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HAMILTONIAN SYSTEMS WITH POSITIVE TOPOLOGICAL ENTROPY AND CONJUGATE POINTS*

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Abstract In this article, we consider the geodesic flows induced by the natural Hamiltonian systems $H(x,p) = \frac{1}{2}g^{ij}(x)p_ip_j + V(x)$ defined on a smooth Riemannian manifold $(M = \mathbb{S}^1 \times N, g)$, where $\mathbb{S}^1$ is the one dimensional torus, N is a compact manifold, g is the Riemannian metric on M and V is a potential function satisfying $V \leq 0$. We prove that under suitable conditions, if the fundamental group $\pi_1(N)$ has sub-exponential growth rate, then the Riemannian manifold M with the Jacobi metric $(h-V)g$, i.e., $(M,(h-V)g)$, is a manifold with conjugate points for all h with $0 < h < \delta$, where δ is a small number.

Keywords Hamiltonian systems, topological entropy, fundamental group, conjugate points.

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1. Introduction

Let M be an n-dimensional smooth (here "smooth" means C^∞) complete manifold with the Riemannian metric g, and $\gamma_{(x,v)}(t)$ be the unique geodesics starting from $x \in M$ with an initial tangent vector $v \in T_xM$. The geodesic flow ϕ_t on the tangent bundle TM is defined as:

$$\phi_t : TM \to TM, \qquad (x,v) \mapsto (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)).$$

Let SM be the unit tangent bundle on M. Then ϕ_t leaves SM invariant. Usually when we say geodesic flows we mean geodesic flows on the unit tangent bundle.

It is well-known that, in local coordinates, the geodesic flow can be generated by the Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L(x,\dot{x})}{\partial \dot{x}} = \frac{\partial L(x,\dot{x})}{\partial x}, \tag{1.1}$$

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with the Lagrangian $L(x, v) = g_x(v, v)$, where g is the Riemannian metric. By the Legendre transformation, one can consider the geodesic flows as Hamiltonian flows on the cotangent bundle T^*M with the Hamiltonian

$$H(x,p) = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij}(x)p_i p_j, \tag{1.2}$$

where (g^{ij}) is the inverse matrix of the Riemannian metric $g = (g_{ij})$. More generally, we consider the natural Hamiltonian system with the Hamiltonian

$$H(x,p) = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij}(x)p_i p_j + V(x), \tag{1.3}$$

where $V(x)$ is a potential function defined on M. For $h > \max_{x\in M}\{V(x)\}$, the Maupertuis principle shows that the Hamiltonian flows restricted to the iso-energy level $\{H(x,p) = h\}$ coincide with (up to a re-parametrization) the geodesic flows of M with the Jacobi metric $\widetilde{g} = (\widetilde{g}_{ij}(x)) = ((h - V(x))g_{ij}(x)) = (h - V)g$. For more details, see [1].

The topological entropy is an important invariance which describes the complexity of the dynamical systems. In this paper, we use $h_{top}(g)$ to denote the topological entropy of the geodesic flow generated by the Riemannian metric g. It is well-known that the topological entropy of a geodesic flow is strongly related to the geometry and topology of the Riemannian manifold with a metric g where the geodesic flow is defined. In 1971, Dinaburg proved that, if the fundamental group of M has exponential growth then $h_{top}(g) > 0$ (cf. [5]). Eight years later, in [8], Manning established the following inequality:

$$h_{top}(g) \geq h(M, g),$$

where $h(M, g)$ denotes the volume entropy of the Riemannian manifold (M, g). When (M, g) has non-positive curvature, the equality holds. Furthermore, Freire-Mañé (cf. [6]) proved that the equality holds for the manifold without conjugate points when the Riemannian metric is Hölder C^3. From the definition we know that $h(M, g) > 0$ if and only if the volume has positively exponential growth rate on the Riemannian universal covering manifold of (M, g).

It is natural to ask the following question: if $h_{top}(g) > 0$, what can we say about the geometry and topology of the manifold on which the geodesic flow is defined? In this paper, we will give a partial answer to this problem. More precisely, using the Maupertuis' principle we study the geodesic flows induced by natural Hamiltonian systems. We will show that, under some conditions, which will be discussed in the next section, if the fundamental group $\pi_1(N)$ has sub-exponential growth rate, then the Riemannian manifold M with the Jacobi metric $(h - V)g$, is a manifold with conjugate points for all h with $0 < h < \delta$, for some small number δ.

2. Assumptions and Main Results

Suppose N is a smooth compact manifold, $\mathbb{S}^1$ is the unite circle. Let $M = \mathbb{S}^1 \times N$, and g be a smooth Riemannian metric on M. Let us consider the Hamiltonian system whose Hamiltonian function is defined in 1.3, with $V(x) \leq 0$. This Hamiltonian system is often called *the natural Hamiltonian system.*

In [2], Bolotin and Rabinowitz studied the chaotic properties of a class of Hamiltonian systems, which is called the reversible Hamiltonian systems. The natural Hamiltonian systems we are considering in this paper is a special case of the reversible Hamiltonian systems. The Riemannian manifold M in [2] is also a product one, saying $(M = \mathbb{S}^1 \times N, g)$, where $\mathbb{S}^1$ is a smooth circle, N is a smooth compact manifold and g is a smooth Riemannian metric on M. By using a variational methods, they discovered the chaotic dynamics on the energy levels with small energies. To obtain their results, Bolotin and Rabinowitz imposed six conditions on H, M and V. The first two of which are automatically satisfied by the natural Hamiltonian systems. Thus we only impose the last four conditions in this paper, which are **(H1)** to **(H4)** in the following.

We write $\mathbb{S}^1 = \mathbb{R}/2\mathbb{Z}$. Put $q = (x, y)$, where $x \in \mathbb{S}^1$ and $y \in N$, $p = (p_x, p_y)$, where $p_x \in \mathbb{R}$ and $p_y \in T^*_y N$. Let $I : M \to M$ be the involution $I(x, y) = (-x, y)$. The fixed point set of I has two components, namely $N_0 = \{0\} \times N$ and $N_1 = \{1\} \times N$. The involution I defines the involution $I_* : T^*M \to T^*M$ by $I_*(p_x, p_y, x, y) = (-p_x, p_y, -x, y)$. Suppose that

(H1) V has a unique maximum point $q_0 \in M$, and it is non-degenerate.

(H2) H is invariant under the involution I_*.

By the uniqueness of maximum point and **(H2)**, it's easy to see that q_0 belongs to the set of fixed points of I. It can be assumed that $q_0 \in N_0$. The Hamiltonian system has an invariant symplectic manifold $Q = T^*N_0$ which contains the equilibrium z_0. Let $\pm\lambda_1$ be the eigenvalues corresponding to the eigenvectors transversal to $T_{z_0}Q$, and the exponents $\pm\lambda_2, ..., \pm\lambda_n$ correspond to the eigenvectors in $T_{z_0}Q$. Suppose that

(H3) $\lambda_1 < \lambda_k$ for $k \geq 2$.

(H4) $d(N_0, N_1) < d(q_0, N_1)$, where d is the distance induced by the Riemannian metric.

For convenience, we call the four conditions stated above the *BR conditions* and call the natural Hamiltonian systems satisfying the BR conditions *BR-type natural Hamiltonian systems.* Bolotin and Rabinowitz [2] proved the following result.

Theorem 2.1 (Bolotin and Rabinowitz)**.** *For the BR-type natural Hamiltonian system H on $(M = \mathbb{S}^1 \times N, g)$ as stated in* (1.3) *with $V(x) \leq 0$, there exists a constant $\delta > 0$, such that the system has positive topological entropy on each energy level $\{H(x,p) = h\}$ with $0 < h < \delta$.*

Remark 2.1. In fact, Bolotin and Rabinowitz proved the above theorem for all h satisfying $0 < |h| < \delta$. But in this paper, we only need the case $0 < h < \delta$. Moreover, by this theorem, we know that the BR-type natural Hamiltonian systems with $V(x) \leq 0$ always have positive topological entropy.

In this paper, we extended the above result and consider the relation between the growth rate of the fundamental group of N and the existence of the conjugate points. Our main results are the following.

Theorem 2.2 (Main Theorem)**.** *Let $(M = \mathbb{S}^1 \times N, g)$ and $\delta > 0$ be the ones stated above. Suppose that H is a BR-type natural Hamiltonian defined on (M, g) given in* (1.3)*, with $V(x) \leq 0$ on M, then If the fundamental group $\pi_1(N)$ of N has a* *sub-exponential growth rate, then for* $0 < h < \delta$, $(M, (h - V)g)$ *is a manifold with conjugate points.*

The definitions of the growth rate of a group and the conjugate point on a Riemannian manifold with be given in the next section. We remark that: the following result is an easy consequence of Theorem 2.2.

Corollary 2.1. *Assume that the conditions given in Theorem 2.2 hold. If* N *is simply connected, or* $\pi_1(N)$ *is finite, or* $\pi_1(N)$ *has polynomial growth rate, equivalently the volume growth rate of the Riemannian universal covering of* N *is sub-exponential, then the Riemannian manifold* $(M, (h - V)g)$ *is a manifold with conjugate points.*

3. Proof of the Main Theorem

Before proving our results, we need to introduce the concepts of growth rate (cf. [7]) and conjugate points (cf. [4]).

Let Γ be a finitely generated group with a given set of generators $\{\gamma_1, ..., \gamma_k\}$. For each element $\gamma \in \Gamma$, we associate it a norm, saying $\|\gamma\|$, to be the minimal length of γ, as a word in the given set of generators $\{\gamma_1, ..., \gamma_k\}$. Then for the given generators above, we denote by $B(r) \subset \Gamma$, $r \geq 0$, the ball of radius r centered at the identity element e, and denote by $\sharp B(r)$ the number of elements in $B(r)$.

Definition 2.1. We say that Γ has *exponential growth* if

$$\overline{\lim}_{r\to+\infty} \frac{\ln \sharp B(r)}{r} > 0.$$

Otherwise, we refer Γ to have *sub-exponential growth rate*.

We say that Γ has *polynomial growth rate* if there are two positive numbers d and C such that for all balls $B(r) \subseteq \Gamma$ with $r \geq 1$, one has

$$\sharp B(r) \leq C r^d.$$

Obviously, the polynomial growth rate is a special case of the sub-exponential growth rate.

Let γ be a geodesic of (M, g), two points $x = \gamma(t_1)$ and $y = \gamma(t_2)$ are called *conjugate* if there exists a non-identically zero Jacobi field Y along γ such that $Y(t_1) = 0 = Y(t_2)$. We say that the manifold M *without conjugate points* if on each geodesic no two points are conjugate. It's easy to see that if a manifold has non-positive curvature, then it has no conjugate points.

Next, we give two lemmas below, which is useful in our proof of theorem 2.2. The first lemma shows the relations between the growth rate of the fundamental group of N and the fundamental group of $M = \mathbb{S}^1 \times N$.

Lemma 2.1. *If both the fundamental groups of manifolds* N_1 *and* N_2 *have sub-exponential growth rates, then the fundamental group of the product manifold* $M = N_1 \times N_2$ *also has a sub-exponential growth rate.*

Proof. From the properties of the fundamental groups on product manifolds, it follows that $\pi_1(M) = \pi_1(N_1 \times N_2)$ is isomorphic to $\pi_1(N_1) \times \pi_1(N_2)$, denoted by $\approx$ the isomorphism between groups, so $\pi_1(M) \approx \pi_1(N_1) \times \pi_1(N_2)$. Choose

$\{\alpha_1, ..., \alpha_k\}$ as the generators of $\pi_1(N_1)$ and $\beta_1, ..., \beta_l$ as the generator of $\pi_1(N_2)$. Then $\{\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_l\}$ are the generators of $\pi_1(M) = \pi_1(N_1 \times N_2)$. Moreover it follows from the property of the direct product of groups that α_i commutes with each β_j for $1 \le i \le k$ and $1 \le j \le l$.

Let $B_T(r)$ be the ball in $\pi_1(T)$ of radius r centered at the identity element e, where $T = M, N_1, N_2$. By some easy calculations, we can show that,

$$\begin{aligned}\sharp B_M(r) &\le \sharp B_{N_1}(r) + \sharp B_{N_2}(r) + \sum_{i=0}^{[r]} \sharp B_{N_1}(i) \times B_{N_2}(r-i) \\ &\le \sharp B_{N_1}(r) + \sharp B_{N_2}(r) + \sharp B_{N_1}(r) \times \sharp B_{N_2}(r) \times ([r]+1).\end{aligned}$$

Where $[r]$ denotes the integral part of r. Hence we have

$$\begin{aligned}&\overline{\lim}_{r\to\infty} \frac{\ln \sharp B_M(r)}{r} \\ \le &\overline{\lim}_{r\to\infty} \frac{\ln((\sharp B_{N_1}(r) + \sharp B_{N_2}(r) + \sharp B_{N_1}(r) \times \sharp B_{N_2}(r) \times ([r]+1))}{r} \\ \le &\overline{\lim}_{r\to\infty} \frac{\ln((\sharp B_{N_1}(r)+1) \times (\sharp B_{N_2}(r)+1) \times ([r]+2))}{r} = 0.\end{aligned}$$

In the last equality, we have shown the fact that $\pi_1(M)$ has a sub-exponential growth rate. This proves the lemma. □

The next lemma gives a sufficient and necessary condition for the sub-exponential growth of $\pi_1(M)$. This is a geometric condition. This condition is crucial in the proof of our main theorem.

Lemma 2.2. *The fundamental group of a compact Riemannian manifold M has sub-exponential growth if and only if its corresponding Riemannian universal covering manifold, denoted by $\widetilde{M}$, has sub-exponential volume growth.*

Proof. In [7], Gromov got the following inequality:

$$\mathtt{Vol}(B_{(g,y)}(cr+c)) \ge \sharp B_M(r) \ge \mathtt{Vol}(B_{(g,y)}(c^{-1}r)), \tag{2.1}$$

where c is a positive constant and $\mathtt{Vol}(B_{(g,y)}(r))$ denotes the Riemannian volume of the ball of radius r centered at point $y \in \widetilde{M}$.

We note that Manning (cf. [8]) had shown that $\lim_{r\to\infty} \frac{\ln \mathtt{Vol}(B_{(g,y)}(r))}{r}$ exists, no less than 0 and independent of y. Using the inequality (2.1), it's easy to see that

$$\overline{\lim}_{r\to\infty} \frac{\ln \sharp B_M(r)}{r} = 0 \text{ if and only if } \lim_{r\to\infty} \frac{\ln \mathtt{Vol}(B_{(g,y)}(r))}{r} = 0.$$

This proves the lemma. □

Now we are ready to prove Theorem 2.2:

Proof. By contradiction we suppose that $(M = \mathbb{S}^1 \times N, (h-V)g)$ is a manifold without conjugate points, then by [6], we have that

$$h_{top}((h-V)g) = \lim_{r\to\infty} \frac{\ln \mathtt{Vol}(B_{((h-V)g,y)}(r))}{r} = h(M, (h-V)g). \tag{2.2}$$

As mentioned in lemma 2.2, the limit in (2.2) exists and does not depend on the choice of the point y, we call the limit *the volume entropy* and denote it by $h(M, (h-V)g)$, (cf. [8]).

Since $\pi_1(\mathbb{S}^1) \approx \mathbb{Z}$, it has polynomial growth rate ($\sharp B_{\mathbb{S}^1}(r) = r + 1$), so it also has sub-exponential growth rate. By lemma 2.1, $\pi_1(M)$ has sub-exponential growth rate. Following by lemma 2.2, we obtain that $\mathtt{Vol}(B_{((h-V)g,y)}(r))$ has sub-exponential growth rate, thus $h(M, (h - V)g) = 0$.

By the Maupertuis principle(cf. [1]), we know that the Hamiltonian flow of H restricted to the energy level

$$Q_h = \{H(x, p) = h\}, \qquad 0 < h < \delta,$$

coincide with (up to a re-parametrization) the geodesic flows of $(M = \mathbb{S}^1 \times N, (h - V)g)$ on the energy level

$$\widetilde{Q}_h = \{\widetilde{H}(x, p) := \frac{1}{2(h - V)} g^{ij} p_i p_j = 1\}, \qquad 0 < h < \delta.$$

Since the topological entropy of the Hamiltonian flow on Q_h is positive, we get that the topological entropy of the geodesic flow of $(M, (h - V)g)$ on $\widetilde{Q}_h$ is positive, i.e, $h_{top}((h-V)g) > 0$. By (2.2), this contradicts to the result that $h(M, (h-V)g) = 0$. So, this contradiction implies that $(M, (h-V)g)$ is a manifold with conjugate points.

We complete the proof of the theorem. □

3. Further Discussion

Another approach characterizing the topological entropy of the geodesic flows via the geometry and topology of the manifold was suggested by Paternain in [9]. His idea is to find topological obstructions to the integrability of geodesic flows based on topological entropy. The main progress along this approach was made by Bolsinov and Taimanov (cf. [3]). They discovered a smoothly integrable geodesic flow which has positive topological entropy. But it is still an open problem that whether a real analytically integrable geodesic flow defined on a Riemannian manifold of dimension larger than 2 has only zero topological entropy. We give the following conjecture.

Conjecture 3.1. *BR-type analytically natural Hamiltonian systems cannot be analytically integrable.*

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