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INTUITIONISTIC UNCERTAIN LINGUISTIC POWERED EINSTEIN AGGREGATION OPERATORS AND THEIR APPLICATION TO MULTI-ATTRIBUTE GROUP DECISION MAKING

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Abstract The intuitionistic uncertain fuzzy linguistic variable can easily express the fuzzy information, and the power average (PA) operator is a useful tool which provides more versatility in the information aggregation procedure. At the same time, Einstein operations are a kind of various t-norms and t-conorms families which can be used to perform the corresponding intersections and unions of intuitionistic fuzzy sets (IFSs). In this paper, we will combine the PA operator and Einstein operations to intuitionistic uncertain linguistic environment, and propose some new PA operators. Firstly, the definition and some basic operations of intuitionistic uncertain linguistic number (IULN), power aggregation (PA) operator and Einstein operations are introduced. Then, we propose intuitionistic uncertain linguistic fuzzy powered Einstein averaging (IULFPEA) operator, intuitionistic uncertain linguistic fuzzy powered Einstein weighted (IULFPEWA) operator, intuitionistic uncertain linguistic fuzzy Einstein geometric (IULFPEG) operator and intuitionistic uncertain linguistic fuzzy Einstein weighted geometric (IULFPEWG) operator, and discuss some properties of them in detail. Furthermore, we develop the decision making methods for multi-attribute group decision making (MAGDM) problems with intuitionistic uncertain linguistic information and give the detail decision steps. At last, an illustrate example is given to show the process of decision making and the effectiveness of the proposed method.

Keywords Intuitionistic uncertain linguistic number, power aggregation (PA) operator, intuitionistic uncertain linguistic fuzzy powered Einstein weighted (IULFPEWA) operator, intuitionistic uncertain linguistic fuzzy Einstein weighted geometric (IULFPEWG) operator, Multi-attribute Group Decision making.

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1. Introduction

Fuzzy set (FS) proposed by Zadeh [31] is a very useful tool to process the fuzzy information. However, because FS has only a membership function, it is difficult to describe the more complex fuzzy information. Atanassov [1] further proposed the intuitionistic fuzzy set (IFS) which has a membership function and a non-membership function, so IFS has more advantages than FS on describing the inconsistent information. IFS is with membership (or called truth-membership) $T_A(x)$ and non-membership (or called falsity-membership) $F_A(x)$. However, because the

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membership function and non-membership of IFS are crisp numbers which are difficult to be obtained in real decision making, the scopes of IFS are further extended. Gargov and Atanassov [3], Atanassov [2] proposed the interval-valued intuitionistic fuzzy set (IVIFS) which extended the membership and non-membership to interval numbers; Liu and Zhang [32] gave the definition of the triangular intuitionistic fuzzy numbers; Wang [18] defined intuitionistic trapezoidal fuzzy number and interval intuitionistic trapezoidal fuzzy numbers, then some decision making methods had been proposed [17, 20].

In real decision making, sometimes we can use linguistic terms such as 'good', 'bad ' to describe the state or performance of a car and cannot use some numbers to express some qualitative information. However, when we use the linguistic variables to express the qualitative information, it only means the membership degree belonged to a linguistic term is 1, and the non-membership degree or hesitation degree cannot be expressed. In order to overcome this shortcoming, Wang and Li [19] proposed the concept of intuitionistic linguistic set by combining intuitionistic fuzzy set with linguistic variables. For the above-mentioned example, we can give an evaluation value 'good' for the state of the car, however, for this evaluation, we have the certainty degree of 80 percent and negation degree of 10 percent, then we can use the intuitionistic linguistic set to express the evaluation result. Of course, it cannot be expressed by IFS or linguistic variables. Furthermore, Wang and Li [19] proposed intuitionistic two-semantics and the Hamming distance between two intuitionistic two-semantics, and ranked the alternatives by calculating the comprehensive membership degree to the ideal solution for each alternative.

Furthermore, the information aggregation operators are an important research orientation of decision making problems, and many research results have been achieved [7, 9–16, 21–28]. In general, they are divided into are two types, i.e., the arithmetic aggregation operators and the geometric aggregation operators. About the differences between them, Liu [9] gave the explanations "The arithmetic aggregation operators emphasize the impact of the overall attribute data and the compensation between the different attribute data, and the geometric aggregation operators emphasize the balance of the system and the coordination between the different attribute data". In addition, the whole operators were included in the general concepts of the t-norms and t-conorms [4], which satisfy the requirements of the conjunction and disjunction operators [21]. Einstein operations are a kind of various t-norms and t-conorms families which can be used to perform the corresponding intersections and unions of IFSs. So, based on Einstein operations, Wang and Liu [22] proposed some intuitionistic fuzzy Einstein aggregation operators such as the intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator and the intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operator. Zhao and Wei [23] established intuitionistic fuzzy Einstein hybrid average (IFEHA) operator and intuitionistic fuzzy Einstein hybrid geometric (IFEHG) operator and proposed intuitionistic fuzzy MADM methods based on them. Guo et al. [7] proposed some operators which extended Einstein operators to hesitant fuzzy sets including hesitant fuzzy Einstein weighted geometric (HFEWG) operator, hesitant fuzzy Einstein ordered weighted geometric (HFEOWG) operator, hesitant fuzzy Einstein hybrid geometric (HFEHG) operator, and hesitant fuzzy Einstein induced ordered weighted geometric (HFEIOWG) operator .

Yager [30] developed a power average (PA) operator and a power OWA (POWA) operator to provide more versatility in the information aggregation process. Based on this, Xu and Yager [29] proposed some new geometric aggregation operators, such as the power-geometric (PG) operator, weighted PG operator, and power-ordered-weighted geometric (POWG) operator. Zhou and Chen [33] presented the generalized power average (GPA) operator and the generalized power ordered weighted average (GPOWA) operator. Then they presented the linguistic generalized power average (LGPA) operator and the weighted linguistic generalized power average (WLGPA) operator and the linguistic generalized power ordered weighted average (LGPOWA) operator which extended the GPA operator and the GPOWA operator to linguistic environment. The same character of them is their aggregation functions use linguistic information and generalized mean in the power average (PA) operator. Xu and Cai [28] developed the uncertain power average operators which aggregated interval fuzzy preference relations. Xu and Wang [25] proposed 2-tuple linguistic power average (2TLPA) operator, 2-tuple linguistic weighted PA operator (2TLWPA) and 2TLPOWA operator. Zhou et al. [34] presented an uncertain generalized power average (UGPA) operator, an uncertain generalized power ordered weighted average (UGPOWA) operator to deal with these arguments which take the form of interval numbers. They developed the generalized intuitionistic fuzzy power averaging (GIFPA) operator and the generalized intuitionistic fuzzy power ordered weighted averaging (GIFPOWA) operator which extended the GPA operator and the GPOWA operator to intuitionistic fuzzy environment.

The intuitionistic uncertain fuzzy linguistic variable can easily express the fuzzy information and the power average (PA) operator is a useful tool which provides more versatility in the information aggregation procedure, and Einstein operations are a kind of various t-norms and t-conorms families can be used to perform the corresponding intersections and unions of IFSs. However, there is no research on the combination of PA operator and Einstein operations under intuitionistic uncertain linguistic environment. The main purpose of this paper is to propose some intuitionistic uncertain linguistic fuzzy powered Einstein operators to extend the using scope of PA operator, and to develop some MAGDM methods based on these operators.

In order to achieve this aim, this paper is organized as following. In the second section, we represent some concepts of the linguistic set and uncertain linguistic numbers, the intuitionistic linguistic set, the Power Aggregation (PA) operator, Einstein operations and we define Einstein operations of intuitionistic uncertain linguistic numbers. In section 3, we propose the concept and operations of intuitionistic uncertain linguistic fuzzy powered Einstein averaging (IULFPEA) operator, intuitionistic uncertain linguistic fuzzy powered Einstein weighted (IULFPEWA) operator, intuitionistic uncertain linguistic fuzzy Einstein geometric (IULFPEG) operator and intuitionistic uncertain linguistic fuzzy Einstein weighted geometric (IULFPEWG) operator, and introduce some properties and special cases of them. Section 4 establishes the procedure of the decision-making method based on the IULFPEWA and IULFPEWG operators. Section 5 gives a numerical example according to our approach. Section 6 summarizes the main conclusion of this paper.

2. Preliminaries

2.1. The linguistic set and uncertain linguistic numbers

The linguistic set is regarded as a good tool to express these qualitative information, we can express the linguistic set by $S = (s_0, s_1, \ldots, s_{l-1})$, and $s_\theta(\theta = 1, 2, \ldots, l-1)$

can be called an linguistic number, l is an odd value which can be the values of 3, 5, 7, 9, etc. generally, For example, when $l = 9$, $S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) =$ (extremely poor, very poor, poor, slightly poor, fair, slightly good, good, very good, extremely good).

Let s_i and s_j are any two linguistic numbers in linguistic set S, they have the following characteristics [5,6]:

(i) If $i > j$, then $si \succ sj$.

(ii) There exists negative operator: $neg(s_i) = s_j$, where $j = l - 1 - i$.

(iii) If $s_i \geq s_j$, $\max(s_i, s_j) = s_i$.

(iv) If $s_i \leq s_j$, $\min(s_i, s_j) = s_i$.

In order to overcome the loss of information in the process of calculations, the original discrete linguistic set $S = (s_0, s_1, \ldots, s_{l-1})$ is extended to the continuous linguistic set $\bar{S} = \{s_\alpha | \alpha \in R^+\}$ which is also meet the strictly monotonically increasing condition [6,34]. Some operational rules are defined as follows [5,6].

$$(1)\quad \beta s_i = s_{\beta \times i}\ ;\ \beta \geq 0, \tag{2.1}$$

$$(2)\quad s_i \oplus s_j = s_{i+j}, \tag{2.2}$$

$$(3)\quad s_i \otimes s_j = s_{i \times j}, \tag{2.3}$$

$$(4)\quad (s_i)^n = s_{i^n}; n \geq 0. \tag{2.4}$$

Definition 2.1 ([26]). Suppose $\tilde{s} = [s_a, s_b]$, $s_a, s_b \in \bar{S}$ with $a \leq b$ are the lower limit and the upper limit of $\tilde{s}$, respectively, then $\tilde{s}$ is called an uncertain linguistic variable.

Let $\tilde{S}$ be a set of all uncertain linguistic variables. $\tilde{s}_1 = [s_{a1}, s_{b1}]$ and $\tilde{s}_2 = [s_{a2}, s_{b2}]$ are any two uncertain linguistic variables, the operational rules are defined as follows [26,27]:

$$(1)\quad \tilde{s}_1 \oplus \tilde{s}_2 = [s_{a1}, s_{b1}] \oplus [s_{a2}, s_{b2}] = [s_{a1+a2}, s_{b1+b2}], \tag{2.5}$$

$$(2)\quad \tilde{s}_1 \otimes \tilde{s}_2 = [s_{a1}, s_{b1}] \otimes [s_{a2}, s_{b2}] = [s_{a1\times a2}, s_{b1\times b2}], \tag{2.6}$$

$$(3)\quad \lambda \tilde{s}_1 = \lambda [s_{a1}, s_{b1}] = [s_{\lambda * a1}, s_{\lambda * b1}], \lambda \geq 0, \tag{2.7}$$

$$(4)\quad (\tilde{s}_1)^\lambda = [s_{a1}, s_{b1}]^\lambda = [s_{a1^\lambda}, s_{b1^\lambda}], \lambda \geq 0. \tag{2.8}$$

2.2. The intuitionistic uncertain linguistic set (IULS)

Definition 2.2 ([19]). Let $h_{\theta(x)} \in \bar{S}$, X be the given discourse domain, then

$$A = \{< x[h_{\theta(x),}(u_A(x), v_A(x))] > | x \in X), \tag{2.9}$$

is called an intuitionistic linguistic set (ILS). where $u_A : X \to [0,1]$ and $v_A : X \to [0,1]$ and satisfying $0 \leq u_A(x) + v_A(x) \leq 1$, $\forall x \in X$. The numbers u_A and v_A respectively represent the membership degree and non-membership degree of the element x to the linguistic term $h_{\theta(x)}$.

The degree of indeterminacy of x to the linguistic term $h_{\theta(x)}$ can be written by $\pi(x) = 1 - u_A(x) - v_A(x)$ where $0 \leq \pi(x) \leq 1$, $\forall x \in X$.

Definition 2.3 ([7]). Let $[s_{\theta(x)}, s_{\tau(x)}] \in \tilde{S}$, and X be the given discourse domain, then

$$A = \{\langle x | [s_{\theta(x)}, s_{\tau(x)}], (u_A(x), v_A(x))] \rangle | x \in X\}, \tag{2.10}$$

is called an intuitionistic uncertain linguistic set (IULS) in which $s_{\theta(x)}, s_{\tau(x)} \in \bar{S}, u_A : X \to [0,1]$, and $v_A : X \to [0,1]$ satisfying the condition $0 \le u_A(x) + v_A(x) \le 1$, $\forall x \in X$. The u_A and v_A respectively express the membership degree and non-membership degree of the element x to the uncertain linguistic variable $[s_{\theta(x)}, s_{\tau(x)}] \in \tilde{S}$.

For each IULS in X, if $\pi(x) = 1 - u_A(x) - v_A(x)$, $\forall x \in X$ then $\pi(x)$ is called the degree of uncertainty of x to the uncertain linguistic variable $[s_{\theta(x)}, s_{\tau(x)}]$. Obviously, It meets $0 \le \pi(x) \le 1$, $\forall x \in X$.

Definition 2.4 ([7]). Let $A = \{\langle x|[s_{\theta(x)}, s_{\tau(x)}], (u_A(x), v_A(x))]\rangle | x \in X\}$ be intuitionistic uncertain linguistic set, and $a = \left\langle [s_{\theta(x)}, s_{\tau(x)}], (u_A(x), v_A(x)) \right\rangle$ is called an intuitionistic uncertain linguistic number (IULN).

Suppose $\tilde{a}_1 = \langle [s_{\theta(a_1)}, s_{\tau(a_1)}], (u(a_1), v(a_1))\rangle$ and $\tilde{a}_2 = \langle [s_{\theta(a_2)}, s_{\tau(a_2)}], (u(a_2), v(a_2))\rangle$ be any two intuitionistic uncertain linguistic numbers, the operational laws are defined as follows [7]:

$$(1)\ \ \tilde{a}_1 + \tilde{a}_2 = \langle [s_{\theta(a_1)+\theta(a_2)}, s_{\tau(a_1)+\tau(a_2)}], (1-(1-u(a_1))(1-u(a_2)), v(a_1)v(a_2))\rangle, \tag{2.11}$$

$$(2)\ \ \tilde{a}_1 \otimes \tilde{a}_2 = \langle [s_{\theta(a_1)\times\theta(a_2)}, s_{\tau(a_1)\times\tau(a_2)}], (u(a_1)u(a_2), v(a_1)+v(a_2)-v(a_1)v(a_2))\rangle, \tag{2.12}$$

$$(3)\ \ \lambda\tilde{a}_1 = \langle [s_{\lambda\times\theta(a_1)}, s_{\lambda\times\tau(a_1)}], (1-(1-u(a_1))^{\lambda}, (v(a_1))^{\lambda})\rangle, \lambda \ge 0, \tag{2.13}$$

$$(4)\ \ \tilde{a}_1^{\lambda} = \langle [s_{(\theta(a_1))^{\lambda}}, s_{(\tau(a_1))^{\lambda}}], ((u(a_1))^{\lambda}, 1-(1-v(a_1))^{\lambda})\rangle, \lambda \ge 0. \tag{2.14}$$

Obviously, these operational results are still intuitionistic uncertain linguistic numbers.

Theorem 2.1 ([7]). *Let* $\tilde{a}_1 = \langle [s_{\theta(a_1)}, s_{\tau(a_1)}], (u(a_1), v(a_1))\rangle$ *and* $\tilde{a}_2 = \langle [s_{\theta(a_2)}, s_{\tau(a_2)}], (u(a_2), v(a_2))\rangle$ *be any two intuitionistic uncertain linguistic numbers, the operational laws have the following characteristics.*

$$(1)\ \ \tilde{a}_1 + \tilde{a}_2 = \tilde{a}_2 + \tilde{a}_1; \tag{2.15}$$

$$(2)\ \ \tilde{a}_1 \otimes \tilde{a}_2 = \tilde{a}_2 \otimes \tilde{a}_1; \tag{2.16}$$

$$(3)\ \ \lambda(\tilde{a}_1 + \tilde{a}_2) = \lambda\tilde{a}_1 + \lambda\tilde{a}_2, \lambda \ge 0; \tag{2.17}$$

$$(4)\ \ \lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_1 = (\lambda_1+\lambda_2)\tilde{a}_1, \lambda_1, \lambda_2 \ge 0; \tag{2.18}$$

$$(5)\ \ \tilde{a}_1^{\lambda_1} \otimes \tilde{a}_1^{\lambda_2} = (\tilde{a}_1)^{\lambda_1+\lambda_2}, \lambda_1, \lambda_2 \ge 0; \tag{2.19}$$

$$(6)\ \ \tilde{a}_1^{\lambda_1} \otimes \tilde{a}_2^{\lambda_1} = (\tilde{a}_1 \otimes \tilde{a}_2)^{\lambda_1}, \lambda_1 \ge 0. \tag{2.20}$$

Definition 2.5 ([7]). Suppose $\tilde{a}_1 = \langle [s_{\theta(a_1)}, s_{\tau(a_1)}], (u(a_1), v(a_1))\rangle$ is an intuitionistic uncertain linguistic number, then the expectation value $E(\tilde{a}_1)$ of $\tilde{a}_1$ can be defined as follows.

$$E(\tilde{a}_1) = \frac{1}{2} \times (u(a_1)+1-v(a_1)) \times s_{(\theta(a_1)+\tau(a_1))/2} = s_{((\theta(a_1)+\tau(a_1))\times(u(a_1)+1-v(a_1)))/4}. \tag{2.21}$$

Definition 2.6 ([7]). Let $\tilde{a}_1 = \langle [s_{\theta(a_1)}, s_{\tau(a_1)}], (u(a_1), v(a_1))\rangle$ be an intuitionistic uncertain linguistic number, then the accuracy function $H(\tilde{a}_1)$ of $\tilde{a}_1$ can be defined as follows.

$$H(\tilde{a}_1) = (u(a_1)+v(a_1)) \times s_{(\theta(a_1)+\tau(a_1))/2} = s_{((\theta(a_1)+\tau(a_1))\times(u(a_1)+v(a_1)))/2}. \tag{2.22}$$

Definition 2.7 ([7]). Let $\tilde{a}_1 = \langle [s_{\theta(a_1)}, s_{\tau(a_1)}], (u(a_1), v(a_1)) \rangle$ and $\tilde{a}_2 = \langle [s_{\theta(a_2)}, s_{\tau(a_2)}], (u(a_2), v(a_2)) \rangle$ be any two intuitionistic uncertain linguistic numbers, then

(1) if $E(\tilde{a}_1) > E(\tilde{a}_2)$, then $\tilde{a}_1 \succ \tilde{a}_2$,

(2) if $E(\tilde{a}_1) = E(\tilde{a}_2)$, then:

 (i) if $H(\tilde{a}_1) > H(\tilde{a}_2)$, then $\tilde{a}_1 \succ \tilde{a}_2$,

 (ii) if $H(\tilde{a}_1) = H(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$.

Definition 2.8 ([13]). Let $\tilde{a}_1 = \langle [s_{\theta(a_1)}, s_{\tau(a_1)}], (u(a_1), v(a_1)) \rangle$ and $\tilde{a}_2 = \langle [s_{\theta(a_2)}, s_{\tau(a_2)}], (u(a_2), v(a_2)) \rangle$ be any two intuitionistic uncertain linguistic numbers, then the normalized Hamming distance between $\tilde{a}_1$ and $\tilde{a}_2$ can be defined as follows:

$$d(\tilde{a}_1, \tilde{a}_2) = \frac{1}{4(l-1)} \left| (1+u(a_1)-v(a_1))(\theta(a_1)+\tau(a_1)) - (1+u(a_2)-v(a_2))(\theta(a_2)+\tau(a_2)) \right|, \tag{2.23}$$

which meets the following conditions:

$$(1)\ \ 0 \le d(\tilde{a}_1, \tilde{a}_2) \le 1; \tag{2.24}$$

$$(2)\ \ d(\tilde{a}_1, \tilde{a}_2) = 0; \tag{2.25}$$

$$(3)\ \ d(\tilde{a}_1, \tilde{a}_2) = d(\tilde{a}_2, \tilde{a}_1); \tag{2.26}$$

$$(4)\ \ d(\tilde{a}_1, \tilde{a}_2) + d(\tilde{a}_2, \tilde{a}_3) \ge d(\tilde{a}_1, \tilde{a}_3). \tag{2.27}$$

2.3. The power aggregation (PA) operator

Definition 2.9 ([30]). The Power Aggregation (PA) operator, which is firstly proposed by Yager, is defined as follows:

$$PA(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n} (1 + T(a_i)) \cdot a_i}{\sum_{i=1}^{n} (1 + T(a_i))}, \tag{2.28}$$

where $T(a_i) = \sum_{j=1, j \ne i}^{n} \sup(a_i, a_j)$, and $\sup(a_i, a_j)$ means the support for a_i from a_j, which satisfies the following rules:

$$(1)\ \ \sup(a_i, a_j) = \sup(a_j, a_i); \tag{2.29}$$

$$(2)\ \ \sup(a_i, a_j) \in [0, 1]; \tag{2.30}$$

$$(3)\ \ \sup(a_i, a_j) \ge \sup(a_m, a_n), \text{ if } |a_i - a_j| \le |a_m - a_n|. \tag{2.31}$$

2.4. Einstein operations of intuitionistic uncertain linguistic numbers

Einstein operations are a kind of the t-norms and t-conorms families which can be used to perform the corresponding intersections and unions of IFSs. Einstein operations are defined as follows [8]:

$$(1)\ \ a \otimes_\varepsilon b = \frac{a+b}{1+a \cdot b}, a, b \in [0, 1], \tag{2.32}$$

$$(2)\ \ a \oplus_\varepsilon b = \frac{a \cdot b}{1 + (1-a) \cdot (1-b)}, a, b \in [0, 1], \tag{2.33}$$

in which Einstein product$\otimes_\varepsilon$ is a t-norm and Einstein sum $\oplus_\varepsilon$ is a t-conorm.

Let $\tilde{a}_1 = \langle [s_{\theta(a_1)}, s_{\tau(a_1)}], (u(a_1), v(a_1)) \rangle$ and $\tilde{a}_2 = \langle [s_{\theta(a_2)}, s_{\tau(a_2)}], (u(a_2), v(a_2)) \rangle$, then we can define the operational rules of intuitionistic uncertain linguistic numbers based on Einstein t-norm and t-conorm shown as follows:

$$(1)\ \tilde{a}_1 \otimes_\varepsilon \tilde{a}_2 = \left\langle \left[s_{\theta(a_1)\theta(a_2)}, s_{\tau(a_1)\tau(a_2)} \right], \left(\frac{u(a_1)\, u(a_2)}{1 + (1 - u(a_1))(1 - u(a_2))}, \frac{\nu(a_1) + \nu(a_2)}{1 + \nu(a_1)\,\nu(a_2)} \right) \right\rangle, \tag{2.34}$$

$$(2)\ \tilde{a}_1 \oplus_\varepsilon \tilde{a}_2 = \left\langle \left[s_{\theta(a_1)+\theta(a_2)}, s_{\tau(a_1)+\tau(a_2)} \right], \left(\frac{u(a_1) + u(a_2)}{1 + u(a_1)\, u(a_2)}, \frac{\nu(a_1)\,\nu(a_2)}{1 + (1 - \nu(a_1))(1 - v(a_2))} \right) \right\rangle, \tag{2.35}$$

$$(3)\ \lambda\tilde{a}_1 = \left\langle \left[s_{\lambda\theta(a_1)}, s_{\lambda\tau(a_1)} \right], \left(\frac{(1 + u(a_1))^\lambda - (1 - u(a_1))^\lambda}{(1 + u(a_1))^\lambda + (1 - u(a_1))^\lambda}, \frac{2(\nu(a_1))^\lambda}{(2 - \nu(a_1))^\lambda + (\nu(a_1))^\lambda} \right) \right\rangle, \quad \lambda \geq 0, \tag{2.36}$$

$$(4)\ \tilde{a}_1^\lambda = \left\langle \left[s_{(\theta(a_1))^\lambda}, s_{(\tau(a_1))^\lambda} \right], \left(\frac{2(u(a_1))^\lambda}{(2 - u(a_1)) + (u(a_1))^\lambda}, \frac{(1 + \nu(a_1))^\lambda - (1 - \nu(a_1))^\lambda}{(1 + \nu(a_1))^\lambda + (1 - \nu(a_1))^\lambda} \right) \right\rangle, \quad \lambda \geq 0. \tag{2.37}$$

Theorem 2.2. *Let $\tilde{a}_1 = \langle [s_{\theta(a_1)}, s_{\tau(a_1)}], (u(a_1), v(a_1)) \rangle$ and $\tilde{a}_2 = \langle [s_{\theta(a_2)}, s_{\tau(a_2)}], (u(a_2), v(a_2)) \rangle$ be two intuitionistic uncertain linguistic numbers, then we have the following operation rules.*

$$(1)\ \tilde{a}_1 \otimes_\varepsilon \tilde{a}_2 = \tilde{a}_2 \otimes_\varepsilon \tilde{a}_1, \tag{2.38}$$

$$(2)\ \tilde{a}_1 \oplus_\varepsilon \tilde{a}_2 = \tilde{a}_2 \oplus_\varepsilon \tilde{a}_1, \tag{2.39}$$

$$(3)\ \lambda(\tilde{a}_1 \oplus_\varepsilon \tilde{a}_2) = \lambda\tilde{a}_2 \oplus_\varepsilon \lambda\tilde{a}_1, \lambda \geq 0, \tag{2.40}$$

$$(4)\ \lambda_1\tilde{a}_1 \oplus_\varepsilon \lambda_2\tilde{a}_1 = (\lambda_1 + \lambda_2) \oplus_\varepsilon \tilde{a}_1, \lambda_1, \lambda_2 \geq 0, \tag{2.41}$$

$$(5)\ \tilde{a}_1^{\lambda_1} \otimes_\varepsilon \tilde{a}_1^{\lambda_2} = \tilde{a}_1^{\lambda_1+\lambda_2}, \lambda_1 \geq 0, \lambda_2 \geq 0, \tag{2.42}$$

$$(6)\ \tilde{a}_1^{\lambda_1} \otimes_\varepsilon \tilde{a}_2^{\lambda_1} = (\tilde{a}_1 \otimes_\varepsilon \tilde{a}_2)^{\lambda_1}, \lambda_1 \geq 0. \tag{2.43}$$

Proof.

(1) Formula (2.38) is obviously right according to the operational rule (2.1) expressed by (2.34).

(2) Formula (2.39) is obviously right according to the operational rule (2.2) expressed by (2.35).

(3) For the left hand of (2.40), we have

$$\tilde{a}_1 \oplus_\varepsilon \tilde{a}_2 = \left\langle \left[s_{\theta(a_1)+\theta(a_2)} s_{\tau(a_1)+\tau(a_2)} \right], \left(\frac{u(a_1) + u(a_2)}{1 + u(a_1)\, u(a_2)}, \frac{\nu(a_1)\,\nu(a_2)}{1 + (1 - \nu(a_1))(1 - v(a_2))} \right) \right\rangle;$$

then

$$\lambda\left(\tilde{a}_1 \oplus \tilde{a}_2\right) = \Big\langle \left[s_{\lambda(\theta(a_1)+\theta(a_1))}, s_{\lambda(\tau(a_1)+\tau(a_1))}\right], \left(\frac{\left(\left(1+u(a_1)\right)\left(1+u(a_2)\right)\right)^{\lambda} - \left(\left(1+u(a_1)\right)\left(1+u(a_2)\right)\right)^{\lambda}}{\left(\left(1+u(a_1)\right)\left(1+u(a_2)\right)\right)^{\lambda} + \left(\left(1+u(a_1)\right)\left(1+u(a_2)\right)\right)^{\lambda}}, \frac{2\left(v(a_1)v(a_2)\right)^{\lambda}}{\left(4-2v(a_1)-2v(a_2)+v(a_1)v(a_2)\right)^{\lambda} + \left(v(a_1)v(a_2)\right)^{\lambda}}\right)\Big\rangle;$$

and for the right hand of (2.40), we have

$$\lambda\tilde{a}_1 = \left\langle \left[s_{\lambda\theta(a_1)}, s_{\lambda\tau(a_1)}\right], \left(\frac{\left(1+u(a_1)\right)^{\lambda} - \left(1-u(a_1)\right)^{\lambda}}{\left(1+u(a_1)\right)^{\lambda} + \left(1-u(a_1)\right)^{\lambda}}, \frac{2\left(\nu(a_1)\right)^{\lambda}}{\left(2-\nu(a_1)\right)^{\lambda} + \left(\nu(a_1)\right)^{\lambda}}\right)\right\rangle;$$

$$\lambda\tilde{a}_2 = \left\langle \left[s_{\lambda\theta(a_2)}, s_{\lambda\tau(a_2)}\right], \left(\frac{\left(1+u(a_2)\right)^{\lambda} - \left(1-u(a_2)\right)^{\lambda}}{\left(1+u(a_2)\right)^{\lambda} + \left(1-u(a_2)\right)^{\lambda}}, \frac{2\left(\nu(a_2)\right)^{\lambda}}{\left(2-\nu(a_2)\right)^{\lambda} + \left(\nu(a_2)\right)^{\lambda}}\right)\right\rangle;$$

then

$$\lambda\tilde{a}_1 \oplus_{\varepsilon} \lambda\tilde{a}_2 = \left\langle \left[s_{\lambda\theta(a_1)\theta(a_1)}, s_{\lambda\tau(a_1)\tau(a_1)}\right], \left(\frac{\left(\left(1+u(a_1)\right)\left(1+u(a_2)\right)\right)^{\lambda} - \left(\left(1+u(a_1)\right)\left(1+u(a_2)\right)\right)^{\lambda}}{\left(\left(1+u(a_1)\right)\left(1+u(a_2)\right)\right)^{\lambda} + \left(\left(1+u(a_1)\right)\left(1+u(a_2)\right)\right)^{\lambda}}, \frac{2\left(v(a_1)v(a_2)\right)^{\lambda}}{\left(4-2v(a_1)-2v(a_2)+v(a_1)v(a_2)\right)^{\lambda} + \left(v(a_1)v(a_2)\right)^{\lambda}}\right)\right\rangle;$$

so, we have$\lambda\left(\tilde{a}_1 \oplus_{\varepsilon} \tilde{a}_2\right) = \lambda\tilde{a}_2 \oplus_{\varepsilon} \lambda\tilde{a}_1, \lambda \geq 0$. i.e., formula (2.40) is right.

(4) Similar to the proof of (2.40), it is easy to prove the formula (2.41) is right. The proof is omitted here.

(5) For the left hand of (2.42), we have

$$\tilde{a}_1^{\lambda_1} = \left\langle \left[s_{(\theta(a_1))^{\lambda_1}}, s_{(\tau(a_1))^{\lambda_1}}\right], \left(\frac{2\left(u(a_1)\right)^{\lambda_1}}{\left(2-u(a_1)\right)^{\lambda} + \left(u(a_1)\right)^{\lambda_1}}, \frac{\left(1+\nu(a_1)\right)^{\lambda_1} - \left(1-\nu(a_1)\right)^{\lambda_1}}{\left(1+\nu(a_1)\right)^{\lambda_1} + \left(1-\nu(a_1)\right)^{\lambda_1}}\right)\right\rangle;$$

$$\tilde{a}_1^{\lambda_2} = \left\langle \left[s_{(\theta(a_1))^{\lambda_2}}, s_{(\tau(a_1))^{\lambda_2}}\right], \left(\frac{2\left(u(a_1)\right)^{\lambda_2}}{\left(2-u(a_1)\right)^{\lambda} + \left(u(a_1)\right)^{\lambda_2}}, \frac{\left(1+\nu(a_1)\right)^{\lambda_2} - \left(1-\nu(a_1)\right)^{\lambda_2}}{\left(1+\nu(a_1)\right)^{\lambda_2} + \left(1-\nu(a_1)\right)^{\lambda_2}}\right)\right\rangle;$$

then

$$\tilde{a}_1^{\lambda_1} \otimes_\varepsilon \tilde{a}_1^{\lambda_2} = \left\langle \left[s_{(\theta(a_1))^{\lambda_1+\lambda_2}}, s_{(\tau(a_1))^{\lambda_1+\lambda_2}} \right], \left(\frac{u(a_1)^{\lambda_1+\lambda_2}}{u(a_1)^{\lambda_1}(2-u(a_1))^{\lambda_2} + u(a_1)^{\lambda_2}(2-u(a_1))^{\lambda_1}}, \frac{2(1+v(a_1))^{\lambda_1+\lambda_2}}{\left((1+v(a_1))^{\lambda_1}+(1-v(a_1))^{\lambda_1}\right)\left((1+v(a_1))^{\lambda_2}+(1-v(a_1))^{\lambda_2}\right)} \right) \right\rangle;$$

and for the right hand of (2.42), we have

$$\tilde{a}_1^{\lambda_1+\lambda_2} = \left\langle \left[s_{(\theta(a_1))^{\lambda_1+\lambda_2}}, s_{(\tau(a_1))^{\lambda_1+\lambda_2}} \right], \left(\frac{u(a_1)^{\lambda_1+\lambda_2}}{u(a_1)^{\lambda_1}(2-u(a_1))^{\lambda_2} + u(a_1)^{\lambda_2}(2-u(a_1))^{\lambda_1}}, \frac{2(1+v(a_1))^{\lambda_1+\lambda_2}}{\left((1+v(a_1))^{\lambda_1}+(1-v(a_1))^{\lambda_1}\right)\left((1+v(a_1))^{\lambda_2}+(1-v(a_1))^{\lambda_2}\right)} \right) \right\rangle.$$

So, we have $\tilde{a}_1^{\lambda_1} \oplus_\varepsilon \tilde{a}_1^{\lambda_2} = \tilde{a}_1^{\lambda_1+\lambda_2}, \lambda_1 \geq 0, \lambda_2 \geq 0$. i.e., formula (2.42) is right.

(6) Similar to the proof of (2.42), it is easy to prove the formula (2.43) is right. The proof is omitted here.

□

3. Some intuitionistic uncertain linguistic fuzzy powered Einstein operators

In this section, we will combine the PA operator and Einstein operations to intuitionistic uncertain linguistic environment, and propose intuitionistic uncertain linguistic fuzzy powered Einstein averaging (IULFPEA) operator, intuitionistic uncertain linguistic fuzzy powered Einstein weighted averaging (IULFPEWA) operator, intuitionistic uncertain linguistic fuzzy powered Einstein geometric (IULFPEG) operator and intuitionistic uncertain linguistic fuzzy powered Einstein weighted geometric (IULFPEWG) operator, and discuss the properties of them.

Definition 3.1. Let $\tilde{a}_i = \langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i)) \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of intuitionistic uncertain linguistic fuzzy numbers, and IULFPEA: $\Omega^n \to \Omega$. If

$$IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\overset{n}{\underset{i=1}{\oplus_\varepsilon}} (1+T(\tilde{a}_i))\,\tilde{a}_i}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))} = \overset{n}{\underset{i=1}{\oplus_\varepsilon}} \left(\frac{(1+T(\tilde{a}_i))\,\tilde{a}_i}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))} \right), \quad (3.1)$$

where, Ω is the set of all intuitionistic uncertain linguistic fuzzy numbers, and $T(\tilde{a}_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \sup(\tilde{a}_i, \tilde{a}_j)$,and $\sup(\tilde{a}_i, \tilde{a}_j)$ is the support for $\tilde{a}_i$ from $\tilde{a}_j$, then *IULFPEA* is called the intuitionistic uncertain linguistic fuzzy powered Einstein averaging (IULFPEA) operator.

Theorem 3.1. *Let* $\tilde{a}_i = \langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i)) \rangle$ $(i = 1, 2, \ldots, n)$ *be a collection of intuitionistic uncertain linguistic fuzzy numbers, then the result aggregated from Definition 3.1 is still an intuitionistic uncertain linguistic fuzzy number, and*

$$\begin{aligned}
&IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \\
&= \left\langle \left[s_{\sum_{i=1}^{n} \frac{\theta(a_i)(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}, s_{\sum_{i=1}^{n} \frac{\tau(a_i)(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \right], \right. \\
&\left(\frac{\prod_{i=1}^{n} (1+u(a_i))^{\frac{1+T(\tilde{a}_i)}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} - \prod_{i=1}^{n} (1-u(a_i))^{\frac{1+T(\tilde{a}_i)}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}{\prod_{i=1}^{n} (1+u(a_i))^{\frac{1+T(\tilde{a}_i)}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} + \prod_{i=1}^{n} (1-u(a_i))^{\frac{1+T(\tilde{a}_i)}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}, \right. \qquad (3.2) \\
&\left. \left. \frac{2 \prod_{i=1}^{n} (v(a_i))^{\frac{1+T(\tilde{a}_i)}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}{\prod_{i=1}^{n} (2-v(a_i))^{\frac{1+T(\tilde{a}_i)}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} + \prod_{i=1}^{n} (v(a_i))^{\frac{1+T(\tilde{a}_i)}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}} \right) \right\rangle,
\end{aligned}$$

where $T(\tilde{a}_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \sup(\tilde{a}_i, \tilde{a}_j)$, *and* $\sup(\tilde{a}_i, \tilde{a}_j)$ *is the support for* $\tilde{a}_i$ *from* $\tilde{a}_j$.

Proof. To simplify the Eq. (3.2), we suppose $c_i = \frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}$ $(i = 1, 2, \ldots, n)$, then, the Eq. (3.2) can be expressed as follows:

$$\begin{aligned}
&IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \\
&= \left\langle \left[s_{\sum_{i=1}^{n} \theta(a_i) c_i}, s_{\sum_{i=1}^{n} \tau(a_i) c_i} \right], \right. \qquad (3.3) \\
&\left. \left(\frac{\prod_{i=1}^{n} (1+u(a_i))^{c_i} - \prod_{i=1}^{n} (1-u(a_i))^{c_i}}{\prod_{i=1}^{n} (1+u(a_i))^{c_i} + \prod_{i=1}^{n} (1-u(a_i))^{c_i}}, \frac{2 \prod_{i=1}^{n} (v(a_i))^{c_i}}{\prod_{i=1}^{n} (2-v(a_i))^{c_i} + \prod_{i=1}^{n} (v(a_i))^{c_i}} \right) \right\rangle.
\end{aligned}$$

The Eq. (3.3) can be proved by Mathematical induction on n as follows:

(i) When n=1, the Eq. (3.3) is right obviously.

(ii) Suppose when$n = k$, the Eq.(3.3) is right, i.e.,

$$\begin{aligned}
&IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) \\
&= \left\langle \left[s_{\sum_{i=1}^{k} \theta(a_i) c_i}, s_{\sum_{i=1}^{k} \tau(a_i) c_i} \right], \right.
\end{aligned}$$

$$\left(\frac{\prod_{i=1}^{k}(1+u(a_i))^{c_i}-\prod_{i=1}^{k}(1-u(a_i))^{c_i}}{\prod_{i=1}^{k}(1+u(a_i))^{c_i}+\prod_{i=1}^{k}(1-u(a_i))^{c_i}},\ \frac{2\prod_{i=1}^{k}(v(a_i))^{c_i}}{\prod_{i=1}^{k}(2-v(a_i))^{c_i}+\prod_{i=1}^{k}(v(a_i))^{c_i}}\right)\Bigg\rangle.$$

Then when $n = k+1$, we have

$$\begin{aligned}
&IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{k+1})\\
&= IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) \oplus_\varepsilon (c_{k+1}\tilde{a}_{k+1})\\
&= IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k) \oplus_\varepsilon \Big\langle \left[s_{\theta(a_{k+1})c_{k+1}}, s_{\tau(a_{k+1})c_{k+1}}\right],\\
&\quad \left(\frac{(1+u(a_{k+1}))^{c_{k+1}}-(1-u(a_{k+1}))^{c_{k+1}}}{(1+u(a_{k+1}))^{c_{k+1}}+(1-u(a_{k+1}))^{c_{k+1}}},\right.\\
&\quad \left.\frac{2(v(a_{k+1}))^{c_{k+1}}}{(2-v(a_{k+1}))^{c_{k+1}}+(v(a_{k+1}))^{c_{k+1}}}\right)\Big\rangle\\
&= \left\langle \left[s_{\sum_{i=1}^{k+1}\theta(a_i)c_i}, s_{\sum_{i=1}^{k+1}\tau(a_i)c_i}\right], \left(\frac{\prod_{i=1}^{k+1}(1+u(a_i))^{c_i}-\prod_{i=1}^{k+1}(1-u(a_i))^{c_i}}{\prod_{i=1}^{k+1}(1+u(a_i))^{c_i}+\prod_{i=1}^{k+1}(1-u(a_i))^{c_i}},\right.\right.\\
&\quad \left.\left.\frac{2\prod_{i=1}^{k+1}(v(a_i))^{c_i}}{\prod_{i=1}^{k+1}(2-v(a_i))^{c_i}+\prod_{i=1}^{k+1}(v(a_i))^{c_i}}\right)\right\rangle.
\end{aligned}$$

So, when $n = k+1$, the Eq. (3.3) is also right.

□

According to (i) and (ii), we can get the Eq. (3.3) is right for all n. Then we can get the Eq. (3.2) is also right.

Theorem 3.2. *[Idempotency] Let $\tilde{a}_i = \tilde{a}$ for all i, and $\tilde{a} = \left\langle \left[s_{\theta(a)}, s_{\tau(a)}\right], u(a), v(a)\right\rangle$, then*

$$IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}. \tag{3.4}$$

Proof. Since $\tilde{a}_i = \tilde{a}$ for all i, we have

$$\begin{aligned}
&IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\\
&= \left\langle \left[s_{\sum_{i=1}^{k}\frac{\theta(a)(1+T(\tilde{a}))}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}, s_{\sum_{i=1}^{k}\frac{\tau(a)(1+T(\tilde{a}))}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}\right],\right.\\
&\quad \left(\frac{\prod_{i=1}^{n}(1+u(a))^{\frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}-\prod_{i=1}^{n}(1-u(a))^{\frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}}{\prod_{i=1}^{n}(1+u(a))^{\frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}+\prod_{i=1}^{n}(1-u(a))^{\frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}},\right.\\
&\quad \left.\left.\frac{2\prod_{i=1}^{n}(v(a))^{\frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}}{\prod_{i=1}^{n}(2-v(a))^{\frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}+\prod_{i=1}^{n}(v(a))^{\frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}}\right)\right\rangle
\end{aligned}$$

$$= \left\langle \left[S_{\theta(a) \sum\limits_{i=1}^{n} \frac{(1+T(\tilde{a}))}{\sum\limits_{i=1}^{n}(1+T(\tilde{a}))}}, S_{\tau(a) \sum\limits_{i=1}^{n} \frac{(1+T(\tilde{a}))}{\sum\limits_{i=1}^{n}(1+T(\tilde{a}))}} \right], \right.$$

$$\left(\frac{(1+u(a))^{\sum_{i=1}^{n} \frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}} - (1-u(a))^{\sum_{i=1}^{n} \frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}}{(1+u(a))^{\sum_{i=1}^{n} \frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}} + (1-u(a))^{\sum_{i=1}^{n} \frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}}, \right.$$

$$\left. \left. \frac{2(v(a))^{\sum_{i=1}^{n} \frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}}{(2-v(a))^{\sum_{i=1}^{n} \frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}} + (v(a))^{\sum_{i=1}^{n} \frac{1+T(\tilde{a})}{\sum_{i=1}^{n}(1+T(\tilde{a}))}}} \right) \right\rangle$$

$$= \left\langle \left[s_{\theta(a)}, s_{\tau(a)}\right], u(a), v(a) \right\rangle = \tilde{a}. \tag{3.5}$$

□

Theorem 3.3. (Boundary) *The IULFPEA operator lies between the max and min operators:* $\tilde{a}_{\min} = \min(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n), \tilde{a}_{\max} = \max(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, *then*

$$\tilde{a}_{\min} \le \mathit{IULFPEA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le \tilde{a}_{\max}. \tag{3.6}$$

Proof. Firstly, let $g(u(a_i)) = \frac{1-u(a_i)}{1+u(a_i)}, u(a_i) \in [0,1]$, we can get $g'(u(a_i)) = \frac{-2}{(1+u(a_i))^2} < 0$ by taking a derivative, so $g(u(a_i))$ is a decreasing function. □

Suppose $u(a_{\min}) \le u(a_i) \le u(a_{\max})$ for all j, we can get $\frac{1-u(a_{\max})}{1+u(a_{\max})} \le \frac{1-u(a_i)}{1+u(a_i)} \le \frac{1-u(a_{\min})}{1+u(a_{\min})}$, then

$$\begin{aligned}
\left(\frac{1-u(a_{\max})}{1+u(a_{\max})}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} &\le \left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \\
&\le \left(\frac{1-u(a_{\min})}{1+u(a_{\min})}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}, \\
\prod_{i=1}^{n}\left(\frac{1-u(a_{\max})}{1+u(a_{\max})}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} &\le \prod_{i=1}^{n}\left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \\
&\le \prod_{i=1}^{n}\left(\frac{1-u(a_{\min})}{1+u(a_{\min})}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}, \\
\left(\frac{1-u(a_{\max})}{1+u(a_{\max})}\right)^{\sum\limits_{i=1}^{n}\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} &\le \prod_{i=1}^{n}\left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \\
&\le \left(\frac{1-u(a_{\min})}{1+u(a_{\min})}\right)^{\sum\limits_{i=1}^{n}\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},
\end{aligned}$$

i.e.

$$\frac{1-u(a_{\max})}{1+u(a_{\max})} \le \prod_{i=1}^{n}\left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \le \frac{1-u(a_{\min})}{1+u(a_{\min})},$$

$$\frac{2}{1+u(a_{\max})} \le 1 + \prod_{i=1}^{n}\left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \le \frac{2}{1+u(a_{\min})},$$

$$\frac{1+u(a_{\max})}{2} \geq \frac{1}{1+\prod_{i=1}^{n}\left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}} \geq \frac{1+u(a_{\min})}{2},$$

thus

$$1+u(a_{\max}) \geq \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}} \geq 1+u(a_{\min}),$$

$$u(a_{\max}) \geq \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}} - 1 \geq u(a_{\min}),$$

therefore

$$u(a_{\max}) \geq \frac{\prod_{i=1}^{n}(1+u(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} - \prod_{i=1}^{n}(1-u(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}{\prod_{i=1}^{n}(1+u(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} + \prod_{i=1}^{n}(1-u(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}} \geq u(a_{\min}). \tag{3.7}$$

Secondly, let $f(v(a_i)) = \frac{2-v(a_i)}{v(a_i)}, v(a_i) \in [0,1]$, we can get $f'(v(a_i)) = \frac{-2}{(u(a_i))^2} < 0$ by taking a derivative, so $f(v(a_i))$ is a decreasing function.

Suppose $v(a_{\max}) \leq v(a_i) \leq v(a_{\min})$ for all j, we can get $\frac{2-v(a_{\min})}{v(a_{\min})} \leq \frac{2-v(a_i)}{v(a_i)} \leq \frac{2-v(a_{\max})}{v(a_{\max})}$, so, we have

$$\left(\frac{2-v(a_{\min})}{v(a_{\min})}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq \left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq \left(\frac{2-v(a_{\max})}{v(a_{\max})}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},$$

$$\left(\frac{2-v(a_{\min})}{v(a_{\min})}\right)^{\sum_{i=1}^{n}\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq \prod_{i=1}^{n}\left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq \left(\frac{2-v(a_{\max})}{v(a_{\max})}\right)^{\sum_{i=1}^{n}\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},$$

$$\frac{2-v(a_{\min})}{v(a_{\min})} \leq \prod_{i=1}^{n}\left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq \frac{2-v(a_{\max})}{v(a_{\max})},$$

then

$$\frac{2}{v(a_{\min})} \leq \prod_{i=1}^{n}\left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} + 1 \leq \frac{2}{v(a_{\max})},$$

$$\frac{v(a_{\min})}{2} \geq \frac{1}{\prod_{i=1}^{n}\left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}+1} \geq \frac{v(a_{\max})}{2},$$

$$v(a_{\min}) \geq \frac{2}{\prod_{i=1}^{n}\left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}+1} \geq v(a_{\max}).$$

Therefore

$$v(a_{\min}) \geq \frac{2\prod_{i=1}^{n}(v(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}{\prod_{i=1}^{n}(2-v(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}+\prod_{i=1}^{n}(v(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}} \geq v(a_{\max}).$$

Because $s_{\theta(a_{\min})} \leq s_{\theta(a_i)} \leq s_{\theta(a_{\max})}, s_{\tau(a_{\min})} \leq s_{\tau(a_i)} \leq s_{\tau(a_{\max})}$ for all i, then

$$s_{\theta(a_{\min})\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\theta(a_i)\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\theta(a_{\max})\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},$$

$$s_{\tau(a_{\min})\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\tau(a_i)\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\tau(a_{\max})\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},$$

$$s_{\sum_{i=1}^{n}\theta(a_{\min})\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\sum_{i=1}^{n}\theta(a_i)\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\sum_{i=1}^{n}\theta(a_{\max})\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},$$

$$s_{\sum_{i=1}^{n}\tau(a_{\min})\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\sum_{i=1}^{n}\tau(a_i)\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\sum_{i=1}^{n}\tau(a_{\max})\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},$$

i.e. $s_{\theta(a_{\min})} \leq s_{\sum_{i=1}^{n}\frac{\theta(a_i)(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\theta(a_{\max})}, s_{\tau(a_{\min})}$

$\leq s_{\sum_{i=1}^{n}\frac{\tau(a_i)(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq s_{\tau(a_{\max})},$

If $IULPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a} = \left\langle \left[s_{\theta(a)}, s_{\tau(a)}\right]; (u(a), v(a))\right\rangle$, we know that

$$s_{\theta(a_{\min})} \leq s_{\theta(a)} \leq s_{\theta(a_{\max})}, s_{\tau(a_{\min})} \leq s_{\tau(a)} \leq s_{\tau(a_{\max})}, u(a_{\min}) \leq u(a) \leq u(a_{\max}),$$

and $v(a_{\max}) \leq v(a) \leq v(a_{\min})$, then we can get that as follows:

$$s_{\theta(a_{\min})} + s_{\tau(a_{\min})} \leq s_{\theta(a)} + s_{\tau(a)} \leq s_{\theta(a_{\max})} + s_{\tau(a_{\max})}, \text{ and}$$
$$u(a_{\min}) - v(a_{\min}) \leq u(a) - v(a) \leq u(a_{\max}) - v(a_{\max}),$$

Therefore, $\tilde{a}_{\min} \leq IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}_{\max}$, which complete the proof of Theorem 3.3.

Theorem 3.4. (monotonicity) *Let* $\tilde{a}_i^* = \left\langle [s^*_{\theta(a_i)}, s^*_{\tau(a_i)}], (u^*(a_i), v^*(a_i))\right\rangle$ *and* $\tilde{a}_i = \left\langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i))\right\rangle$ *be two collection of intuitionistic uncertain linguistic fuzzy number, and if* $s_{\theta(a_i)} \leq s^*_{\theta(a_i)}, s_{\tau(a_i)} \leq s^*_{\tau(a_i)}, u(a_i) \leq u^*(a_i)$, *and* $v^*(a_i) \leq v(a_i)$, *for all* $i, i = 1, 2, \ldots, n$, *then* $IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq IULFPEA(\tilde{a}^*_1, \tilde{a}^*_2, \ldots, \tilde{a}^*_n)$.

Proof. Since $s_{\theta(a_i)} + s_{\tau(a_i)} \leq s^*_{\theta(a_i)} + \leq s^*_{\tau(a_i)}$, and $u(a_i) \leq u^*(a_i)$, and $v^*(a_i) \leq v(a_i)$, for all i, we can get

$$\sum_{i=1}^{n}\left(\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}\left(s_{\theta(a_i)}+s_{\tau(a_i)}\right)\right) \leq \sum_{i=1}^{n}\left(\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}\left(s^*_{\theta(a_i)}+s^*_{\tau(a_i)}\right)\right),$$

i.e. $s_{\sum_{i=1}^{n}\left(\theta(a_i)\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}\right)} + s_{\sum_{i=1}^{n}\left(\tau(a_i)\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}\right)} \leq s^*_{\sum_{i=1}^{n}\left(\theta(a_i)\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}\right)} + s^*_{\sum_{i=1}^{n}\left(\tau(a_i)\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}\right)}$.

□

Since $\frac{1-u^*(a_i)}{1+u^*(a_i)} \leq \frac{1-u(a_i)}{1+u(a_i)}, i = 1, 2, \ldots, n$, then

$$\prod_{i=1}^{n}\left(\frac{1-u^*(a_i)}{1+u^*(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \leq \prod_{i=1}^{n}\left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},$$

$$\frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-u^*(a_i)}{1+u^*(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}} \geq \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-u(a_i)}{1+u(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}},$$

so, we have

$$\frac{\prod_{i=1}^{n}(1+u(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} - \prod_{i=1}^{n}(1-u(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}{\prod_{i=1}^{n}(1+u(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} + \prod_{i=1}^{n}(1-u(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}$$

$$\geq \frac{\prod_{i=1}^{n}(1+u^*(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} - \prod_{i=1}^{n}(1-u^*(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}{\prod_{i=1}^{n}(1+u^*(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} + \prod_{i=1}^{n}(1-u^*(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}.$$

Since $\frac{2-v^*(a_i)}{v^*(a_i)} \geq \frac{2-v(a_i)}{v(a_i)}, i = 1, 2, \ldots, n$, then

$$\left(\frac{2-v^*(a_i)}{v^*(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \geq \left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},$$

$$\prod_{i=1}^{n}\left(\frac{2-v^*(a_i)}{v^*(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \geq \prod_{i=1}^{n}\left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}},$$

$$\frac{1}{\prod_{i=1}^{n}\left(\frac{2-v^*(a_i)}{v^*(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}+1} \leq \frac{1}{\prod_{i=1}^{n}\left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}+1},$$

$$\frac{2}{\prod_{i=1}^{n}\left(\frac{2-v^*(a_i)}{v^*(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}+1} \leq \frac{2}{\prod_{i=1}^{n}\left(\frac{2-v(a_i)}{v(a_i)}\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}+1},$$

so

$$\frac{2\prod_{i=1}^{n}(v(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}{\prod_{i=1}^{n}(2-v(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} + \prod_{i=1}^{n}(v(a_i))^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}$$

$$\leq \frac{2\prod_{i=1}^{n}\left(v^*(a_i)\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}{\prod_{i=1}^{n}\left(2-v^*(a_i)\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}+\prod_{i=1}^{n}\left(v^*(a_i)\right)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}}.$$

We can get $IULFPEA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq IULFPEA(\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_n^*)$, which complete the proof of the Theorem 3.4.

Definition 3.2. Let $\tilde{a}_i = \left\langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i))\right\rangle (i = 1, 2, \ldots, n)$ be a collection of intuitionistic uncertain linguistic fuzzy numbers, and IULFPEWA: $\Omega^n \to \Omega$. If

$$IULFPEWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\overset{n}{\underset{i=1}{\oplus_\varepsilon}} w_i\left(1+T\left(\tilde{a}_i\right)\right)\tilde{a}_i}{\sum_{i=1}^{n} w_i\left(1+T\left(\tilde{a}_i\right)\right)} = \overset{n}{\underset{i=1}{\oplus_\varepsilon}}\left(\frac{w_i\left(1+T\left(\tilde{a}_i\right)\right)\tilde{a}_i}{\sum_{i=1}^{n} w_i\left(1+T\left(\tilde{a}_i\right)\right)}\right), \tag{3.8}$$

where, Ω is the set of all intuitionistic uncertain linguistic fuzzy numbers, and $T(\tilde{a}_i) = \sum_{\substack{j=1\\ i\neq j}}^{n} \sup(\tilde{a}_i, \tilde{a}_j)$, and $\sup(\tilde{a}_i, \tilde{a}_j)$ is the support for $\tilde{a}_i$ from $\tilde{a}_j$, and $w = (w_1, w_2, \ldots w_n)^T$ is the weighting vector of the $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ such that $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. Then *IULFPEWA* is called the intuitionistic uncertain linguistic fuzzy powered Einstein weighted averaging (IULFPEWA) operator.

Theorem 3.5. *Let $\tilde{a}_i = \left\langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i))\right\rangle (i = 1, 2, \ldots, n)$ be a collection of intuitionistic uncertain linguistic fuzzy numbers, then the result aggregated from Definition 3.2 is still an intuitionistic uncertain linguistic fuzzy number, and*

$$\begin{aligned}
&IULFPEWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\\
&= \overset{n}{\underset{i=1}{\oplus_\varepsilon}}\left(\frac{w_i\left(1+T\left(\tilde{a}_i\right)\right)\tilde{a}_i}{\sum_{i=1}^{n} w_i\left(1+T\left(\tilde{a}_i\right)\right)}\right)\\
&= \left\langle\left[s_{\sum_{i=1}^{n}\frac{\theta(a_i)w_i(1+T(\tilde{a}_i))}{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))}}, s_{\sum_{i=1}^{n}\frac{\tau(a_i)w_i(1+T(\tilde{a}_i))}{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))}}\right],\right.\\
&\left(\frac{\prod_{i=1}^{n}\left(1+u\left(a_i\right)\right)^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^{n} w_i(1+T(a_i))}}-\prod_{i=1}^{n}\left(1-u\left(a_i\right)\right)^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^{n} w_i(1+T(a_i))}}}{\prod_{i=1}^{n}\left(1+u\left(a_i\right)\right)^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^{n} w_i(1+T(a_i))}}+\prod_{i=1}^{n}\left(1-u\left(a_i\right)\right)^{\frac{w_i(1+T(a_i))}{\sum_{i=1}^{n} w_i(1+T(a_i))}}},\right.\\
&\left.\left.\frac{2\prod_{i=1}^{n}\left(v\left(a_i\right)\right)^{\frac{w_i(1+T(\tilde{a}_i))}{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))}}}{\prod_{i=1}^{n}\left(2-v\left(a_i\right)\right)^{\frac{w_i(1+T(\tilde{a}_i))}{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))}}+\prod_{i=1}^{n}\left(v\left(a_i\right)\right)^{\frac{w_i(1+T(\tilde{a}_i))}{\sum_{i=1}^{n} w_i(1+T(\tilde{a}_i))}}}\right)\right\rangle,
\end{aligned} \tag{3.9}$$

where $T(\tilde{a}_i) = \sum_{\substack{j=1\\ i\neq j}}^{n} \sup(\tilde{a}_i, \tilde{a}_j)$, and $\sup(\tilde{a}_i, \tilde{a}_j)$ is the support for $\tilde{a}_i$ from $\tilde{a}_j$, and $w = (w_1, w_2, \ldots w_n)^T$ is the weighting vector of the $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ such that

$w_i \in [0,1],\ \sum_{i=1}^{n} w_i = 1.$

The proof is similar with the Theorem 3.2, and it is omitted here.

Similar to Theorems 3.3-3.4, it is easy to prove the IULFPEWA operator has the following properties.

Theorem 3.6. (Idempotency) *Let all $\tilde{a}_i = \tilde{a}$ for all i, then*

$$IULFPEWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}. \tag{3.10}$$

Theorem 3.7. (Boundary) *The IULFPEWA operator lies between the max and min operators: $\tilde{a}_{\min} = \min(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n), \tilde{a}_{\max} = \max(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, then*

$$\tilde{a}_{\min} \leq IULFPEWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq \tilde{a}_{\max}.$$

Definition 3.3. Let $\tilde{a}_i = \left\langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i)) \right\rangle (i = 1, 2, \ldots, n)$ be a collection of intuition uncertain linguistic fuzzy number, then intuitionistic uncertain linguistic fuzzy powered Einstein geometric (IULFPEG) operator of dimension n is a mapping, and has

$$IULFPEG(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \overset{n}{\underset{i=1}{\oplus_\varepsilon}} (\tilde{a}_i)^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}}, \tag{3.11}$$

where $T(\tilde{a}_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \sup(\tilde{a}_i, \tilde{a}_j)$, and $\sup(\tilde{a}_i, \tilde{a}_j)$ is the support for $\tilde{a}_i$ from $\tilde{a}_j$.

Theorem 3.8. *Let $\tilde{a}_i = \left\langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i)) \right\rangle (i = 1, 2, \ldots, n)$ be a collection of intuitionistic uncertain linguistic fuzzy number, then the result aggregated from Definition 3.3 is still an intuitionistic uncertain linguistic fuzzy numbers, and*

$$\begin{aligned} & IULFPEG(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \\ & = \overset{n}{\underset{i=1}{\otimes_\varepsilon}} \tilde{a}_i^{\frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))}} \\ & = \left\langle \left[s_{\prod_{i=1}^{n} \left(\theta(a_i)^{b_i}\right)}, s_{\prod_{i=1}^{n} \left(\tau(a_i)^{b_i}\right)} \right], \right. \\ & \left. \left(\frac{2\prod_{i=1}^{n} (u(a_i))^{b_i}}{\prod_{i=1}^{n} (2-u(a_i))^{b_i} + \prod_{i=1}^{n} (u(a_i))^{b_i}}, \frac{\prod_{i=1}^{n} (1+v(a_i))^{b_i} - \prod_{i=1}^{n} (1-v(a_i))^{b_i}}{\prod_{i=1}^{n} (1+v(a_i))^{b_i} + \prod_{i=1}^{n} (1-v(a_i))^{b_i}} \right) \right\rangle, \end{aligned} \tag{3.12}$$

where $b_i = \frac{(1+T(\tilde{a}_i))}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))} (i = 1, 2, \ldots, n)$ and $T(\tilde{a}_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \sup(\tilde{a}_i, \tilde{a}_j)$, and $\sup(\tilde{a}_i, \tilde{a}_j)$ is the support for $\tilde{a}_i$ from $\tilde{a}_j$.

The proof of this theorem is similar with Theorem 3.2, it's omitted here.

Similar to Theorems 3.3-3.6, it is easy to prove the IULFPEG operator has the following properties.

Theorem 3.9. (Idempotency) *Let $\tilde{a}_i = \tilde{a}$ for all i, then*

$$IULFEG(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}. \tag{3.13}$$

Theorem 3.10. (Boundary) *The IULFEWA operator lies between the max and min operators:* $\tilde{a}_{\min} = \min(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n), \tilde{a}_{\max} = \max(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$*, then*

$$\tilde{a}_{\min} \leq IULFEG(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}_{\max}. \tag{3.14}$$

Theorem 3.11. (monotonicity) *Let* $\tilde{a}_i^* = \left\langle [s^*_{\theta(a_i)}, s^*_{\tau(a_i)}], (u^*(a_i), v^*(a_i)) \right\rangle$ *and* $\tilde{a}_i = \left\langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i)) \right\rangle$ $(i = 1, 2, \ldots, n)$ *be two collection of intuitionistic uncertain linguistic fuzzy numbers, and if* $s_{\theta(a_i)} \leq s^*_{\theta(a_i)}, s_{\tau(a_i)} \leq s^*_{\tau(a_i)}, u(a_i) \leq u^*(a_i)$*, and* $u^*(a_i) \leq u(a_i)$*, for all* $i, i = 1, 2, \ldots, n$*, then*

$$IULFEG(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq IULFEG(\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_n^*). \tag{3.15}$$

Definition 3.4. Let $\tilde{a}_i = \left\langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i)) \right\rangle (i = 1, 2, \ldots, n)$ be a collection of intuition uncertain linguistic fuzzy numbers, and IULFPEWG: $\Omega^n \to \Omega$. If

$$IULFPEWG(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \mathop{\oplus_{\varepsilon}}_{i=1}^{n} (\tilde{a}_i)^{\frac{w_i(1+T(\tilde{a}_i))}{\sum_{i=1}^n w_i(1+T(\tilde{a}_i))}}, \tag{3.16}$$

where, Ω is the set of all intuitionistic uncertain linguistic fuzzy numbers, and $T(\tilde{a}_i) = \sum_{\substack{j=1 \\ i \neq j}}^n \sup(\tilde{a}_i, \tilde{a}_j)$, and $\sup(\tilde{a}_i, \tilde{a}_j)$ is the support for $\tilde{a}_i$ from $\tilde{a}_j$, and $w = (w_1, w_2, \ldots w_n)^T$ is the weighting vector of the $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ such that $w_i \in [0, 1]$, $\sum_{i=1}^n w_i = 1$. Then *IULFPEWG* is called the intuitionistic uncertain linguistic fuzzy powered Einstein weighted geometric (IULFPEWG) operator.

Theorem 3.12. *Let* $\tilde{a}_i = \left\langle [s_{\theta(a_i)}, s_{\tau(a_i)}], (u(a_i), v(a_i)) \right\rangle (i = 1, 2, \ldots, n)$ *be a collection of intuitionistic uncertain linguistic fuzzy numbers, then the result aggregated from Definition 3.4 is still an intuitionistic uncertain linguistic fuzzy number, and*

$$\begin{aligned}
&IULFPEWG(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \\
&= \mathop{\otimes}_{i=1}^{n} \tilde{a}_i^{\frac{w_i(1+T(\tilde{a}_i))}{\sum_{i=1}^n w_i(1+T(\tilde{a}_i))}} \\
&= \left\langle \left[s_{\prod_{i=1}^n \theta(a_i)^{b_i}}, s_{\prod_{i=1}^n \tau(a_i)^{b_i}} \right], \right. \\
&\quad \left. \left(\frac{2\prod_{i=1}^n (u(a_i))^{b_i}}{\prod_{i=1}^n (2-u(a_i))^{b_i} + \prod_{i=1}^n (u(a_i))^{b_i}}, \frac{\prod_{i=1}^n (1+v(a_i))^{b_i} - \prod_{i=1}^n (1-v(a_i))^{b_i}}{\prod_{i=1}^n (1+v(a_i))^{b_i} + \prod_{i=1}^n (1-v(a_i))^{b_i}} \right) \right\rangle,
\end{aligned} \tag{3.17}$$

where $b_i = \frac{w_i(1+T(\tilde{a}_i))}{\sum_{i=1}^n w_i(1+T(\tilde{a}_i))} (i = 1, 2, \ldots, n)$ *and* $T(\tilde{a}_i) = \sum_{\substack{j=1 \\ i \neq j}}^n \sup(\tilde{a}_i, \tilde{a}_j)$*, and* $\sup(\tilde{a}_i, \tilde{a}_j)$ *is the support for* $\tilde{a}_i$ *from* $\tilde{a}_j$*, and* $w = (w_1, w_2, \ldots w_n)^T$ *is the weighting vector of the* $\tilde{a}_i (i = 1, 2, \ldots, n)$*,* $w_i \in [0, 1]$*,* $\sum_{i=1}^n w_i = 1$*.*

The proof of this theorem is similar with Theorem 3.2, it's omitted here.

Similar to Theorems 3.3-3.4, it is easy to prove the IULFPWG operator has the following properties.

Theorem 3.13. (Idempotency) *Let all $\tilde{a}_i = \tilde{a}$ for all i, then*

$$IULFEWG(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}. \tag{3.18}$$

Theorem 3.14. (Boundary) *The IULFPEWG operator lies between the max and min operators: $\tilde{a}_{\min} = \min(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n), \tilde{a}_{\max} = \max(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, then*

$$\tilde{a}_{\min} \le IULFEWG(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le \tilde{a}_{\max}. \tag{3.19}$$

4. The decision-making methods based on the IULF-PEWA operator and IULFPEWG operator

In order to strengthen the efficiency of this decision-making, we can make several experts participate in the decision-making under intuitionistic uncertain linguistic fuzzy environment.

Considering the multiple attribute group decision making problems with intuitionistic uncertain linguistic fuzzy information described as follow.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of attributes, $W = \{w_1, w_2, \ldots, w_n\}$ is the weight vector of the attribute C_j $(j = 1, 2, \ldots, n)$, where $w_j \ge 0$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} w_j = 1$. Let $D = \{D_1, D_2, \ldots, D_p\}$ be the set of decision makers, and $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_p)^T$ be the weight vector of decision makers D_q $(q = 1, 2, \ldots, p)$, where $\lambda_q \ge 0$, $\sum\limits_{q=1}^{p} \lambda_q = 1$. Suppose $H^{(q)} = [h_{ij}^{(q)}]_{m\times n}$ are the decision matrices where $h_{ij}^{(q)} = \langle [s_{\theta^{(q)}(h_{ij})}, s_{\tau^{(q)}(h_{ij})}], \left(u\left(h_{ij}^{(q)}\right), v\left(h_{ij}^{(q)}\right)\right)\rangle$ takes the form of the intuitionistic uncertain linguistic variables given by the decision maker D_q for alternative A_i with respect to attribute C_j, and $u\left(h_{ij}^{(q)}\right) \ge 0$, $v\left(h_{ij}^{(q)}\right) \ge 0$, $0 \le u\left(h_{ij}^{(q)}\right) + v\left(h_{ij}^{(q)}\right) \le 1$, $s_{\theta^{(q)}(h_{ij})}, s_{\tau^{(q)}(h_{ij})} \in S$. Then, the ranking of alternatives is finally acquired.

The methods involve the following steps:

Step 1. Calculate the supports.

$$\sup(h_{ij}^{(q)}, h_{ij}^{(t)}) = 1 - d(h_{ij}^{(q)}, h_{ij}^{(t)}), q, t = 1, 2, \ldots, p;\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{4.1}$$

which satisfies the support conditions expressed by formulas (2.29)-(2.31), where $d(h_{ij}^{(q)}, h_{ij}^{(t)})$ is the distance between two IULNs $h_{ij}^{(q)}$ and $h_{ij}^{(t)}$, which are defined by Definition 2.8.

Step 2. Calculate $T\left(h_{ij}^{(q)}\right)$.

$$T\left(h_{ij}^{(q)}\right) = \sum_{\substack{t=1\\ t\neq q}}^{p} \sup(h_{ij}^{(q)}, h_{ij}^{(t)}), q,\ t = 1, 2, \ldots, p;\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n. \tag{4.2}$$

Step 3. Utilize the IULFPEWA operator or the IULFPEWG operator to aggregate all the individual intuitionitic uncertain linguistic fuzzy decision matrices $H^{(q)} = \left(h_{ij}^{(q)}\right)_{m\times n}$ $(q = 1, 2, \ldots, p)$ into the collective intuitionitic uncertain linguistic fuzzy decision matrix $\tilde{H} = (h_{ij})_{m\times n}$ where $h_{ij} = \langle [s_{\theta(h_{ij})}, s_{\tau(h_{ij})}], \left(u\left(h_{ij}\right), v\left(h_{ij}\right)\right)\rangle$ $i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n.$

$$\begin{aligned} h_{ij} =& IULFPEWA(h_{ij}^{(1)}, h_{ij}^{(2)}, \ldots, h_{ij}^{(p)}) \\ =& \left\langle \left[s_{\sum_{q=1}^{p} \frac{\theta^{(q)}(h_{ij})\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}, s_{\sum_{q=1}^{p} \frac{\tau^{(q)}(h_{ij})\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}} \right], \right. \\ & \left(\frac{\prod_{q=1}^{p}\left(1+u\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}} - \prod_{q=1}^{p}\left(1-u\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}}{\prod_{q=1}^{p}\left(1+u\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}} + \prod_{q=1}^{p}\left(1-u\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}}, \right. \\ & \left. \left. \frac{2\prod_{q=1}^{p}\left(v\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}}{\prod_{q=1}^{p}\left(2-v\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}} + \prod_{q=1}^{p}\left(v\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}} \right) \right\rangle \end{aligned} \tag{4.3}$$

or

$$\begin{aligned} h_{ij} =& IULFPEWG(h_{ij}^{(1)}, h_{ij}^{(2)}, \ldots, h_{ij}^{(p)}) \\ =& \left\langle \left[s_{\prod_{q=1}^{p} \theta(h_{ij}^{(q)})^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}}, s_{\prod_{q=1}^{p} \tau(h_{ij}^{(q)})^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}} \right], \right. \\ & \left(\frac{2\prod_{q=1}^{p}\left(u\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}}{\prod_{q=1}^{p}\left(2-u\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}} + \prod_{q=1}^{p}\left(u\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}}, \right. \\ & \left. \left. \frac{\prod_{q=1}^{p}\left(1+v\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}} - \prod_{q=1}^{p}\left(1-v\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}}{\prod_{q=1}^{p}\left(1+v\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}} + \prod_{q=1}^{p}\left(1-v\left(h_{ij}^{(q)}\right)\right)^{\frac{\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}{\sum_{q=1}^{p}\lambda_q\left(1+T\left(h_{ij}^{(q)}\right)\right)}}} \right) \right\rangle \end{aligned} \tag{4.4}$$

Step 4. Calculate the supports.

$$\sup\left(h_{ij}, h_{ik}\right) = 1 - d\left(h_{ij}, h_{ik}\right),\ i = 1, \ldots, m;\ j, k = 1, \ldots, n. \tag{4.5}$$

Step 5. Calculate $T(h_{ij})$.

$$T(h_{ij}) = \sum_{\substack{k=1\\k\neq j}}^{n} \sup(h_{ij}, h_{ik}),\ i=1,\ldots,m;\ j,k=1,\ldots,n. \tag{4.6}$$

Step 6. Aggregate the intuitionistic uncertain linguistic fuzzy numbers for each alternative by the IULFPEWA (or IULFPEWG) operator:

$$\begin{aligned} h_i =& IULFPEWA(h_{i1}, h_{i2}, \ldots, h_{in}) \\ =& \left\langle \left[s_{\sum_{j=1}^{n} \frac{\theta(h_{ij}) w_j (1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}, s_{\sum_{j=1}^{n} \frac{\tau(h_{ij}) w_j (1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}} \right], \right. \\ & \left(\frac{\prod_{j=1}^{n} (1+u(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}} - \prod_{j=1}^{n} (1-u(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}}{\prod_{j=1}^{n} (1+u(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}} + \prod_{j=1}^{n} (1-u(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}}, \right. \\ & \left. \left. \frac{2\prod_{j=1}^{n} (v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}}{\prod_{j=1}^{n} (2-v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}} + \prod_{j=1}^{n} (v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}} \right) \right\rangle \end{aligned} \tag{4.7}$$

or

$$\begin{aligned} h_i =& IULFPEWG(h_{i1}, h_{i2}, \ldots, h_{in}) \\ =& \left\langle \left[s_{\prod_{j=1}^{n} \theta(h_{ij})^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}}, s_{\prod_{j=1}^{n} \tau(h_{ij})^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}} \right], \right. \\ & \left(\frac{2\prod_{j=1}^{n} (u(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}}{\prod_{j=1}^{n} (2-u(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}} + \prod_{j=1}^{n} (u(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}}, \right. \\ & \left. \left. \frac{\prod_{j=1}^{n} (1+v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}} - \prod_{j=1}^{n} (1-v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}}{\prod_{j=1}^{n} (1+v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}} + \prod_{j=1}^{n} (1-v(h_{ij}))^{\frac{w_j(1+T(h_{ij}))}{\sum_{j=1}^{n} w_j (1+T(h_{ij}))}}} \right) \right\rangle \end{aligned} \tag{4.8}$$

Step 7. Calculate the value $E(h_i)$ of h_i.

Step 8. Rank $h_i\,(i=1,2,\ldots,m)$ in descending order according to the comparison method of IULNs described in Definition 2.7.

Step 9. End.

5. An numerical example

In this section, we will provide an example to illustrate the application of IULFPEWA and IULFPEWG operator (Cited from [7]). Suppose that an investment company wants to invest an amount of money to a company. There are four candidate companies $A_i(i = 1, 2, 3, 4)$ evaluated by three decision makers $\{D_1, D_2, D_3\}$. The weight vector of the decision makers is $\lambda = (0.4, 0.32, 0.28)^T$, and the attributes considered include: C_1 (the risk index), C_2 (the growth index), C_3 (the social-political impact index), and C_4 (the environmental impact index). Suppose the attribute weight vector is $w = (0.32, 0.26, 0.18, 0.24)^T$.The three decision makers $\{D_1, D_2, D_3\}$ evaluate the four companies $A_i(i = 1, 2, 3, 4)$ with respect to the attributes$C_j(j = 1, 2, 3, 4)$ by using the intuitionistic uncertain linguistic variables (suppose that the decision makers use linguistic term set $S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6)$ to express their evaluation results) and construct the following decision matrices $H^{(q)} = [h_{ij}^{(q)}]_{4\times 4}$ $(q = 1, 2, 3)$ listed in Tables 1-3.

Table 1. Decision matrix $H^{(1)}$.

	C1	C2	C3	C4
A1	$\langle[S_5, S_5](0.2, 0.7)\rangle$	$\langle[S_2, S_3](0.4, 0.6)\rangle$	$\langle[S_5, S_6](0.5, 0.5)\rangle$	$\langle[S_3, S_4](0.2, 0.6)\rangle$
A2	$\langle[S_4, S_5](0.4, 0.6)\rangle$	$\langle[S_5, S_5](0.4, 0.5)\rangle$	$\langle[S_3, S_4](0.1, 0.8)\rangle$	$\langle[S_4, S_4](0.5, 0.5)\rangle$
A3	$\langle[S_3, S_4](0.2, 0.7)\rangle$	$\langle[S_4, S_4](0.2, 0.7)\rangle$	$\langle[S_4, S_5](0.3, 0.7)\rangle$	$\langle[S_4, S_5](0.2, 0.7)\rangle$
A4	$\langle[S_6, S_6](0.5, 0.4)\rangle$	$\langle[S_2, S_3](0.2, 0.8)\rangle$	$\langle[S_3, S_4](0.2, 0.6)\rangle$	$\langle[S_3, S_3](0.3, 0.6)\rangle$

Table 2. Decision matrix $H^{(2)}$.

	C1	C2	C3	C4
A1	$\langle[S_3, S_4](0.1, 0.7)\rangle$	$\langle[S_3, S_4](0.2, 0.7)\rangle$	$\langle[S_3, S_4](0.2, 0.8)\rangle$	$\langle[S_6, S_6](0.4, 0.5)\rangle$
A2	$\langle[S_5, S_6](0.4, 0.5)\rangle$	$\langle[S_3, S_4](0.3, 0.6)\rangle$	$\langle[S_4, S_5](0.2, 0.6)\rangle$	$\langle[S_3, S_4](0.2, 0.7)\rangle$
A3	$\langle[S_4, S_5](0.2, 0.6)\rangle$	$\langle[S_4, S_4](0.2, 0.7)\rangle$	$\langle[S_2, S_3](0.4, 0.6)\rangle$	$\langle[S_3, S_4](0.3, 0.7)\rangle$
A4	$\langle[S_5, S_5](0.3, 0.6)\rangle$	$\langle[S_4, S_5](0.4, 0.5)\rangle$	$\langle[S_2, S_3](0.3, 0.6)\rangle$	$\langle[S_4, S_4](0.2, 0.6)\rangle$

Table 3. Decision matrix $H^{(3)}$.

	C1	C2	C3	C4
A1	$\langle[S_5, S_5](0.2, 0.6)\rangle$	$\langle[S_3, S_4](0.3, 0.7)\rangle$	$\langle[S_4, S_5](0.4, 0.5)\rangle$	$\langle[S_4, S_4](0.2, 0.7)\rangle$
A2	$\langle[S_4, S_5](0.3, 0.7)\rangle$	$\langle[S_5, S_5](0.3, 0.6)\rangle$	$\langle[S_2, S_3](0.1, 0.8)\rangle$	$\langle[S_3, S_4](0.4, 0.6)\rangle$
A3	$\langle[S_4, S_4](0.2, 0.7)\rangle$	$\langle[S_5, S_5](0.3, 0.6)\rangle$	$\langle[S_1, S_3](0.1, 0.8)\rangle$	$\langle[S_4, S_4](0.2, 0.7)\rangle$
A4	$\langle[S_3, S_4](0.2, 0.7)\rangle$	$\langle[S_3, S_4](0.1, 0.7)\rangle$	$\langle[S_4, S_5](0.3, 0.6)\rangle$	$\langle[S_5, S_5](0.4, 0.5)\rangle$

5.1. Ranking four candidate companies by the IULFPEWA operator

Step 1. Calculate the supports $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$.

$$\sup(h_{ij}^{(1)}, h_{ij}^{(2)}) = \begin{pmatrix} 0.908 & 0.979 & 0.658 & 0.725 \\ 0.888 & 0.829 & 0.863 & 0.813 \\ 0.921 & 1.000 & 0.942 & 0.988 \\ 0.742 & 0.746 & 0.971 & 0.975 \end{pmatrix}$$

Table 4. Decision matrix $\tilde{H}$.

	C1	C2
A1	$\langle[S_{4.371}, S_{4.686}](0.169, 0.671)\rangle$	$\langle[S_{2.599}, S_{3.599}](0.311, 0.659)\rangle$
A2	$\langle[S_{4.319}, S_{5.319}](0.373, 0.593)\rangle$	$\langle[S_{4.366}, S_{4.683}](0.341, 0.559)\rangle$
A3	$\langle[S_{3.599}, S_{4.317}](0.200, 0.668)\rangle$	$\langle[S_{4.271}, S_{4.271}](0.228, 0.672)\rangle$
A4	$\langle[S_{4.822}, S_{5.102}](0.355, 0.542)\rangle$	$\langle[S_{2.896}, S_{3.896}](0.236, 0.674)\rangle$

Table 4. Decision matrix $\tilde{H}$. (continues)

	C3	C4
A1	$\langle[S_{4.064}, S_{5.064}](0.368, 0.587)\rangle$	$\langle[S_{4.181}, S_{4.595}](0.277, 0.596)\rangle$
A2	$\langle[S_{3.027}, S_{4.027}](0.133, 0.735)\rangle$	$\langle[S_{3.395}, S_{4.000}](0.458, 0.589)\rangle$
A3	$\langle[S_{2.525}, S_{3.799}](0.296, 0.693)\rangle$	$\langle[S_{3.679}, S_{4.399}](0.258, 0.700)\rangle$
A4	$\langle[S_{2.954}, S_{3.954}](0.260, 0.600)\rangle$	$\langle[S_{3.863}, S_{3.863}](0.342, 0.572)\rangle$

$$\sup(h_{ij}^{(1)}, h_{ij}^{(3)}) = \begin{pmatrix} 0.958 & 0.992 & 0.879 & 0.992 \\ 0.925 & 0.917 & 0.975 & 0.900 \\ 0.979 & 0.875 & 0.825 & 0.979 \\ 0.596 & 0.967 & 0.913 & 0.800 \end{pmatrix}$$

$$\sup(h_{ij}^{(2)}, h_{ij}^{(3)}) = \begin{pmatrix} 0.867 & 0.971 & 0.982 & 0.717 \\ 0.813 & 0.913 & 0.838 & 0.913 \\ 0.942 & 0.875 & 0.883 & 0.992 \\ 0.854 & 0.779 & 0.883 & 0.825 \end{pmatrix}$$

Step 2. Calculate $T\left(h_{ij}^{(q)}\right)$ $(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$.

$$T\left(h_{ij}^{(1)}\right) = \begin{pmatrix} 1.840 & 1.965 & 1.445 & 1.660 \\ 1.775 & 1.695 & 1.805 & 1.655 \\ 1.880 & 1.850 & 1.720 & 1.960 \\ 1.205 & 1.655 & 1.860 & 1.730 \end{pmatrix}$$

$$T\left(h_{ij}^{(2)}\right) = \begin{pmatrix} 1.730 & 1.940 & 1.568 & 1.330 \\ 1.725 & 1.690 & 1.640 & 1.670 \\ 1.835 & 1.850 & 1.790 & 1.975 \\ 1.515 & 1.430 & 1.825 & 1.760 \end{pmatrix}$$

$$T\left(h_{ij}^{(3)}\right) = \begin{pmatrix} 1.790 & 1.955 & 1.833 & 1.650 \\ 1.685 & 1.795 & 1.775 & 1.775 \\ 1.905 & 1.700 & 1.650 & 1.965 \\ 1.340 & 1.695 & 1.755 & 1.550 \end{pmatrix}$$

Step 3. Utilize the IULFPEWA operator (Eq. (4.3)) to aggregate all the three decision matrices mentioned above into the following decision matrix showing in the Table 4.

Step 4. Calculate the supports according to the Eq. (4.5) $(k, j = 1, 2, 3, 4)$.

$$\sup(h_{1j}, h_{1k}) = \begin{pmatrix} 1 & 0.980 & 0.891 & 0.940 \\ 0.980 & 1 & 0.871 & 0.919 \\ 0.891 & 0.871 & 1 & 0.952 \\ 0.940 & 0.919 & 0.952 & 1 \end{pmatrix}$$

$$\sup(h_{2j}, h_{2k}) = \begin{pmatrix} 1 & 0.981 & 0.804 & 0.954 \\ 0.981 & 1 & 0.822 & 0.937 \\ 0.804 & 0.822 & 1 & 0.849 \\ 0.954 & 0.937 & 0.849 & 1 \end{pmatrix}$$

$$\sup(h_{3j}, h_{3k}) = \begin{pmatrix} 1 & 0.978 & 0.983 & 0.988 \\ 0.978 & 1 & 0.869 & 0.920 \\ 0.983 & 0.869 & 1 & 0.971 \\ 0.988 & 0.920 & 0.971 & 1 \end{pmatrix}$$

$$\sup(h_{4j}, h_{4k}) = \begin{pmatrix} 1 & 0.823 & 0.856 & 0.912 \\ 0.823 & 1 & 0.969 & 0.911 \\ 0.856 & 0.969 & 1 & 0.942 \\ 0.912 & 0.911 & 0.942 & 1 \end{pmatrix}$$

Step 5. Calculate $T(h_{ij})$ $(i, j = 1, 2, 3, 4)$.

$$T(h_{ij}) = \begin{pmatrix} 2.810 & 2.771 & 2.713 & 2.810 \\ 2.739 & 2.777 & 2.475 & 2.777 \\ 2.949 & 2.929 & 2.916 & 2.949 \\ 2.589 & 2.703 & 2.765 & 2.765 \end{pmatrix}$$

Step 6. Aggregate the intuitionistic uncertain linguistic fuzzy numbers for each alternative by the IULFPEWA1 operator:

$$h_1 = ([s_{3.812}, s_{4.449}](0.707, 0.634));\ h_2 = ([s_{3.888}, s_{4.611}](0.738, 0.608));$$
$$h_3 = ([s_{3.601}, s_{4.232}](0.695, 0.677));\ h_4 = ([s_{3.742}, s_{4.274}](0.719, 0.596)).$$

Step 7. Calculate the value $E(h_i)$ of h_i.

$$E(h_1) = s_{2.214}, \quad E(h_2) = s_{2.401}, \quad E(h_3) = s_{1.994}, \quad E(h_4) = s_{2.251}.$$

Step 8. Rank $E(h_i)$ in descending order, we can get the best alternative.
Because$E(h_2) > E(h_4) > E(h_1) > E(h_3)$. A_2 is the best choice.

Step 9. End.

5.2. Ranking four candidate companies by the IULFPEWG operator

Step 1'. Calculate the supports and the result is same with Step 1.

Step 2'. Calculate$T\left(h_{ij}^{(q)}\right)$and the result is same with Step 2.

Step 3'. Utilize the IULFPEWG operator (Eq. (4.4)) to aggregate all the three decision matrices mentioned above into the following decision matrix showing in the table 5.

Table 5. Decision matrix $\tilde{H}$.

	C1	C2
A1	$\langle[S_{4.258}, S_{4.661}](0.162, 0.674)\rangle$	$\langle[S_{2.550}, S_{3.564}](0.298, 0.663)\rangle$
A2	$\langle[S_{4.295}, S_{5.300}](0.370, 0.601)\rangle$	$\langle[S_{4.366}, S_{4.659}](0.337, 0.562)\rangle$
A3	$\langle[S_{3.564}, S_{4.293}](0.200, 0.671)\rangle$	$\langle[S_{4.271}, S_{4.249}](0.224, 0.675)\rangle$
A4	$\langle[S_{4.646}, S_{5.036}](0.330, 0.564)\rangle$	$\langle[S_{2.896}, S_{3.807}](0.206, 0.700)\rangle$

Table 5. Decision matrix $\tilde{H}$. (continues)

	C3	C4
A1	$\langle[S_{3.975}, S_{4.994}](0.355, 0.649)\rangle$	$\langle[S_{4.006}, S_{4.513}](0.248, 0.620)\rangle$
A2	$\langle[S_{2.925}, S_{3.952}](0.124, 0.726)\rangle$	$\langle[S_{3.361}, S_{4.373}](0.356, 0.585)\rangle$
A3	$\langle[S_{2.182}, S_{3.679}](0.249, 0.681)\rangle$	$\langle[S_{3.647}, S_{4.373}](0.228, 0.681)\rangle$
A4	$\langle[S_{2.851}, S_{3.878}](0.255, 0.600)\rangle$	$\langle[S_{3.780}, S_{3.780}](0.286, 0.574)\rangle$

Step 4'. Calculate the supports according to the Eq. (4.5) $(k, j = 1, 2, 3, 4)$.

$$\sup(h_{1j}, h_{1k}) = \begin{pmatrix} 1 & 0.981 & 0.917 & 0.958 \\ 0.981 & 1 & 0.898 & 0.939 \\ 0.917 & 0.898 & 1 & 0.959 \\ 0.958 & 0.939 & 0.959 & 1 \end{pmatrix}$$

$$\sup(h_{2j}, h_{2k}) = \begin{pmatrix} 1 & 0.984 & 0.807 & 0.929 \\ 0.984 & 1 & 0.823 & 0.945 \\ 0.807 & 0.823 & 1 & 0.956 \\ 0.929 & 0.945 & 0.956 & 1 \end{pmatrix}$$

$$\sup(h_{3j}, h_{3k}) = \begin{pmatrix} 1 & 0.978 & 0.965 & 0.990 \\ 0.978 & 1 & 0.944 & 0.988 \\ 0.965 & 0.944 & 1 & 0.956 \\ 0.990 & 0.988 & 0.956 & 1 \end{pmatrix}$$

$$\sup(h_{4j}, h_{4k}) = \begin{pmatrix} 1 & 0.833 & 0.875 & 0.915 \\ 0.833 & 1 & 0.958 & 0.917 \\ 0.875 & 0.958 & 1 & 0.917 \\ 0.915 & 0.917 & 0.917 & 1 \end{pmatrix}$$

Step 5'. Calculate $T(h_{ij})\,(i, j = 1, 2, 3, 4)$.

$$T(h_{ij}) = \begin{pmatrix} 2.856 & 2.818 & 2.774 & 2.856 \\ 2.720 & 2.752 & 2.585 & 2.830 \\ 2.934 & 2.911 & 2.865 & 2.934 \\ 2.623 & 2.708 & 2.750 & 2.750 \end{pmatrix}$$

Step 6'. Aggregate the intuitionistic uncertain linguistic fuzzy numbers for each alternative by the IULFPEWG operator.

$$h_1 = ([s_{3.629}, s_{4.367}](0.243, 0.858));\ h_2 = ([s_{3.800}, s_{4.544}](0.299, 0.842));$$
$$h_3 = ([s_{3.443}, s_{4.184}](0.221, 0.867));\ h_4 = ([s_{3.584}, s_{4.172}](0.270, 0.841)).$$

Step 7'. Calculate the value $E(h_i)$ of h_i.

$$E(h_1) = s_{0.766}, \quad E(h_2) = s_{0.954}, \quad E(h_3) = s_{0.675}, \quad E(h_4) = s_{0.831}.$$

Step 8'. Rank $E(h_i)$ in descending order, we can get the best alternative.

Because $E(h_2) > E(h_4) > E(h_1) > E(h_3)$. A_2 is the best choice.

Step 9'. End.

In order to analyze the effectiveness of the methods proposed above, we can compare them with that developed by Liu and Jin [9]. We acquire same ranking results for these methods. But the methods represented in this paper are based on Einstein operations, and combine the intuitionistic uncertain language and PA operators. Absolutely, the method proposed by Liu and Jin [9] can give the comprehensive evaluation values of all alternatives and the ranking results by generalized aggregation operators, and the methods in this paper can only provide the ranking results. However, these methods are the solutions of the MAGDM problems.

6. Conclusion

The intuitionistic uncertain linguistic numbers are a useful tool to convey the fuzzy information. This paper focuses on multi-attribute group decision making (MAGDM) problems in which the attribute values are expressed by intuitionistic uncertain linguistic numbers. The definition and some basic operations of intuitionistic uncertain linguistic numbers, power aggregation(PA) operators and Einstein operations are introduced. Then, we apply the Einstein operations to the PA operators under intuitionistic uncertain linguistic environment and put forward some new aggregation operators such as intuitionistic uncertain linguistic fuzzy powered Einstein averaging (IULFPEA) operator, intuitionistic uncertain linguistic fuzzy powered Einstein weighted averaging (IULFPEWA) operator, intuitionistic uncertain linguistic fuzzy Einstein geometric (IULFEG) operator ,intuitionistic uncertain linguistic fuzzy Einstein weighted geometric (IULFEWG) operator .We also discussed some properties of them in detail. Further, we propose the decision making method for MAGDM problems with intuitionistic uncertain linguistic information and show the detail decision steps. In the future, we should try our best to use the proposed operators to extend the scope of application.

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