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ON THE PROPERTIES OF A CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS *

Wenjuan Li and Qinghua Xu†

Abstract In the present paper, we introduce an interesting subclass $\mathcal{K}_s^p(h)$ of analytic functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{K}_s^p(h)$, basic properties such as the coefficient bounds, the distortion and growth theorems are derived. The results presented here would provide extensions of those given by Q.-H. Xu et al. [2].

Keywords Starlike functions, coefficient bounds, distortion and growth theorems, subordination principle.

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1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers, $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of positive integers, $\mathbb{N}_o = \{1, 3, 5, ...\}$ be the set of odd numbers and $\mathbb{N}_e = \{2, 4, 6, ...\}$ be the set of even numbers. We also let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{S}_p$ the subclass of $\mathcal{A}(p)$, consisting of all univalent functions in $\mathcal{A}(p)$. A function $f(z)$ in $\mathcal{A}(p)$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies (see [4]):

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \le \alpha < p;\ p \in \mathbb{N})$$

or equivalently

$$\frac{zf'(z)}{f(z)} \prec \frac{p + (p - 2\alpha)z}{1 - z} \quad (z \in \mathbb{U};\ 0 \le \alpha < p;\ p \in \mathbb{N}),$$

for some real α $(0 \le \alpha < p)$. We denote by $\mathcal{S}_p^*(\alpha)$ the subclass of $\mathcal{A}(p)$ consisting of all starlike functions of order α in $\mathbb{U}$.

†the corresponding author. Email address: xuqhjxsd@163.com(Q. Xu)
College of Mathematics and Information Science, JiangXi Normal University, NanChang 330022, P. R. China

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Definition 1.1. Let the function $f(z)$ be analytic in $\mathbb{U}$ and defined by (1.1). We say that $f \in \mathcal{K}_s^p$ if there exists a function $g \in \mathcal{S}^*(\frac{1}{2})$ such that

$$\Re\left(-\frac{z^2 f'(z)}{z^{p-1}\cdot g(z)\cdot g(-z)}\right) > 0 \quad (z \in \mathbb{U};\ p \in \mathbb{N}).$$

Remark 1.1. For $p = 1$, we obtain the class $\mathcal{K}_s^1 = \mathcal{K}_s$, studied by Gao and Zhou [6].

Definition 1.2. Let the function $f(z)$ be analytic in $\mathbb{U}$ and defined by (1.1). We say that $f \in \mathcal{K}_s^p(\alpha)$ $(0 \leq \alpha < p)$ if there exists a function $g \in \mathcal{S}^*(\frac{1}{2})$ such that

$$\Re\left(-\frac{z^2 f'(z)}{z^{p-1}\cdot g(z)\cdot g(-z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leq \alpha < p;\ p \in \mathbb{N}).$$

Remark 1.2. For $p = 1$, we obtain the class $\mathcal{K}_s^1(\alpha) = \mathcal{K}_s(\alpha)$, proved by Kowalczyk and Leś -Bomba [7].

In many earlier investigations, various interesting subclasses of the analytic function class $\mathcal{A}(\mathcal{A}(1) = \mathcal{A})$ has been studied from a number of different viewpoint. We choose to recall here the investigations by(for example) Altintas etc. [1, 2], Breaz etc.[3], Owa etc.[9], Robertson[10], Srivastava and Owa [13].

Here, on the basis of the above-cited works (especially [6, 7]), we introduce the following subclasses of analytic functions.

Definition 1.3. Let

$$h : \mathbb{U} \to \mathbb{C}$$

be a convex function such that

$$h(0) = p \text{ and } h(\overline{z}) = \overline{h(z)} \quad (z \in \mathbb{U};\ \Re(h(z)) > 0).$$

Suppose also that the function h satisfies the following conditions for $r \in (0, 1)$:

$$\begin{aligned} \min_{|z|=r} |h(z)| &= \min\{h(r), h(-r)\} \quad (0 < r < 1), \\ \max_{|z|=r} |h(z)| &= \max\{h(r), h(-r)\} \quad (0 < r < 1). \end{aligned} \tag{1.2}$$

Let the function $f(z)$ be analytic in $\mathbb{U}$ and defined by (1.1). We say that $f \in \mathcal{K}_s^p(h)$ if there exists a function $g \in \mathcal{S}^*(\frac{1}{2})$ such that

$$-\frac{z^2 f'(z)}{z^{p-1}\cdot g(z)\cdot g(-z)} \in h(\mathbb{U}) \quad (z \in \mathbb{U};\ p \in \mathbb{N}). \tag{1.3}$$

In particular, for $p = 1$, we obtain the class $\mathcal{K}_s^1(h) = \mathcal{K}_s(h)$, introduced by Xu etc. [14].

Remark 1.3. Various special cases of the functions h would provide interesting subclasses of analytic functions. For the case

$$h(z) = \frac{p + (p - 2\alpha)z}{1 - z} \quad (z \in \mathbb{U}; 0 \leq \alpha < p; p \in \mathbb{N}),$$

we easily verified that h is a convex function in $\mathbb{U}$ and satisfies the hypotheses of Definition 1.3. If $f \in \mathcal{K}_s^p(h)$, then

$$\Re\left(-\frac{z^2 f'(z)}{z^{p-1}\cdot g(z)\cdot g(-z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \le \alpha < p;\ p \in \mathbb{N}),$$

that is, $f \in \mathcal{K}_s^p(\alpha)$. In particular, if we let

$$h(z) = p\frac{1+z}{1-z}$$

in Definition 1.3, then $f \in \mathcal{K}_s^p$.

Definition 1.4. (see [8,12]). For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\varpi(z)$, analytic in $\mathbb{U}$ with

$$\varpi(0) = 0 \text{ and } |\varpi(z)| < 1 \ \ (z \in \mathbb{U}),$$

such that

$$f(z) = g(\varpi(z)) \quad (z \in \mathbb{U}).$$

In particular, if the functions g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In this paper, by using the above principle of subordination between analytic functions, we obtain the coefficient bounds as well as several distortion and growth theorems for functions in the function class $f \in \mathcal{K}_s^p(h)$. Our results generalize the related works of some authors.

2. Main result and their demonstrations

The following lemmas have an important application in proving the desired result for the class $f \in \mathcal{K}_s^p(h)$.

Lemma 2.1 (see [11])**.** *Let the function $h(z)$ given by*

$$h(z) = \sum_{n=1}^{\infty} h_n z^n$$

be convex in $\mathbb{U}$. Suppose also that the function $f(z)$ given by

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

is holomorphic in $\mathbb{U}$. If $f(z) \prec g(z)$ $(z \in \mathbb{U})$, then

$$|a_n| \le |h_1| \quad (n \in \mathbb{N}).$$

Lemma 2.2 (See [6]). *Let* $g \in \mathcal{S}^*(\frac{1}{2})$. *Then*

$$G(z) = -\frac{g(z) \cdot g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \quad (z \in \mathbb{U}) \tag{2.1}$$

is an odd starlike function and

$$|B_{2n-1}| \leq 1 \quad (n \in \mathbb{N}^* \ := \mathbb{N} \backslash \{1\} = \{2, 3, 4, ...\}).$$

We now state and prove the main results of our present investigation.

Theorem 2.1. *An analytic function* $f \in \mathcal{K}_s^p(h)$ *if and only if there exists a function* $g \in \mathcal{S}^*(\frac{1}{2})$ *such that*

$$-\frac{z^2 f'(z)}{z^{p-1} \cdot g(z) \cdot g(-z)} \prec h(z) \quad (z \in \mathbb{U}; p \in \mathbb{N}).$$

Proof. Theorem 2.1 can be proven easily, thus we choose to omit the details involved. □

In view of Remark 1.3, if we let

$$h(z) = \frac{p + (p - 2\alpha)z}{1 - z} \quad (z \in \mathbb{U};\ 0 \leq \alpha < p;\ p \in \mathbb{N})$$

in Theorem 2.1, we can reduce the following corollary.

Corollary 2.1. *An analytic function* $f \in \mathcal{K}_s^p(\alpha)$ $(0 \leq \alpha < p)$ *if there exists a function* $g \in \mathcal{S}^*(\frac{1}{2})$ *such that*

$$-\frac{z^2 f'(z)}{z^{p-1} \cdot g(z) \cdot g(-z)} \prec \frac{p + (p - 2\alpha)z}{1 - z} \quad (z \in \mathbb{U};\ p \in \mathbb{N}).$$

Our proposed coefficient bounds and distortion inequalities for functions in the class $\mathcal{K}_s^p(h)$ are given below.

Theorem 2.2. *Let the function* $f(z)$ *be defined by (1.1). If* $f \in \mathcal{K}_s^p(h)$, *then*

$$\begin{aligned} |a_{p+n}| &\leq \frac{n+1}{2(p+n)} |h'(0)| \qquad (n \in \mathbb{N}_o), \\ |a_{p+n}| &\leq \frac{1}{p+n} (\frac{n}{2} |h'(0)| + p) \quad (n \in \mathbb{N}_e). \end{aligned} \tag{2.2}$$

Proof. Suppose that $f \in \mathcal{K}_s^p(h)$, then there exists a function $g \in \mathcal{S}^*(\frac{1}{2})$ such that (1.3) holds true. Define a function $G(z)$ by

$$G(z) = -\frac{g(z) \cdot g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \quad (z \in \mathbb{U}). \tag{2.3}$$

By virtue of Lemma 2.2, we know that $G(z)$ is an odd starlike function and

$$|B_{2n-1}| \leq 1 \quad (n \in \mathbb{N}^*). \tag{2.4}$$

Applying (1.3) together with (2.3), we obtain

$$\frac{zf'(z)}{z^{p-1} \cdot G(z)} \in h(\mathbb{U}). \tag{2.5}$$

By putting

$$p(z) = \frac{zf'(z)}{z^{p-1}G(z)} = \frac{pz^p + \sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}}{z^p + ...} \quad (z \in \mathbb{U}), \tag{2.6}$$

then we see that

$$p(0) = h(0) = p, p(z) \in h(\mathbb{U}) \quad (z \in \mathbb{U}).$$

Therefore, we have

$$p(z) \prec h(z) \quad (z \in \mathbb{U}).$$

By using Lemma 2.1, we have

$$|p_n| = |\frac{p^{(n)}(0)}{n!}| \leq |h'(0)| \quad (n \in \mathbb{N}). \tag{2.7}$$

On the other hand, in view of (2.6), we get

$$zf'(z) = z^{p-1}G(z)p(z) \quad (z \in \mathbb{U}). \tag{2.8}$$

Since $p(0) = p$, let

$$p(z) = p + p_1 z + p_2 z^2 + ... \quad (z \in \mathbb{U}). \tag{2.9}$$

Thus, substituting (2.3) and (2.9) in (2.8), we deduce that

$$\begin{aligned} &pz^p + \sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n} \\ =&(z^p + B_3 z^{p+2} + B_5 z^{p+4} + ...)(p + p_1 z + p_2 z^2 + ...) \quad (n \in \mathbb{N}). \end{aligned} \tag{2.10}$$

Comparing the coefficients of z^{p+n} on the both sides of (2.10), we obtain the following relations:

$$(p+n)a_{p+n} = p_n + B_3 p_{n-2} + B_5 p_{n-4} + ... + B_{n-2}p_3 + B_n p_1 \quad (n \in \mathbb{N}_o) \tag{2.11}$$

and

$$(p+n)a_{p+n} = p_n + B_3 p_{n-2} + B_5 p_{n-4} + ... + B_{n-1}p_2 + B_{n+1}p \quad (n \in \mathbb{N}_e). \tag{2.12}$$

Finally, by using Lemma 2.2 and (2.7), we obtain

$$|a_{p+n}| \leq \frac{n+1}{2(p+n)}|h'(0)| \qquad (n \in \mathbb{N}_o)$$

and

$$|a_{p+n}| \leq \frac{1}{p+n}(\frac{n}{2}|h'(0)| + p) \qquad (n \in \mathbb{N}_e).$$

This evidently completes the proof of Theorem 2.2. □

In view of Remark 1.3, upon setting

$$h(z) = p\frac{1+z}{1-z} \quad (z \in \mathbb{U})$$

in Theorem 2.2, we have the following coefficient bounds belonging to the class $\mathcal{K}_s^p$.

Corollary 2.2. *Let the function $f(z)$ be defined by (1.1). If $f(z) \in \mathcal{K}_s^p$, then*

$$|a_{p+n}| \leq \frac{p(n+1)}{p+n} \quad (n \in \mathbb{N}).$$

Remark 2.1. For $p = 1$, we obtain the classes $\mathcal{K}_s^1(h) = \mathcal{K}_s(h)$, $\mathcal{K}_s^1 = \mathcal{K}_s$. The coefficient bounds of class $\mathcal{K}_s(h)$ and $\mathcal{K}_s$ have been proved by Xu etc. [14].

Theorem 2.3. *Let the function $f(z) \in \mathcal{A}(p)$ be defined by (1.1). If $f \in \mathcal{K}_s^p(h)$, then*

$$\begin{aligned} &\frac{r^{p-1}\min\{h(r),h(-r)\}}{1+r^2} \leq |f'(z)| \leq \frac{r^{p-1}\max\{h(r),h(-r)\}}{1-r^2} \qquad (n \in \mathbb{N}_o), \\ &\int_0^r \frac{t^{p-1}\min\{h(t),h(-t)\}}{1+t^2}d_t \leq |f(z)| \leq \int_0^r \frac{t^{p-1}\max\{h(t),h(-t)\}}{1-t^2}d_t, \end{aligned} \tag{2.13}$$

where $|z| = r$ and $0 \leq r < 1$.

Proof. Suppose that $f \in \mathcal{K}_s^p(h)$, then there exists a function $g \in \mathcal{S}^*(\frac{1}{2})$ such that (1.3) holds true. From Lemma 2.2, the function $G(z)$ given by (2.1) is an odd starlike function. Thus, we have (see [5]):

$$\frac{r}{1+r^2} \leq |G(z)| \leq \frac{r}{1-r^2} \quad (|z| = r; 0 \leq r < 1). \tag{2.14}$$

According to Theorem 2.1, we find that

$$\frac{zf'(z)}{z^{p-1} \cdot G(z)} \prec h(z) \quad (z \in \mathbb{U}).$$

Also, by using (1.2), we have

$$\min\{h(r),h(-r)\} \leq |\frac{zf'(z)}{z^{p-1} \cdot G(z)}| \leq \max\{h(r),h(-r)\} \quad (|z| = r; 0 \leq r < 1). \tag{2.15}$$

Combining (2.14) with (2.15), the upper and lower bounds of $|f'(z)|$ have been proved.

Let

$$z = re^{i\theta} \quad (0 < r < 1).$$

If τ denotes the closed line-segment between 0 and z, by noting that

$$\begin{aligned} f(z) &= \int_\tau f'(\xi)d_\xi \\ &= \int_0^r f'(te^{i\theta})e^{i\theta}d_t \quad (|z| = r; 0 \leq r < 1), \end{aligned}$$

we thus deduce that

$$\begin{aligned} |f(z)| &= |\int_0^z f'(\xi)d_\xi| \\ &\leq \int_0^r |f'(te^{i\theta})|d_t \\ &\leq \int_0^r \frac{t^{p-1}\max\{h(t),h(-t)\}}{1-t^2}d_t \quad (|z|=r; 0 \leq r < 1) \end{aligned}$$

making use of the upper estimate of $|f'(z)|$. The upper bound of $|f(z)|$ has been proved. Next, we will prove the lower bound of $|f(z)|$. For this purpose, it is sufficient to show that it holds true for z_0 nearest to zero, where

$$|z_0| = r \qquad (0 < r < 1).$$

Moreover, we have

$$|f(z)| \geq |f(z_0)| \qquad (|z|=r;\ 0 \leq r < 1).$$

Since $f(z)$ is a close-to-convex function in the open unit disk $\mathbb{U}$. We deduce that the original image of the closed line-segment τ_0 between 0 and $f(z_0)$ is a piece of arc Γ in the disk $\mathbb{U}_r$ given by

$$\mathbb{U}_r = \{z : z \in \mathbb{C} \text{ and } |z| \leq r\ (0 \leq r < 1)\}.$$

Hence, we have

$$\begin{aligned} |f(z_0)| &= \int_{f(\Gamma)} d_{|}\omega| = \int_\Gamma |f'(z)||d_z| \\ &\geq \int_0^r \frac{t^{p-1}\min\{h(t),h(-t)\}}{1+t^2}d_t \quad (|z|=r;\ 0 \leq r < 1) \end{aligned}$$

making use of the lower estimate of $|f'(z)|$. This completes the proof of the Theorem 2.3. □

In view of Remark 1.3, by setting

$$h(z) = p\frac{1+z}{1-z} \text{ and } h(z) = \frac{p+(p-2\alpha)z}{1-z} \qquad (z \in \mathbb{U};\ 0 \leq \alpha < p)$$

in Theorem 2.3, we have the following corollaries, respectively.

Corollary 2.3. *Let the function $f(z) \in \mathcal{A}(p)$ be defined by (1.1). If $f \in \mathcal{K}_s^p$, then*

$$\begin{aligned} &\frac{pr^{p-1}(1-r)}{(1+r)(1+r^2)} \leq |f'(z)| \leq \frac{pr^{p-1}}{(1-r)^2} \qquad (|z|=r;\ 0 \leq r < 1), \\ &\int_0^r \frac{pt^{p-1}(1-t)}{(1+t)(1+t^2)}d_t \leq |f(z)| \leq \int_0^r \frac{pt^{p-1}}{(1-t)^2}d_t \qquad (|z|=r;\ 0 \leq r < 1). \end{aligned} \tag{2.16}$$

Corollary 2.4. *Let the function $f(z) \in \mathcal{A}(p)$ be defined by (1.1). If $f \in \mathcal{K}_s^p(\alpha)$, then*

$$\begin{aligned} &\frac{r^{p-1}(p-(p-2\alpha)r)}{(1+r)(1+r^2)} \leq |f'(z)| \leq \frac{r^{p-1}(p+(p-2\alpha)r)}{(1-r^2)(1-r)} \qquad (|z|=r; 0 \leq r < 1), \\ &\int_0^r \frac{t^{p-1}(p-(p-2\alpha)t)}{(1+t)(1+t^2)}d_t \leq |f(z)| \leq \int_0^r \frac{t^{p-1}(p+(p-2\alpha)t)}{(1-t^2)(1-t)}d_t \ (|z|=r; 0 \leq r < 1). \end{aligned} \tag{2.17}$$

Remark 2.2. For $p = 1$, we obtain the classes $\mathcal{K}_s^1 = \mathcal{K}_s$, $\mathcal{K}_s^1(\alpha) = \mathcal{K}_s(\alpha)$. The distortion inequalities of class $\mathcal{K}_s$ and $\mathcal{K}_s(\alpha)$ have been proved by Xu etc. [14].

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