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THE DIRICHLET PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS ON RIEMANNIAN MANIFOLDS*

Lifeng Guo

Abstract In this paper, after discussing the properties of the Nemytsky operator, we obtain the existence of weak solutions for Dirichlet problemss of non-homogeneous $p(m)$-harmonic equations.

Keywords Variable exponent, Riemannian manifold, Nemytsky operator.

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1. Introduction

After Kováčik and Rákosnik first discussed the $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ spaces in [11], a lot of research has been done concerning these kinds of variable exponent spaces (see [1, 6, 7, 9] and the references therein). The existence of solutions for $p(x)-$Laplacian Dirichlet problems on bounded domains in $\mathbb{R}^n$ have been greatly discussed. For example, Chabrowski and Fu [2] and Fan and Zhang [8] established some results about the existence of solutions under some conditions. More informations about the theory of variable exponential function space can be found in [4, 5]. In recent years, the theory on problems with variable exponential growth conditions has important applications in nonlinear elastic mechanics (see [17]), electrorheological fluids (see [12, 14]).

Let (M, g) be a Riemannian manifold. For $u \in C^\infty(M)$, ∇u denotes the covariant derivative of u. The components of ∇u in local coordinates are given by $(\nabla u)_i = \nabla^i u,\ i = 1, 2 \cdots, n$. By definition one has that

$$|\nabla u| = \sum_{i,j=1}^{n} g^{ij} \nabla^i u \nabla^j u.$$

In this article we will always assume (M, g) is a connected n-dimensional smooth orientable complete Riemannian manifold ($n \geq 3$). $d\mu = \sqrt{\det(g_{ij})}dx$ is the Riemannian volume element on (M, g), where the g_{ij} are the components of the Riemannian metric g in the chart and dx is the Lebesgue volume element of $\mathbb{R}^n$. Let $\gamma : [a, b] \to M$ be a curve of class C^1, the length of γ is

$$L(\gamma) = \int_a^b \sqrt{g\big(\gamma(t)\big)\Big((\frac{d\gamma}{dt})(t), (\frac{d\gamma}{dt})(t)\Big)}d\mu.$$

Email address: lfguo1981@126.com
School of Mathematics and Statistics, Northeast Petroleum University, Heilongjiang, 163318 Daqing, China
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For $m_1, m_2 \in M$, let $C^1_{m_1,m_2}$ be the space of piecewise C^1 curves $\gamma : [a,b] \to M$ such that $\gamma(a) = m_1$ and $\gamma(b) = m_2$. One can define a distance $d_g(m_1, m_2) = \inf_{C^1_{m_1,m_2}} L(\gamma)$ on M.

We denote by $L^1_{loc}(M)$ the space of locally integrable functions on M, denote by $C^\infty_c(M)$ the vector space of smooth functions with compact support on M.

The Riemannian measure and the characteristic function of a set $A \subseteq M$ will be denoted by $\mu(A)$ and χ_A, respectively.

Let $\mathcal{P}(M)$ be the set of all measurable functions $p : M \to [1, \infty]$. For $p \in \mathcal{P}(M)$ we put $M_1 = M^p_1 = \{m \in M : p(m) = 1\}$, $M_\infty = M^p_\infty = \{m \in M : p(m) = \infty\}$, $M_0 = M \backslash (M_1 \cup M_\infty)$, $p^- = \operatorname{essinf}_{M_0} p(m)$ and $p^+ = \operatorname{esssup}_{M_0} p(m)$ if $\mu(M_0) > 0$, $p^- = p^+ = 1$ if $\mu(M_0) = 0$. We always assume that $p \in \mathcal{P}(M)$, $\mathcal{P}_1(M) = \mathcal{P}(M) \cap L^\infty(M)$ and $\mathcal{P}_2(M) = \{p \in \mathcal{P}_1(M) : 1 < \operatorname{essinf}_M p(m)\}$. We use the convention $1/\infty = 0$.

In 2012, Fu and Guo first introduced variable exponent function spaces on Riemannian manifolds in [10]. Also motivated by [10], we are interested in the following Dirichlet problems:

$$\begin{cases} -\operatorname{div}(\nabla u |\nabla u|^{p(m)-2}) + \lambda u |u|^{p(m)-2} = f(m, u), & m \in M, \\ u(m) = 0, & m \in \partial M. \end{cases}$$

2. Preliminaries and Nemytsky Operator

For a function u on M we define the functional $\rho_{p(m),M}$ by

$$\rho_{p(m),M}(u) = \int_{M \backslash M_\infty} |u|^{p(m)} d\mu + \operatorname{esssup}_{M_\infty} |u|.$$

Definition 2.1. The Lebesgue space $L^{p(m)}(M)$ is the class of functions u such that

$$\rho_{p(m),M}(\lambda u) < \infty \text{ for some } \lambda = \lambda(u) > 0$$

with the following norm

$$||u||_{L^{p(m)}(M)} = \inf\{\lambda > 0 : \rho_{p(m),M}(u\lambda) \leq 1\}.$$

Definition 2.2. The Sobolev space $W^{1,p(m)}(M)$ consists of such functions $u \in L^{p(m)}(M)$ for which $\nabla^i u \in L^{p(m)}(M),\ i = 1, 2, ..., n$. The norm is defined by

$$||u||_{W^{1,p(m)}(M)} = ||u||_{L^{p(m)}(M)} + \sum_{i=1}^{n} ||\nabla^i u||_{L^{p(m)}(M)}.$$

The space $W^{1,p(m)}_0(M)$ is defined as the closure of $C^\infty_c(M)$ in $W^{1,p(m)}(M)$.

Given $p \in \mathcal{P}(M)$ we define the conjugate function $p'(m) \in \mathcal{P}(M)$ by

$$p'(m) = \begin{cases} \infty, & \text{if } m \in M_1, \\ 1, & \text{if } m \in M_\infty, \\ \frac{p(m)}{p(m)-1}, & \text{if } m \in M_0. \end{cases}$$

Lemma 2.1 (see [10]). *If $p(m) \in \mathcal{P}(M)$, then the inequality*

$$\int_M |\langle u, v\rangle| d\mu \leq 2||u||_{L^{p(m)}(M)}||v||_{L^{p'(m)}(M)}$$

holds for every $u \in L^{p(m)}(M)$, $v \in L^{p'(m)}(M)$.

Lemma 2.2 (see [10]). *Let $p \in \mathcal{P}_1(M)$. Then*

(i) If $||u||_{L^{p(m)}(M)} \geq 1$, we have $||u||^{p^-}_{L^{p(m)}(M)} \leq \rho_{p(m),M}(u) \leq ||u||^{p^+}_{L^{p(m)}(M)}$.

(ii) If $||u||_{L^{p(m)}(M)} < 1$, we have $||u||^{p^-}_{L^{p(m)}(M)} \geq \rho_{p(m),M}(u) \geq ||u||^{p^+}_{L^{p(m)}(M)}$.

Lemma 2.3 (see [10]). *If $p \in \mathcal{P}_1(M)$, $u_t, u \in L^{p(m)}(M)$, then the following conditions are equivalent:*

(i) $\lim_{t\to\infty} \rho_{p(m),M}(u_t - u) = 0$;

(ii) $\lim_{t\to\infty} ||u_t - u||_{L^{p(m)}(M)} = 0$;

(iii) u_t converges to u on M in measure and

$$\lim_{t\to\infty} \rho_{p(m),M}(u_t) = \rho_{p(m),M}(u).$$

Lemma 2.4 (see [10]). *If $p \in \mathcal{P}_1(M)$, $u \in L^{p(m)}(M)$ is absolutely continuous with respect to the norm $|| \cdot ||_{L^{p(m)}(M)}$.*

Lemma 2.5 (see [10]). *If $p \in \mathcal{P}_2(M)$, then $L^{p(m)}(M)$ and $W^{1,p(m)}(M)$ are separable, reflexive Banach spaces.*

Given two Banach spaces X and Y, the symbol $X \curvearrowright Y$ means that X is continuously embedded in Y.

Lemma 2.6 (see [10]). *Let $0 < \mu(M) < \infty$. If $p(m), q(m) \in \mathcal{P}(M)$ and $p(m) \leq q(m)$ a.e. $m \in M$, then*

$$L^{q(m)}(M) \curvearrowright L^{p(m)}(M). \tag{2.1}$$

The norm of the embedding operator (2.1) does not exceed $\mu(M) + 1$.

Lemma 2.7 (see [10]). *Let M be a compact smooth Riemannian manifold with a boundary or without boundary and $p(m), q(m) \in C(\overline{M}) \cap \mathcal{P}_2(M)$. Assume that*

$$p(m) < n,\ q(m) < \frac{np(m)}{n - p(m)},\ \text{for } m \in \overline{M}.$$

Then

$$W^{1,p(m)}(M) \curvearrowright L^{q(m)}(M)$$

is a continuous and compact imbedding.

Let $f(m, u)$ $(m \in M,\ u \in \mathbb{R})$ be a Carachéodory function, and N_f be the Nemytsky operator defined by f, i.e. $N_f u(m) = f(m, u)$.

Theorem 2.1. *Let M be a compact Riemannian manifold and $p_1, p_2 \in \mathcal{P}_1(M)$. If N_f maps $L^{p_1(m)}(M)$ into $L^{p_2(m)}(M)$, then N_f is continuous, bounded and there is a constant $\beta \geq 0$ and a non-negative function $\alpha(m) \in L^{p_2(m)}(M)$ such that for $m \in M$ and $u \in \mathbb{R}$, the following inequality holds*

$$|f(m, u)| \leq \alpha(m) + \beta|u|^{p_1(m)/p_2(m)}. \tag{2.2}$$

On the other hand, if f satisfies (2.2), then N_f maps $L^{p_1(m)}(M)$ into $L^{p_2(m)}(M)$, and thus N_f is continuous and bounded.

Proof. Since M is compact, M can be covered by a finite number of charts $(U_\alpha, f_\alpha), \alpha = 1, 2 \cdots, k$. For $v \in L^1(M)$, we define

$$Hv(m) = h(m, v) = |N_f(\text{sgn}v|v|^{1/p_1(m)})|^{p_2(m)},$$

then H maps $L^1(M)$ into $L^1(M)$. By [16], we have that H is bounded, continuous and

$$\begin{aligned} \big|H\big(v(m)\chi_{U_\alpha}\big)\big| &= \big|H\big(v(f_\alpha^{-1}(x))\big)\big| \\ &\leq a_\alpha(x) + b_\alpha\big|v(f_\alpha^{-1}(x))\big| \\ &= a_\alpha(f_\alpha(m)) + b_\alpha\big|v(m)\chi_{U_\alpha}\big| \ , \end{aligned}$$

for any $\alpha = 1, 2, \cdots, k$, where $a_\alpha(f_\alpha(m)) \in L^1(U_\alpha)$ is a non-negative function and constant $b_\alpha > 0$. Let $a(m) = \Sigma_{\alpha=1}^k a_\alpha(f_\alpha(m)) \in L^1(M)$ and $b = \max\{b_1, b_2, \cdots, b_k\}$, then $|Hv(m)| \leq a(m) + b|v(m)|$.

We assume that $f(m, 0) = 0$, otherwise we can consider $f(m, u) - f(m, 0)$ instead.

First, we only need to prove N_f is continuous at 0 when $f(m, 0) = 0$. If this is not true, we can find a sequence $\{u_t\} \subset L^{p_1(m)}(M)$ satisfies $\lim_{t\to\infty} ||u_t||_{L^{p_1(m)}(M)} = 0$, but $||N_f u_t||_{L^{p_2(m)}(M)} > \sigma$ where σ is some positive constant. Without loss of generality, we can suppose that $||u_t||_{L^{p_1(m)}(M)} < 1$, thus by Lemma 2.2 we have $\rho_{p_1(m),M}(u_t) \leq ||u_t||_{L^{p_1(m)}(M)}^{p_1^-}$, and hence

$$\lim_{t\to\infty} \int_M |u_t|^{p_1(m)} d\mu = 0.$$

Let $v_t = \text{sgn}u_t|u_t|^{p_1(m)}$. Then $\lim_{t\to\infty} ||v_t||_{L^1(M)} = 0$, and hence $\lim_{t\to\infty} ||Hv_t||_{L^1(M)} = 0$. Thus,

$$\lim_{t\to\infty} \int_M |N_f u_t|^{p_2(m)} d\mu = \lim_{t\to\infty} \int_M |Hv_t| d\mu = 0.$$

By Lemma 2.3, we have $\lim_{t\to\infty} ||N_f u_t||_{L^{p_2(m)}(M)} = 0$, which is a contradiction.

Next, let $\mathcal{A}$ be a bounded set in $L^{p_1(m)}(M)$. By Lemma 2.2, we have that $\mathcal{A}$ is bounded in modular. For $v \in L^1(M)$, let H be defined as above, then $H : L^1(M) \to L^1(M)$ is bounded. For $u \in \mathcal{A}$, taking $v = \text{sgn}u|u|^{p_1(m)} \in L^1(M)$, then $\{||v||_{L^1(M)}\}$ is uniformly bounded. Then there is a constant $C > 0$ such that $||H(\text{sgn}u|u|^{p_1(m)})||_{L^1(M)} \leq C$, thus $\int_M |N_f u|^{p_2(m)} d\mu \leq C$. Therefore, $N_f(\mathcal{A})$ is bounded in $L^{p_2(m)}(M)$.

Since M is compact, M can be covered by a finite number of charts (U_α, f_α). By Lemma 2.4, we can assume $\{U_\alpha\}$ such that $\{u\chi_{U_\alpha}\}$ are uniformly bounded in $L^{p_1(m)}(M)$ for $u \in L^{p_1(m)}(M)$. Writing $u^{(\alpha)} = u\chi_{U_\alpha}$ and $K = \sup\{\sum_\alpha \chi_{U_\alpha}(m) : m \in M\}$, then

$$|N_f u| \leq \sum_\alpha |N_f u^{(\alpha)}| \leq K|N_f u| \text{ and } N_f u^{(\alpha)} \in L^{p_2(m)}(M).$$

Since

$$\int_M |Hv| d\mu \leq \sum_\alpha \int_{U_\alpha} |Hv| d\mu \leq K \int_M |Hv| d\mu,$$

where $v = \text{sgn}u|u|^{p_1(m)} \in L^1(M)$, we have $N_f u \in L^{p_2(m)}(M)$.

For $u \in L^{p_1(m)}(M)$, set $v = \text{sgn}u|u|^{p_1(m)}$, then $v \in L^1(M)$ and thus

$$|N_f u(m)|^{p_2(m)} = |Hv(m)| \le a(m) + b|u(m)|^{p_1(m)}.$$

We can deduce that

$$\begin{aligned} |N_f u(m)| &\le (a(m) + b|u|^{p_1(m)})^{1/p_2(m)} \\ &\le a(m)^{1/p_2(m)} + b^{1/p_2(m)}|u|^{p_1(m)/p_2(m)} \\ &\le \alpha(m) + \beta|u|^{p_1(m)/p_2(m)}, \end{aligned}$$

where $\alpha(m) = a(m)^{1/p_2(m)} \ge 0$, $\alpha(m) \in L^{p_2(m)}(M)$, and $\beta = \max\{1, b\}$.

On the other hand, if (2.2) holds, we let $u \in L^{p_1(m)}(M)$. It is obvious that

$$\alpha(m) + \beta|u|^{p_1(m)/p_2(m)} \in L^{p_2(m)}(M).$$

Therefore

$$\int_M |N_f u|^{p_2(m)} d\mu \le \int_M |\alpha(m) + \beta|u|^{p_1(m)/p_2(m)}|^{p_2(m)} < \infty,$$

i.e. N_f maps $L^{p_1(m)}(M)$ into $L^{p_2(m)}(M)$. □

3. Existence of weak solutions

In this section, we shall show some applications of the Sobolev space to Dirichlet problems of the $p(m)$-harmonic equations on Riemannian manifolds. We shall assume that (M, g) is a connected n-dimensional smooth compact Riemannian manifold with smooth boundary $(n \ge 3)$ and $p(m) \in C(\overline{M}) \cap \mathcal{P}_2(M)$.

Definition 3.1. A fuction u is a weak solution for the following Dirichlet problems

$$\begin{cases} -\text{div}(\nabla u|\nabla u|^{p(m)-2}) + \lambda u|u|^{p(m)-2} = f(m, u), & m \in M, \\ u(m) = 0, & m \in \partial M, \end{cases} \tag{3.1}$$

where $f(m, u) \in L^{p'(m)}(M), \lambda > 0$, if $u \in W_0^{1,p(m)}(M)$ satisfies

$$\int_M \langle \nabla u|\nabla u|^{p(m)-2}, \nabla v\rangle + \lambda uv|u|^{p(m)-2} d\mu = \int_M f(m, u)v d\mu \tag{3.2}$$

for every $v \in W_0^{1,p(m)}(M)$.

Let $(\cdot, \cdot)$ denote a dual between $X := W_0^{1,p(m)}(M)$ and X'. First we define the energy functional on $W_0^{1,p(m)}(M)$ by

$$\Psi(u) = \int_M \frac{1}{p(m)}(|\nabla u|^{p(m)} + \lambda|u|^{p(m)})d\mu - \int_M F(m, u)d\mu := I(u) - K(u),$$

where $F(m, t) = \int_0^t f(m, s)ds$. Then for $u, v \in W_0^{1,p(m)}(M)$, we have

$$\begin{aligned} (\Psi'(u), v) &= (I'(u), v) - (K'(u), v) \\ &= \int_M \langle \nabla u|\nabla u|^{p(m)-2}, \nabla v\rangle d\mu + \int_M \lambda vu|u|^{p(m)-2} d\mu - \int_M f(m, u)v d\mu. \end{aligned}$$

We denote $J = I' : X \to X'$, then

$$(J(u), v) = \int_M \langle \nabla u|\nabla u|^{p(m)-2}, \nabla v\rangle d\mu + \int_M \lambda uv|u|^{p(m)-2} d\mu := (J_1(u), v) + (J_2(u), v),$$

where $u, v \in X$.

Lemma 3.1. *$J = I' : X \to X'$ is a continuous, bounded and strictly monotone operator.*

Proof. It is obvious that J is continuous and bounded. For any $y, z \in \mathbb{R}^N$, we have the following inequalities (see [15]) from which we can get the strictly monotonicity of J:

(h_1) $(|z|^{p-2}z - |y|^{p-2}y) \cdot (z - y) \geq (\frac{1}{2})^p |z - y|^p,\ p \in [2, \infty)$,

(h_2) $[(|z|^{p-2}z - |y|^{p-2}y) \cdot (z - y)](|z|^p + |y^p|)^{(p-2)/2} \geq (p-1)|z - y|^p,\ p \in (1, 2)$. □

By Theorem 2.1 and Lemma 3.1, we can get the following Lemma 3.2.

Lemma 3.2. *The functional $\Psi \in C^1(W_0^{1,p(m)}(M), \mathbb{R})$.*

Therefore, the weak solution to Dirichlet problems (3.1) is a critical point of Ψ and vise versa (see [3]).

Next, we suppose $f(m, s)$ satisfies the following assumption:

(N): Let $f : M \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory condition and

$$|f(m, s)| \leq C_1 + C_2|s|^{\theta(m)-1} \text{ for any } (m, s) \in M \times \mathbb{R},$$

where $\theta(m) \in C(\overline{M}) \cap \mathcal{P}_2(M)$ and $\theta(m) \leq p(m)$.

Lemma 3.3. *The functional Ψ is weakly lower semi-continuous in $W_0^{1,p(m)}(M)$.*

Proof. Let $u_t \rightharpoonup u$ weakly in $W_0^{1,p(m)}(M)$. Since J is a convex functional, we deduced that the following inequality holds

$$J(u_t) \geq J(u) + (J'(u), u_t - u).$$

Then we get that $\liminf_{t\to\infty} J(u_t) \geq J(u)$. Then J is weakly lower semi-continuous.

Let $u_t \rightharpoonup u$ weakly in $W_0^{1,p(m)}(M)$. By Lemma 2.6 and 2.7, we get that $u_t \to u$ strongly in $L^{\theta(m)}(M)$ and $L^1(M)$. Without loss of generality, we assume that $u_t \to u$ a.e. in M, and hence $F(m, u_t) \to F(m, u)$ a.e. $m \in M$. From (N) we have

$$|F(m, s)| \leq C_1|s| + C_2|s|^{\theta(m)},$$

then the integrals of the functions $|F(m, u_t) - F(m, u)|$ possess absolutely equicontinuity on M. By Vitali convergence Theorem (see [13]),

$$\int_M |F(m, u_t) - F(m, u)| d\mu \to 0, \text{ as } t \to \infty.$$

Therefore, Ψ is weakly lower semi-continuous in $W_0^{1,p(m)}(M)$. □

Theorem 3.1. *Let $f(m, s)$ satisfies the condition (N). Then Dirichlet problems (3.1) has a weak solution in $W_0^{1,p(m)}(M)$.*

Proof. From the condition (N) we can obtain $|F(m,s)| \le C_1|s| + C_2|s|^{\theta(m)}$, then by Lemma 2.2 and Young inequality, we have

$$\begin{aligned}\Psi(u) &= \int_M \frac{1}{p(m)}|\nabla u|^{p(m)}d\mu + \int_M \frac{\lambda}{p(m)}|u|^{p(m)}d\mu - \int_M F(x,u)d\mu \\ &\ge \int_M \frac{1}{p(m)}|\nabla u|^{p(m)}d\mu + \int_M \frac{\lambda}{p(m)}|u|^{p(m)}d\mu - \int_M \big(\varepsilon|u|^{p(m)} + C(\varepsilon,\theta)\big)d\mu \\ &\ge \int_M \frac{\min\{\lambda,1\}}{2p^+}\big(|\nabla u|^{p(m)}d\mu + |u|^{p(m)}\big)d\mu - C(\varepsilon,\theta) \\ &\to \infty\end{aligned}$$

as $||u||_{W^{1,p(m)}(M)} \to \infty$, where $\varepsilon = \frac{\min\{\lambda,1\}}{2p^+}$. Since Ψ is weakly lower semi-continuous, Ψ has a minimum point u_0 in $W_0^{1,p(m)}(M)$, and u_0 is a weak solution of Dirichlet problems (3.1). □

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