A NOVEL REGULARIZATION METHOD AND APPLICATION TO LOAD IDENTIFICATION OF COMPOSITE LAMINATED CYLINDRICAL SHELL *

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Abstract In this paper, a novel regularization method (MRO) is suggested to identify the multi-source dynamic loads on a surface of composite laminated cylindrical shell. Regularization methods can solve the difficulty of the solution of ill-conditioned inverse problems by the approximation of a family of neighbouring well-posed problems. Based on the construction of a new regularization operator, corresponding regularization method is established. We prove the stability of the proposed method according to suitable parameter choice strategy that leads to optimal convergence rate toward the minimal-norm and least square solution of an ill-posed linear operator equation in the presence of noisy data. Furthermore, numerical simulations show that the multi-source dynamic loads on a surface of composite laminated cylindrical shell are successfully identified, and demonstrate the effectiveness and robustness of the present method.

Keywords Load identification, ill-posed problems, regularization, general source conditions.

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1. Introduction

It is expected that structural damage prognosis as a promising technology will be applied to aerospace systems in the future. Farrar et al. assisted that structural damage prognosis can be defined as the estimation of a system's remaining useful life based on behavioral prediction models [7]. In fact, the interaction forces between the system structure are very important in the optimum design. It is expensive and subjected to bias for the direct measurement of the forces using the appropriate instrument. Moreover, the results we obtain by computational simulations are subjected to modelling errors. But we cannot directly measure the external loads in the most cases of many practical applications as a result of extremely large magnitudes of loads for a short-time period and the difficulties during the installation of force-measurement devices. So we'd better indirectly reconstruct the applied loads by exploiting the structural dynamic response data. It is necessary to develop

a technique to identify the loads acting on the system structure by the vibration responses, and the cost involved will be much less than that by direct measurement.

Recently, many scientific workers have developed different kinds of inverse approaches for the identification of the loads acting on the system. It is worth noting that inverse analysis has been used in numerous fields of science and technology, such as radar tracking, oil reservoir identification, medical tomography, residual stress determination, non-destructive testing and material property estimation [1, 3, 10, 11, 13]. Liu and Han presented an inverse procedure for identifying both concentrated and extended line load using Greens function and Heaviside step function in time domain [12, 14]. Zhu et al. developed an inverse method based on modal superposition and regularization technique to identify moving loads [22]. Dolye reconstructed the impact force by the dynamic response in the bimaterial beam system. Moreover, we may encounter some difficulties. For instance, the model of multiple dof and the multi-point excitation model fail to work, and direct measurement for distributed dynamic loads is not available. Especially in developing structural health monitoring systems, it has received much attention to establishing a method for identifying the distributed dynamic loads on a continuum so that the evolution of induced damage can be predicted [2, 15, 17-19, 21]. Unfortunately, load identification problems discussed in above references are complex inverse problems with inherent ill-posedness. Meanwhile, from these studies mentioned above, we should pay much attention to the complicated technical problems in mathematics, especially in the ill-posedness and regularization methods [5, 6, 8, 9, 20]. In this paper, we propose a modified regularization operator, and establish a corresponding stable regularization method, and prove that the regularized approximations provide order optimal error bound on the appropriate set than Tikhonov regularization method, then apply the present method to the reconstruction of the distributed dynamic loads acting on the composite laminated cylindrical shell.

This paper is organized as follows. In Section 2 a new regularization method is established. In Section 3 we derive the error bounds between the true solution and the regularized solution by our method, and prove that the regularization approximation provides optimal error bound if the regularization parameter is properly chosen. In Section 4, this present method is applied to load identification of composite laminated cylindrical shell and we make a conclusion in Section 5.

2. The establishment of regularization method

Let X and Y be real Hilbert spaces and $K \in L(X, Y)$, i.e. $K : X \to Y$ is a bounded linear operator. We consider the equation

$$Kx = y, \quad (y \in R(K)). \tag{2.1}$$

Throughout this paper we assume:
(H1) $y^\delta \in X$ is the available noisy data with

$$\|y - y^\delta\| \leq \delta$$

and known noise level δ.

Then we solve

$$Kx = y^\delta. \tag{2.2}$$

If $R(K)$ is not closed, the problem (2.1) is ill-posed, then we have to use a regularization method for solving it. In general terms, regularization is the approximation of an ill-posed problem by a family of neighboring well-posed problems. A regularization method consists of a regularization operator and a parameter choice rule which is convergent in the sense that if the regularization parameter is chosen according that rule, then the regularized solutions will converge to the true solution in the norm as the noise level approaches zero. One of the most famous regularization methods is Tikhonov regularization method which exploits the following regularization operator [4, 16]:

$$q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2}, \alpha > 0, \mu > 0, \tag{2.3}$$

where regularization parameter and singular value will always be defined by α, μ, respectively, and then x_α^δ can be obtained by

$$x_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*y^\delta. \tag{2.4}$$

Theorem 2.1. *Let $(\mu_j, x_j, y_j)_{j \in N}$ be a singular system for the linear operator K : $X \to Y$, and let $q : (0, +\infty) \times (0, \|K\|) \to R$. Then $q(\alpha, \mu)$ is called a regularization operator, and corresponding regularization method can be given by*

$$x_\alpha := R_\alpha y = \sum_{j=1}^{\infty} \frac{q(\alpha, \mu_j)}{\mu_j}(y, y_j)x_j, \tag{2.5}$$

$$\lim_{\alpha \to 0} R_\alpha K x = x, \quad x \in X, \tag{2.6}$$

if the following conditions hold:

(i) $|q(\alpha, \mu)| \le 1$ for $\alpha > 0$ and $\mu \in (0, \|K\|)$.

(ii) For any $\alpha > 0$, there exists $c(\alpha) > 0$ such that

$$|q(\alpha, \mu)| \le c(\alpha)\mu, \quad \mu \in (0, \|K\|).$$

(iii) $\lim_{\alpha \to 0} q(\alpha, \mu) = 1$ for $\mu \in (0, \|K\|)$.

  (H2)   Let $K : X \to Y$ is a biunivocal compact operator and $y \in R(K)$.

It is easy to check that under the condition of (H2), the equation (2.1) has unique solution x. Exploiting singular system, we obtain

$$x = \sum_{j=1}^{\infty} \frac{1}{\mu_j}(y, y_j)x_j. \tag{2.7}$$

Since $\mu_j \to 0$ as $j \to +\infty$, as well as (2.5) and (2.7) we obtain the convergent approximate solution by the attenuation of $q(\alpha, \mu)$ to $1/\mu$, which can be performed by regularization operator. So we can obtain corresponding regularization method if a proper regularization operator is established.

In the following we will propose a new regularization operator and prove its regular property.

We define $q(\alpha, \mu) : R^+ \times (0, \|K\|) \to R^+$ given by

$$q(\alpha, \mu) = 1 - e^{-\frac{\mu}{\alpha}}. \tag{2.8}$$

Theorem 2.2. *The function $q(\alpha, \mu)$ (2.8) is a regularization operator. Moreover,*

$$1 - q(\alpha, \mu) \leq \frac{\alpha}{e\mu}. \tag{2.9}$$

Proof. It is easy to check that $q(\alpha, \mu) \leq 1$ and $\lim_{\alpha \to 0} q(\alpha, \mu) = 1$. Since

$$\frac{1 - x}{1 + x} < e^{-2x}, \quad x > 0,$$

we have

$$1 - e^{-\frac{\mu}{\alpha}} < 1 - \frac{1 - \frac{\mu}{2\alpha}}{1 + \frac{\mu}{2\alpha}} = \frac{2\mu}{2\alpha + \mu} < \frac{\mu}{\alpha}.$$

So the first result of the assertion follows from Lemma 2.1.

Next we prove the second result. By virtue of

$$e^x \geq ex, \quad x > 0,$$

we have

$$1 - q(\alpha, \mu) = 1 - (1 - e^{-\mu/\alpha}) = e^{-\mu/\alpha} \leq \frac{\alpha}{e\mu}.$$

Now the assertion can be proved easily. □

Remark 2.1. Using the results of Theorem 2.2, we obtain that the approximate solution of the equation (2.2) is given by

$$x_\alpha^\delta = R_\alpha y^\delta = \sum_{j=1}^{\infty} \frac{q(\alpha, \mu_j)}{\mu_j} (y^\delta, y_j) x_j, \tag{2.10}$$

where $R_\alpha : Y \to X$ and

$$R_\alpha = [I - e^{-\frac{(K^*K)^{\frac{1}{2}}}{\alpha}}](K^*K)^{-\frac{1}{2}}, \alpha > 0. \tag{2.11}$$

3. Parameter choice and convergence

For obtaining the stable approximate solution of problem (2.1), some regularization technique is usually performed. Also, in order to guarantee certain convergence rates for $\|x_\alpha^\delta - x\|$, the set of solutions of problem (2.1) has to be restricted to some source sets. For operator equations (2.1), exploiting singular system theory, we define a subspace of X :

$$X_\alpha := R((K^*K)^{\frac{1}{2}}) := \{x \in X : \|X\|_\alpha < \infty\},$$

where

$$\|X\|_\alpha = (\sum_{j=1}^{\infty} \mu_j^{-2} |(x, x_j)|^2)^{\frac{1}{2}}. \tag{3.1}$$

We further make the following assumption:
(H3) Source conditions of the type $x \in M_{\alpha, E}$ with $M_{\alpha, E}$ are given by

$$M_{\alpha, E} = \{x \in X, : \|x\|_\alpha \leq E\}. \tag{3.2}$$

Theorem 3.1. *Let (H1)-(H3) be satisfied. Then,*

$$\|x_\alpha^\delta - x\| \leq \frac{1}{\alpha}\delta + \frac{\alpha E}{e}. \tag{3.3}$$

Proof.   Using the triangle inequality, as well as (H1), (H2) and (H3) we have

$$\begin{aligned}
\|x_\alpha^\delta - x\| &\leq \|R_\alpha(y^\delta - y)\| + \|R_\alpha y - x\| \\
&\leq \|R_\alpha\|\delta + \|R_\alpha Kx - x\| \\
&< \frac{\delta}{\alpha} + \|R_\alpha Kx - x\|.
\end{aligned} \tag{3.4}$$

In order to estimate $\|R_\alpha Kx - x\|$ in terms of μ, exploiting (2.9), (H3), and singular system theory, we obtain

$$\begin{aligned}
\|R_\alpha Kx - x\|^2 &= \sum_{j=1}^\infty [1 - q(\alpha, \mu_j)]^2 |(x, x_j)|^2 \\
&\leq \sum_{j=1}^\infty (\frac{\alpha}{e})^2 \mu_j^{-2} |(x, x_j)|^2 \\
&< (\frac{\alpha}{e})^2 E^2.
\end{aligned} \tag{3.5}$$

Now the desired estimate (3.3) follows from (3.4) and (3.5). $\square$

In our next theorem, we will provide the order optimal error bound for $\|x_\alpha^\delta - x\|$ provided that α is chosen properly.

Theorem 3.2. *Let the assumptions in Theorem 3.1 be fulfilled, and α be determined by*

$$\alpha = \alpha^*(\delta) = (\frac{e}{E})^{\frac{1}{2}} \delta^{\frac{1}{2}}. \tag{3.6}$$

Then,

$$\|x_\alpha^\delta - x\| \leq O(\delta^{\frac{1}{2}}). \tag{3.7}$$

Proof.   By virtue of (3.3), we define the function $f : R^+ \to R^+$,

$$f(\alpha) = \frac{1}{\alpha}\delta + \frac{\alpha E}{e}.$$

If we choose the regularization parameter α by (3.6), then we can derive the minimum value of function $f(\alpha)$. Then the assertion can be easily proved. $\square$

Remark 3.1. Theorem 3.2 yields a new regularization parameter choice that leads to the optimal convergence rate. Unfortunately, due to prior information in actual computations of engineering problems, it is difficult or impossible to determine an appropriate value for the regularization parameter. As the errors in the measurement are unknown, L-curve method is usually adopted to perform it. In fact, we may obtain the same optimal convergence if we choose the proper regularization parameter by the L-curve method. So we will choose the regularization parameter by L-curve method in the following numerical simulations of engineering example.

4. Application

In this section we consider the multi-source dynamic load identification problem for a linear and time-invariant dynamic system, and the response at an arbitrary receiving point in a structure can be expressed as a convolution integral of the forcing time-history and the corresponding Green's kernel in time domain:

$$y(t) = \int_0^t G(t - \tau)p(\tau)d\tau, \tag{4.1}$$

where $y(t)$ is the response which can be displacement, velocity, acceleration, strain, etc. $G(t)$ is the corresponding Green's function, which is the kernel of impulse response. $p(t)$ is the desired unknown dynamic load acting on the structure.

By discretizing this convolution integral, the whole concerned time period is separated into equally spaced intervals, and the equation (4.1) is transformed into a system of algebraic equation:

$$Y(t) = G(t)P(t), \tag{4.2}$$

or equivalently,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} g_1 & & & \\ g_2 & g_1 & & \\ \vdots & \vdots & \ddots & \\ g_m & g_{m-1} & \cdots & g_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix} \triangle t,$$

where y_i, g_i, and p_i are response, Green's function matrix and input force at time $t = i\triangle t$, respectively. $\triangle t$ is the discrete time interval. Since the structure without applied force is static before force is applied, y_0 and g_0 are equal to zero. All the elements in the upper triangular part of G are zeros and are not shown. The special form of the Green's function matrix reflects the characteristic of the convolution integral.

To recover the time history $P(t)$, the knowledge of $y(t)$ and $G(t)$ are required. The response at a receiving point can be obtained by instrument measurement. The Green's function of a structure is obtained by finite element method (FEM). A practical engineering problem is to determine radial forces of composite laminated cylindrical shell, as shown in Figure 1. Thin-walled cylindrical shell structure has been widely used in the aerospace structures. The cylindrical shell size is 200.0 mm in middle radius, 10.0 mm in thickness, and 500.0 mm in length. It consists of one carbon/epoxy layer and one glass/epoxy layer. Its stacking sequence is denoted by $[C90/G + 45/G - 45]_s$, where C and G stand for the carbon/epoxy and the glass/epoxy layer, respectively, and $90, +45$, and -45 stand for the angle of fiber-orientation to the center axis. The subscript of "s" means that it is symmetrically stacked. The material properties of the carbon/epoxy and glass/epoxy are listed in Table 1.

The radial concentrated load is applied to the outside surface and the measured response is the radial displacement. One side of the shell is free, and the other side is fixed. We establish its finite element model which can be seen in Figure 1. The arrow in Figure 1 denotes the acting point of dynamic force.
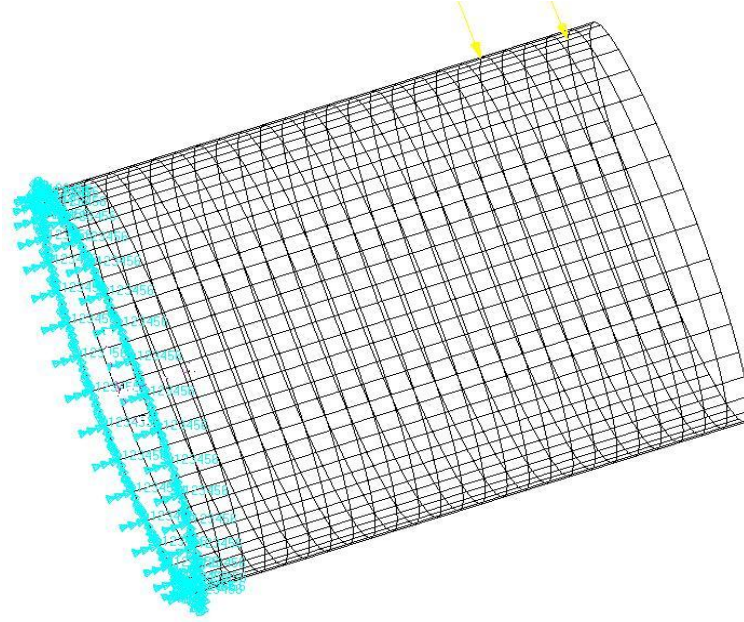
Figure 1. The finite element model of composite laminated cylindrical shell

The concentrated loads are defined as follows:

$$F_1(t) = \begin{cases} q_1 \sin(\frac{2\pi t}{t_d}), & 0 \leq t \leq 2t_d, \\ 0, & t < 0 \quad \text{and} \quad t > 2t_d, \end{cases}$$

$$F_2(t) = \begin{cases} 4q_2 t/t_d, & 0 \leq t \leq t_d/4, \\ 2q_2 - 4q_2 t/t_d, & t_d/4 < t \leq 3t_d/4, \\ 4q_2 t/t_d - 4q_2, & 3t_d/4 < t \leq t_d, \\ 0, & t > t_d, \end{cases}$$

where t_d is the time cycle of sine force, and $q_i(i = 1, 2)$ is a constant amplitude of the force. When $t_d = 0.004s, q_1 = 1000N$, and $q_2 = 800N$, the sine force and triangle force are shown in Figures 2-3. The experimental data of response is simulated by the computed numerical solution, and the corresponding radial displacement response can be obtained by FEM as shown in Figure 4-5. Furthermore, a noise is directly added to the computer-generated response to simulate the noise-contaminated measurement, and the noisy response is defined as follows:

$$Y_{err} = Y_{cal} + l_{noise} \cdot std(Y_{cal}) \cdot rand(-1, 1),$$

where Y_{cal} is the computer-generated response; $std(Y_{cal})$ is the standard deviation of Y_{cal}; $rand(-1, 1)$ denotes the random number between -1 and $+1$; l_{noise} is a parameter to control the level of the noise contamination. Herein, we consider the case of noise level namely 5%, and our method is adopted to determine the identified force. To evaluate the performances of the present method, five time points are selected, and the identified force for each point will be compared to the corresponding actual force.

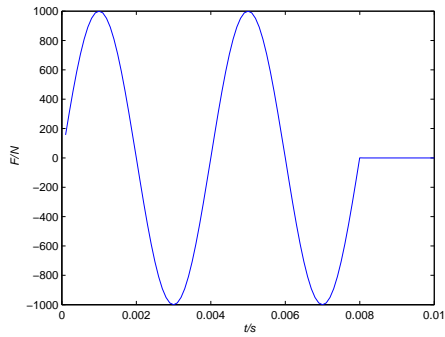The results of numerical simulations are as follows:

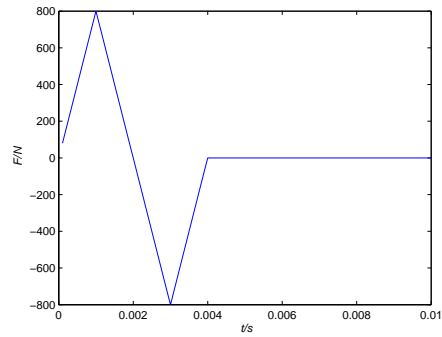**Figure 2.** The radial concentrated sine load acting on the outside surface

Figure 3. The radial concentrated triangle load acting on the outside surface
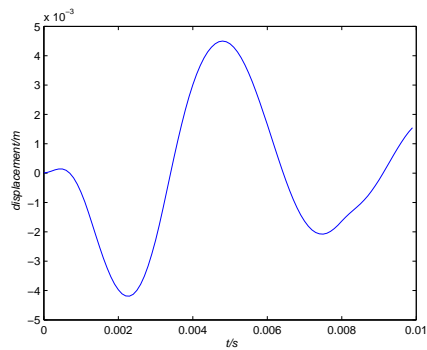

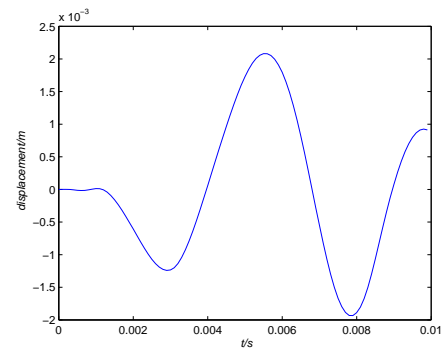**Figure 4.** The corresponding radial displacement response at one point

Figure 5. The corresponding radial displacement response at the other point
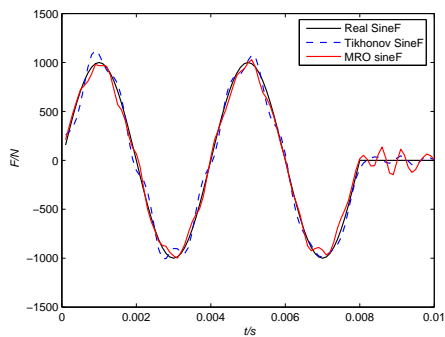

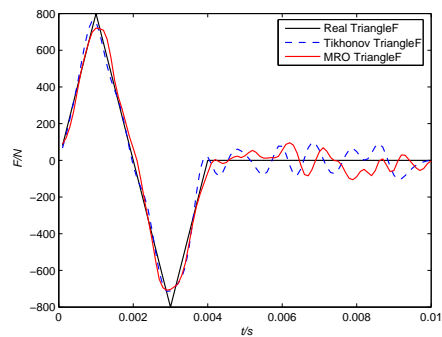**Figure 6.** The identified sine force at noise level 5%

Figure 7. The identified triangle force at noise level 5%

From Figures 6-7, it can be shown that the present method (MRO) can both stably and effectively identify the multi-source dynamic loads by the measured noisy responses. Moreover, the more detailed results by them at the five time points are listed in Table 2. It can be found that at these five time points for noise level ±5%, the deviations of the identified loads by the present method are smaller than

Tikhonov regularization method because of better efficient identification. It can be also seen that the most deviations by Tikhonov regularization method and the present method concentrate in the range of $15.5\%, 14.8\%$, respectively. In addition, for the identification of sine force, the maximal deviation and average deviation by the present method are $14.78\%, 4.84\%$, respectively, obviously smaller than Tikhonov regularization method. Besides, the maximal deviation and average deviation of the identification of triangle force by the present method are $13.2\%, 5.02\%$, respectively, both smaller than Tikhonov regularization method. The numerical results show that the present algorithm performs well when recovering the loading time function, and also gives satisfactory results.

Table 1. The material properties of composite laminated cylindrical shell

Material Constants	$E_1(GPa)$	$E_2(GPa)$	$G_{12}(GPa)$	ν_{12}	ν_{23}	$\rho(g/cm^3)$
Glass/epoxy	38.49	9.367	3.414	0.2912	0.5071	2.66
Carbon/epoxy	142.17	9.255	4.795	0.3340	0.4862	1.90

Table 2. The identified force at five time points at noise level 5%.

	Time point	Real force	Tikhonov method Identified force	Error (%)	Present method Identified force	Error (%)
Sine	0.001	1000	1081.9	8.19	968.41	3.16
Triangle	0.0006	480	521.98	5.25	482.94	0.3675
Sine	0.003	-1000	-900.95	9.91	-975.39	2.46
Triangle	0.001	800	752.84	5.90	720.69	9.91
Sine	0.0045	707.11	775.59	6.85	635.11	7.2
Triangle	0.0016	320	328.41	1.05	345.98	3.25
Sine	0.0063	-453.99	-549.44	9.55	-435.53	1.85
Triangle	0.0033	-560	-639.79	9.97	-560.21	0.03
Sine	0.0073	-891.01	-905.15	1.41	-839.22	5.18
Triangle	0.0038	-160	-176.15	2.02	-181.99	2.75
Error (%)			Maximum	Average	Maximum	Average
Sine			15.41	5.57	14.78	4.84
Triangle			13.72	5.56	13.20	5.02

5. Conclusion

A novel regularization method is proposed, proved theoretically and applied to the multi-source dynamic load identification of composite laminated cylindrical shell. Additionally, in the application to engineering example, comparing with the traditional Tikhonov regularization method, the present method can provide more efficient and numerically stable approximation of the expected loads. In one word, our method is effective and accurate for solving the load identification problems of the practical structural engineering.

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