers of the first kind*, J. Number Theory, 133(7)(2013), 2307–2319.
- [33] F. Qi, *Properties of modified Bessel functions and completely monotonic degrees of differences between exponential and trigamma functions*, Math. Inequal. Appl., 18(2)(2015), 493–518.
- [34] F. Qi, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, Integral Transforms Spec. Funct., 18(7)(2007), 503–509.

- [35] F. Qi and C. Berg, *Complete monotonicity of a difference between the exponential and trigamma functions and properties related to a modified Bessel function*, *Mediterr. J. Math.*, 10(4)(2013), 1685–1696.
- [36] F. Qi, P. Cerone, and S. S. Dragomir, *Complete monotonicity of a function involving the divided difference of psi functions*, *Bull. Aust. Math. Soc.*, 88(2)(2013), 309–319.
- [37] F. Qi and C.-P. Chen, *A complete monotonicity property of the gamma function*, *J. Math. Anal. Appl.*, 296(2004), 603–607.
- [38] F. Qi and B.-N. Guo, *A logarithmically completely monotonic function involving the gamma function*, *Taiwanese J. Math.*, 14(4)(2010), 1623–1628.
- [39] F. Qi and B.-N. Guo, *Complete monotonicity of divided differences of the di- and tri-gamma functions and applications*, *Georgian Math. J.*, 22(2015), in press.
- [40] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, *RGMI Res. Rep. Coll.*, 7(1)(2004), Art. 8, 63–72.
- [41] F. Qi and B.-N. Guo, *Completely monotonic functions involving divided differences of the di- and tri-gamma functions and some applications*, *Commun. Pure Appl. Anal.*, 8(6)(2009), 1975–1989.
- [42] F. Qi and B.-N. Guo, *Monotonicity and convexity of ratio between gamma functions to different powers*, *J. Indones. Math. Soc. (MIHMI)*, 11(1)(2005), 39–49.
- [43] F. Qi and B.-N. Guo, *Some logarithmically completely monotonic functions related to the gamma function*, *J. Korean Math. Soc.*, 47(6)(2010), 1283–1297.
- [44] F. Qi, B.-N. Guo, and C.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, *RGMI Res. Rep. Coll.*, 7(1)(2004), Art. 5, 31–36.
- [45] F. Qi, B.-N. Guo, and C.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, *J. Aust. Math. Soc.*, 80(2006), 81–88.
- [46] F. Qi, B.-N. Guo, S. Guo, and S.-X. Chen, *A function involving gamma function and having logarithmically absolute convexity*, *Integral Transforms Spec. Funct.*, 18(11)(2007), 837–843.
- [47] F. Qi, S. Guo, and B.-N. Guo, *Complete monotonicity of some functions involving polygamma functions*, *J. Comput. Appl. Math.*, 233(9)(2010), 2149–2160.
- [48] F. Qi, S. Guo, B.-N. Guo, and S.-X. Chen, *A class of k -log-convex functions and their applications to some special functions*, *Integral Transforms Spec. Funct.*, 19(3)(2008), 195–200.
- [49] F. Qi and W.-H. Li, *A logarithmically completely monotonic function involving the ratio of two gamma functions and originating from the coding gain*, <http://arxiv.org/abs/1303.1877>.
- [50] F. Qi and Q.-M. Luo, *Bounds for the ratio of two gamma functions: from Wendel's asymptotic relation to Elezović-Giordano-Pečarić's theorem*, *J. Inequal. Appl.*, 2013, 2013:542, 20 pages; Available online at <http://dx.doi.org/10.1186/1029-242X-2013-542>.

- [51] F. Qi and Q.-M. Luo, *Bounds for the ratio of two gamma functions—From Wendel’s and related inequalities to logarithmically completely monotonic functions*, Banach J. Math. Anal., 6(2)(2012), 132–158.
- [52] F. Qi and Q.-M. Luo, *Complete monotonicity of a function involving the gamma function and applications*, Period. Math. Hungar., 69(2)(2014), 159–169.
- [53] F. Qi, Q.-M. Luo, and B.-N. Guo, *Complete monotonicity of a function involving the divided difference of digamma functions*, Sci. China Math., 56(11)(2013), 2315–2325.
- [54] F. Qi, Q.-M. Luo, and B.-N. Guo, *The function $(b^x - a^x)/x$: Ratio’s properties*, In: *Analytic number theory, approximation theory, and special functions*, G. V. Milovanović and M. Th. Rassias (Eds), Springer, 2014, 485–494.
- [55] F. Qi, D.-W. Niu, J. Cao, and S.-X. Chen, *Four logarithmically completely monotonic functions involving gamma function*, J. Korean Math. Soc., 45(2)(2008), 559–573.
- [56] F. Qi and S.-H. Wang, *Complete monotonicity, completely monotonic degree, integral representations, and an inequality related to the exponential, trigamma, and modified Bessel functions*, Glob. J. Math. Anal., 2(3)(2014), 91–97.
- [57] F. Qi, C.-F. Wei, and B.-N. Guo, *Complete monotonicity of a function involving the ratio of gamma functions and applications*, Banach J. Math. Anal., 6(1)(2012), 35–44.
- [58] F. Qi and X.-J. Zhang, *Complete monotonicity of a difference between the exponential and trigamma functions*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math., 21(2)(2014), 141–145.
- [59] F. Qi, X.-J. Zhang, and W.-H. Li, *An integral representation for the weighted geometric mean and its applications*, Acta Math. Sin. (Engl. Ser.), 30(1)(2014), 61–68.
- [60] F. Qi, X.-J. Zhang, and W.-H. Li, *Lévy-Khintchine representation of the geometric mean of many positive numbers and applications*, Math. Inequal. Appl., 17(2)(2014), 719–729.
- [61] F. Qi, X.-J. Zhang, and W.-H. Li, *Lévy-Khintchine representations of the weighted geometric mean and the logarithmic mean*, Mediterr. J. Math., 11(2)(2014), 315–327.
- [62] Y.-M. Yu, *An inequality for ratios of gamma functions*, J. Math. Anal. Appl., 352(2)(2009), 967–970.
- [63] T.-H. Zhao, Y.-M. Chu, and Y.-P. Jiang, *Monotonic and logarithmically convex properties of a function involving gamma functions*, J. Inequal. Appl., 2009(2009), Article ID 728612, 13 pages.
- [64] J.-L. Zhao, B.-N. Guo, and F. Qi, *A refinement of a double inequality for the gamma function*, Publ. Math. Debrecen, 80(3-4)(2012), 333–342.
- [65] J.-L. Zhao, B.-N. Guo, and F. Qi, *Complete monotonicity of two functions involving the tri- and tetra-gamma functions*, Period. Math. Hungar., 65(1)(2012), 147–155.