THE DIRICHLET PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS ON RIEMANNIAN MANIFOLDS

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Abstract In this paper, after discussing the properties of the Nemytsky operator, we obtain the existence of weak solutions for Dirichlet problems of non-homogeneous \( p(m) \)-harmonic equations.

Keywords Variable exponent, Riemannian manifold, Nemytsky operator.


1. Introduction

After Kováčik and Rákosník first discussed the \( L^{p(x)}(\Omega) \) and \( W^{k,p(x)}(\Omega) \) spaces in [11], a lot of research has been done concerning these kinds of variable exponent spaces (see [1, 6, 7, 9] and the references therein). The existence of solutions for \( p(x) \)-Laplacian Dirichlet problems on bounded domains in \( \mathbb{R}^n \) have been greatly discussed. For example, Chabrowski and Fu [2] and Fan and Zhang [8] established some results about the existence of solutions under some conditions. More informations about the theory of variable exponential function space can be found in [4, 5].

In recent years, the theory on problems with variable exponential growth conditions has important applications in nonlinear elastic mechanics (see [17]), electrorheological fluids (see [12, 14]).

Let \((M, g)\) be a Riemannian manifold. For \( u \in C^\infty(M) \), \( \nabla u \) denotes the covariant derivative of \( u \). The components of \( \nabla u \) in local coordinates are given by \( (\nabla u)_i = \nabla^i u \), \( i = 1, 2 \cdots, n \). By definition one has that

\[
|\nabla u| = \sum_{i,j=1}^{n} g^{ij} \nabla^i u \nabla^j u.
\]

In this article we will always assume \((M, g)\) is a connected \( n \)-dimensional smooth orientable complete Riemannian manifold \((n \geq 3)\). \( d\mu = \sqrt{\det(g_{ij})} dx \) is the Riemannian volume element on \((M, g)\), where the \( g_{ij} \) are the components of the Riemannian metric \( g \) in the chart and \( dx \) is the Lebesgue volume element of \( \mathbb{R}^n \). Let \( \gamma : [a, b] \to M \) be a curve of class \( C^1 \), the length of \( \gamma \) is

\[
L(\gamma) = \int_a^b \sqrt{g(\gamma(t)) \left( \frac{d\gamma}{dt}(t), \frac{d\gamma}{dt}(t) \right)} \, d\mu.
\]
The Sobolev space $\mathcal{C}^1_{c}(\mathbb{R}^N)$ is the space of piecewise $\mathcal{C}^1$ functions $\gamma : [a, b] \to M$ such that $\gamma(a) = m_1$ and $\gamma(b) = m_2$. One can define a distance $d_\gamma(m_1, m_2) = \inf_{C^1_{c,m_1,m_2}} L(\gamma)$ on $M$.

We denote by $L^1_{loc}(M)$ the space of locally integrable functions on $M$, denote by $C^\infty_c(M)$ the vector space of smooth functions with compact support on $M$.

The Riemannian measure and the characteristic function of a set $A \subseteq M$ will be denoted by $\mu(A)$ and $\chi_A$, respectively.

Let $\mathcal{P}(M)$ be the set of all measurable functions $p : M \to [1, \infty]$. For $p \in \mathcal{P}(M)$ we put $M_1 = M_1^p = \{ m \in M : p(m) = 1 \}$, $M_\infty = M_\infty^p = \{ m \in M : p(m) = \infty \}$, $M_0 = M \setminus (M_1 \cup M_\infty)$, $p^- = \operatorname{essinf}_{M_0} p(m)$ and $p^+ = \operatorname{esssup}_{M_0} p(m)$ if $\mu(M_0) > 0$, $p^- = p^+ = 1$ if $\mu(M_0) = 0$. We always assume that $p \in \mathcal{P}(M)$, $\mathcal{P}_1(M) = \mathcal{P}(M) \cap L^\infty(M)$ and $\mathcal{P}_2(M) = \{ p \in \mathcal{P}_1(M) : 1 < \operatorname{essinf}_M p(m) \}$. We use the convention $1/\infty = 0$.

In 2012, Fu and Guo first introduced variable exponent function spaces on Riemannian manifolds in [10]. Also motivated by [10], we are interested in the following Dirichlet problems:

\[
\begin{cases}
-\nabla(u)|\nabla u|^{p(m)-2} + \lambda u|u|^{p(m)-2} &= f(m, u), \quad m \in M, \\
u(m) &= 0, \quad m \in \partial M.
\end{cases}
\]

### 2. Preliminaries and Nemytskky Operator

For a function $u$ on $M$ we define the functional $\rho_{p(m), M}$ by

\[
\rho_{p(m), M}(u) = \int_{M \setminus M_\infty} |u|^{p(m)} d\mu + \operatorname{esssup}_{M_\infty} |u|.
\]

**Definition 2.1.** The Lebesgue space $L^{p(m)}(M)$ is the class of functions $u$ such that $\rho_{p(m), M}(\lambda u) < \infty$ for some $\lambda = \lambda(u) > 0$ with the following norm

\[
\|u\|_{L^{p(m)}(M)} = \inf\{ \lambda > 0 : \rho_{p(m), M}(u\lambda) \leq 1 \}.
\]

**Definition 2.2.** The Sobolev space $W^{1,p(m)}(M)$ consists of such functions $u \in L^{p(m)}(M)$ for which $\nabla^i u \in L^{p(m)}(M)$, $i = 1, 2, \ldots, n$. The norm is defined by

\[
\|u\|_{W^{1,p(m)}(M)} = \|u\|_{L^{p(m)}(M)} + \sum_{i=1}^n \|\nabla^i u\|_{L^{p(m)}(M)}.
\]

The space $W^{1,p(m)}_0(M)$ is defined as the closure of $C^\infty_c(M)$ in $W^{1,p(m)}(M)$.

Given $p \in \mathcal{P}(M)$ we define the conjugate function $p'(m) \in \mathcal{P}(M)$ by

\[
p'(m) = \begin{cases} 
\infty, & \text{if } m \in M_1, \\
1, & \text{if } m \in M_\infty, \\
p(m)^{-1}, & \text{if } m \in M_0.
\end{cases}
\]
Lemma 2.1 (see [10]). If \( p(m) \in \mathcal{P}(M) \), then the inequality
\[
\int_M |(u, v)| d\mu \leq 2 \|u\|_{L^{p(m)}(M)} \|v\|_{L^{p'(m)}(M)}
\]
holds for every \( u \in L^{p(m)}(M) \), \( v \in L^{p'(m)}(M) \).

Lemma 2.2 (see [10]). Let \( p \in \mathcal{P}(M) \). Then
(i) If \( \|u\|_{L^{p(m)}(M)} \geq 1 \), we have \( \|u\|_{L^{p(m)}(M)}^{p^+} \leq \rho_{p(m)}(u) \leq \|u\|_{L^{p(m)}(M)}^{p^+} \).
(ii) If \( \|u\|_{L^{p(m)}(M)} < 1 \), we have \( \|u\|_{L^{p(m)}(M)}^{p^-} \geq \rho_{p(m)}(u) \geq \|u\|_{L^{p(m)}(M)}^{p^-} \).

Lemma 2.3 (see [10]). If \( p \in \mathcal{P}(M) \), \( u, u \in L^{p(m)}(M) \), then the following conditions are equivalent:
(i) \( \lim_{t \to \infty} \rho_{p(m)}(u_t - u) = 0 \);
(ii) \( \lim_{t \to \infty} \|u_t - u\|_{L^{p(m)}(M)} = 0 \);
(iii) \( u_t \) converges to \( u \) on \( M \) in measure and
\[
\lim_{t \to \infty} \rho_{p(m)}(u_t) = \rho_{p(m)}(u).
\]

Lemma 2.4 (see [10]). If \( p \in \mathcal{P}(M) \), \( u \in L^{p(m)}(M) \) is absolutely continuous with respect to the norm \( \| \cdot \|_{L^{p(m)}(M)} \).

Lemma 2.5 (see [10]). If \( p \in \mathcal{P}(M) \), then \( L^{p(m)}(M) \) and \( W^{1,p(m)}(M) \) are separable, reflexive Banach spaces.

Given two Banach spaces \( X \) and \( Y \), the symbol \( X \subset Y \) means that \( X \) is continuously embedded in \( Y \).

Lemma 2.6 (see [10]). Let \( 0 < \mu(M) < \infty \). If \( p(m), q(m) \in \mathcal{P}(M) \) and \( p(m) \leq q(m) \) a.e. \( m \in M \), then
\[
L^{q(m)}(M) \subset L^{p(m)}(M),
\]
The norm of the embedding operator (2.1) does not exceed \( \mu(M) + 1 \).

Lemma 2.7 (see [10]). Let \( M \) be a compact smooth Riemannian manifold with a boundary or without boundary and \( p(m), q(m) \in C(\overline{M}) \cap \mathcal{P}(M) \). Assume that
\[
p(m) < n, q(m) < \frac{np(m)}{n - p(m)}, \text{ for } m \in \overline{M}.
\]
Then
\[
W^{1,p(m)}(M) \subset L^{q(m)}(M)
\]
is a continuous and compact imbedding.

Let \( f(m, u) \) (\( m \in M, u \in \mathbb{R} \)) be a Carachéodory function, and \( N_f \) be the Nemytsky operator defined by \( f \), i.e. \( N_f u(m) = f(m, u) \).

Theorem 2.1. Let \( M \) be a compact Riemannian manifold and \( p_1, p_2 \in \mathcal{P}(M) \). If \( N_f \) maps \( L^{p_1(m)}(M) \) into \( L^{p_2(m)}(M) \), then \( N_f \) is continuous, bounded and there is a constant \( \beta \geq 0 \) and a non-negative function \( \alpha(m) \in L^{p_2(m)}(M) \) such that for \( m \in M \) and \( u \in \mathbb{R} \), the following inequality holds
\[
|f(m, u)| \leq \alpha(m) + \beta |u|^{p_1(m)/p_2(m)}.
\]
On the other hand, if \( f \) satisfies (2.2), then \( N_f \) maps \( L^{p_1(m)}(M) \) into \( L^{p_2(m)}(M) \), and thus \( N_f \) is continuous and bounded.
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Proof. Since $M$ is compact, $M$ can be covered by a finite number of charts $(U_\alpha, f_\alpha), \alpha = 1, 2, \ldots, k$. For $v \in L^1(M)$, we define

$$Hv(m) = h(m, v) = |N_f(\text{sgn} |u|^{1/p_1(m)})|^{p_2(m)},$$

then $H$ maps $L^1(M)$ into $L^1(M)$. By [16], we have that $H$ is bounded, continuous and

$$|H(v(m)\chi_{U_\alpha})| = |H(v(f_\alpha^{-1}(x)))| \leq a_\alpha(x) + b_\alpha|v(f_\alpha^{-1}(x))| = a_\alpha(f_\alpha(m)) + b_\alpha|v(m)\chi_{U_\alpha}|,$$

for any $\alpha = 1, 2, \ldots, k$, where $a_\alpha(f_\alpha(m)) \in L^1(U_\alpha)$ is a non-negative function and constant $b_\alpha > 0$. Let $a(m) = \sum_{\alpha=1}^k a_\alpha(f_\alpha(m))$. For $v \in L^1(U_\alpha)$, then $|Hv(m)| \leq a(m) + b|v(m)|$.

We assume that $v(m, 0) = 0$, otherwise we can consider $f(m, u) - f(m, 0)$ instead.

First, we only need to prove $N_f$ is continuous at 0 when $f(m, 0) = 0$. If this is not true, we can find a sequence $\{u_t\} \subset L^{p_1(m)}(M)$ satisfies $\lim_{t \to \infty} \|u_t\|_{L^{p_1(m)}(M)} = 0$, but $\|N_fu_t\|_{L^{p_2(m)}(M)} > \sigma$ where $\sigma$ is some positive constant. Without loss of generality, we can suppose that $\|u_t\|_{L^{p_1(m)}(M)} < 1$, thus by Lemma 2.2 we have

$$\rho_{p_1(m), M}(u_t) \leq \|u_t\|_{L^{p_1(m)}(M)}^{p_1(m)},$$

and hence

$$\lim_{t \to \infty} \int_M |u_t|^{p_1(m)}d\mu = 0.$$

Let $v_t = \text{sgn} u_t|u_t|^{p_1(m)}$. Then $\lim_{t \to \infty} \|v_t\|_{L^1(M)} = 0$, and hence $\lim_{t \to \infty} \|Hv_t\|_{L^1(M)} = 0$. Thus,

$$\lim_{t \to \infty} \int_M |N_fu_t|^{p_2(m)}d\mu = \lim_{t \to \infty} \int_M |Hv_t|d\mu = 0.$$

By Lemma 2.3, we have $\lim_{t \to \infty} \|N_fu_t\|_{L^{p_2(m)}(M)} = 0$, which is a contradiction.

Next, let $A$ be a bounded set in $L^{p_1(m)}(M)$. By Lemma 2.2, we have that $A$ is bounded in modular. For $v \in L^1(M)$, let $H$ be defined as above, then $H : L^1(M) \to L^1(M)$ is bounded. For $u \in A$, taking $v = \text{sgn} u|u|^{p_1(m)} \in L^1(M)$, then $\{\|v\|_{L^1(M)}\}$ is uniformly bounded. Then there is a constant $C > 0$ such that $\|H(\text{sgn} u|u|^{p_1(m)})\|_{L^1(M)} \leq C$, thus $\int_M |N_fu|^{p_2(m)}d\mu \leq C$. Therefore, $N_f(A)$ is bounded in $L^{p_2(m)}(M)$.

Since $M$ is compact, $M$ can be covered by a finite number of charts $(U_\alpha, f_\alpha)$. By Lemma 2.4, we can assume $\{U_\alpha\}$ such that $\{u\chi_{U_\alpha}\}$ are uniformly bounded in $L^{p_1(m)}(M)$ for $u \in L^{p_1(m)}(M)$. Writing $u^{(\alpha)} = u\chi_{U_\alpha}$ and $K = \sup\{\sum_{\alpha} \chi_{U_\alpha}(m) : m \in M\}$, then

$$|N_fu| \leq \sum_{\alpha} |N_fu^{(\alpha)}| \leq K|N_fu|$$

and $N_fu^{(\alpha)} \in L^{p_2(m)}(M)$.

Since

$$\int_M |Hv|d\mu \leq \sum_{\alpha} \int_{U_\alpha} |Hv|d\mu \leq K \int_M |Hv|d\mu,$$

where $v = \text{sgn} u|u|^{p_1(m)} \in L^1(M)$, we have $N_fu \in L^{p_2(m)}(M)$. 
For $u \in L^{p_1(m)}(M)$, set $v = \text{sgn} u |u|^{p_1(m)}$, then $v \in L^1(M)$ and thus
$$|N_f u(m)|^{p_2(m)} = |H v(m)| \leq a(m) + b |u(m)|^{p_1(m)}.$$ 

We can deduce that
$$|N_f u(m)| \leq (a(m) + b |u|^{p_1(m)})^{1/p_2(m)}$$
$$\leq a(m)^{1/p_2(m)} + b^{1/p_2(m)} |u|^{p_1(m)/p_2(m)}$$
$$\leq \alpha(m) + \beta |u|^{p_1(m)/p_2(m)},$$
where $\alpha(m) = a(m)^{1/p_2(m)} \geq 0$, $\alpha(m) \in L^{p_2(m)}(M)$, and $\beta = \max\{1, b\}$.

On the other hand, if (2.2) holds, we let $u \in L^{p_1(m)}(M)$. It is obvious that
$$\alpha(m) + \beta |u|^{p_1(m)/p_2(m)} \in L^{p_2(m)}(M).$$

Therefore
$$\int_M |N_f u|^{p_2(m)} d\mu \leq \int_M |\alpha(m) + \beta |u|^{p_1(m)/p_2(m)}|^{p_2(m)} < \infty,$$
i.e. $N_f$ maps $L^{p_1(m)}(M)$ into $L^{p_2(m)}(M)$.

\section{Existence of weak solutions}

In this section, we shall show some applications of the Sobolev space to Dirichlet problems of the $p(m)$-harmonic equations on Riemannian manifolds. We shall assume that $(M, g)$ is a connected $n$-dimensional smooth compact Riemannian manifold with smooth boundary ($n \geq 3$) and $p(m) \in C(\overline{M}) \cap P_2(M)$.

\textbf{Definition 3.1.} A function $u$ is a weak solution for the following Dirichlet problems
\begin{equation}
\begin{cases}
-\text{div}(\nabla u|\nabla u|^{p(m)-2}) + \lambda u |u|^{p(m)-2} = f(m, u), & m \in M, \\
u(m) = 0, & m \in \partial M,
\end{cases}
\end{equation}

where $f(m, u) \in L^{p'(m)}(M), \lambda > 0$, if $u \in W^{1,p(m)}_0(M)$ satisfies
\begin{equation}
\int_M \langle \nabla u|\nabla u|^{p(m)-2}, \nabla v \rangle + \lambda uv |u|^{p(m)-2}d\mu = \int_M f(m, u)v d\mu
\end{equation}
for every $v \in W^{1,p(m)}_0(M)$.

Let $(\cdot, \cdot)$ denote a dual between $X := W^{1,p(m)}_0(M)$ and $X'$. First we define the energy functional on $W^{1,p(m)}_0(M)$ by
$$\Psi(u) = \int_M \frac{1}{p(m)}(|\nabla u|^{p(m)} + \lambda |u|^{p(m)})d\mu - \int_M F(m, u)d\mu := I(u) - K(u),$$
where $F(m, t) = \int_0^t f(m, s)ds$. Then for $u, v \in W^{1,p(m)}_0(M)$, we have
$$\langle \Psi'(u), v \rangle = \langle I'(u), v \rangle - \langle K'(u), v \rangle$$
$$= \int_M \langle \nabla u|\nabla u|^{p(m)-2}, \nabla v \rangle d\mu + \int_M \lambda vu |u|^{p(m)-2}d\mu - \int_M f(m, u)v d\mu.$$
Let \( J = I' : X \to X' \), then
\[
(J(u), v) = \int_M \langle \nabla u | \nabla u \rangle^{p(m)-2} \nabla u \rangle d\mu + \int_M \lambda \nu |u|^{p(m)-2} d\mu := (J_1(u), v) + (J_2(u), v),
\]
where \( u, v \in X \).

**Lemma 3.1.** \( J = I' : X \to X' \) is a continuous, bounded and strictly monotone operator.

**Proof.** It is obvious that \( J \) is continuous and bounded. For any \( y, z \in \mathbb{R}^N \), we have the following inequalities (see [15]) from which we can get the strictly monotonicity of \( J \):
\[
\begin{align*}
(h_1) \quad & |z|^{p-2}z - |y|^{p-2}y \cdot (z - y) \geq (\frac{1}{2})^p |z - y|^p, \quad p \in [2, \infty), \\
(h_2) \quad & |z|^{p-2}z - |y|^{p-2}y \cdot (z - y) \geq (|z|^p + |y|^p) (p-2)/2 \geq (p-1)|z - y|^p, \quad p \in (1, 2).
\end{align*}
\]

By Theorem 2.1 and Lemma 3.1, we can get the following Lemma 3.2.

**Lemma 3.2.** The functional \( \Psi \in C^1(W^{1,p(m)}_0(M), \mathbb{R}) \).

Therefore, the weak solution to Dirichlet problems (3.1) is a critical point of \( \Psi \) and vice versa (see [3]).

Next, we suppose \( f(m, s) \) satisfies the following assumption:

\( (N): \) Let \( f : M \times \mathbb{R} \to \mathbb{R} \) satisfy Carathéodory condition and
\[
|f(m, s)| \leq C_1 + C_2 |s|^\theta(m) - 1 \quad \text{for any } (m, s) \in M \times \mathbb{R},
\]
where \( \theta(m) \in C(M) \cap P_2(M) \) and \( \theta(m) \leq p(m) \).

**Lemma 3.3.** The functional \( \Psi \) is weakly lower semi-continuous in \( W^{1,p(m)}_0(M) \).

**Proof.** Let \( u_t \rightharpoonup u \) weakly in \( W^{1,p(m)}_0(M) \). Since \( J \) is a convex functional, we deduced that the following inequality holds
\[
J(u_t) \geq J(u) + (J'(u), u_t - u).
\]

Then we get that \( \liminf_{t \to \infty} J(u_t) \geq J(u) \). Then \( J \) is weakly lower semi-continuous.

Let \( u_t \rightharpoonup u \) weakly in \( W^{1,p(m)}_0(M) \). By Lemma 2.6 and 2.7, we get that \( u_t \to u \) strongly in \( L^{\theta(m)}(M) \) and \( L^1(M) \). Without loss of generality, we assume that \( u_t \to u \) a.e. in \( M \), and hence \( F(m, u_t) \to F(m, u) \) a.e. \( m \in M \). From (\( N \)) we have
\[
|F(m, s)| \leq C_1 |s| + C_2 |s|^\theta(m),
\]
then the integrals of the functions \( |F(m, u_t) - F(m, u)| \) possess absolutely equicontinuity on \( M \). By Vitali convergence Theorem (see [13]),
\[
\int_M |F(m, u_t) - F(m, u)| d\mu \to 0, \quad \text{as } t \to \infty.
\]

Therefore, \( \Psi \) is weakly lower semi-continuous in \( W^{1,p(m)}_0(M) \).

**Theorem 3.1.** Let \( f(m, s) \) satisfies the condition (\( N \)). Then Dirichlet problems (3.1) has a weak solution in \( W^{1,p(m)}_0(M) \).
**Proof.** From the condition \((N)\) we can obtain \(|F(m, s)| \leq C_1 |s| + C_2 |s|^\theta(m)\), then by Lemma 2.2 and Young inequality, we have

\[
\Psi(u) = \int_M \frac{1}{p(m)} |\nabla u|^{p(m)} \, d\mu + \int_M \frac{\lambda}{p(m)} |u|^{p(m)} \, d\mu - \int_M F(x, u) \, d\mu \\
\geq \int_M \frac{1}{p(m)} |\nabla u|^{p(m)} \, d\mu + \int_M \frac{\lambda}{p(m)} |u|^{p(m)} \, d\mu - \int_M (\varepsilon |u|^{p(m)} + C(\varepsilon, \theta)) \, d\mu \\
\geq \int_M \frac{\min\{\lambda, 1\}}{2p^+} (|\nabla u|^{p(m)} \, d\mu + |u|^{p(m)} \, d\mu) - C(\varepsilon, \theta) \\
\to \infty
\]

as \(|u|_{W^{1,p(m)}(M)} \to \infty\), where \(\varepsilon = \frac{\min\{\lambda, 1\}}{2p^+}\). Since \(\Psi\) is weakly lower semi-continuous, \(\Psi\) has a minimum point \(u_0 \in W^{1,p(m)}(M)\), and \(u_0\) is a weak solution of Dirichlet problems (3.1). \(\square\)

**References**


