Journal of Applied Analysis and Computation
Volume 5, Number 3, August 2015, 515–526
Website:http://jaac-online.com/
doi:10.11948/2015041

LIMIT CYCLE BIFURCATIONS OF A KIND OF LIÉNARD SYSTEM WITH A HYPOBOLIC SADDLE AND A NILPOTENT CUSP*

Junmin Yang[†,1,2], Feng Liang[3]

Abstract This paper gives a general theorem on the number of limit cycles of a near Hamiltonian system with a heteroclinic loop passing through a hyperbolic saddle and a nilpotent cusp. Then we study a kind of Liénard systems of type $(n, 4)$ for $3 \le n \le 27$ and obtain the lower bound of the maximal number of limit cycles for this kind of system.

Keywords limit cycle, Liénard system, nilpotent cusp, heteroclinic loop.

MSC(2000) 35D, 35C.

1. Introduction and main results

Consider a Liénard system of the form

$$\dot{x} = y, \quad \dot{y} = -g(x) - \varepsilon f(x)y, \tag{1.1}$$

where $\varepsilon \ge 0$ is a small parameter, $f(x)$ and $g(x)$ are polynomials of x and $\deg f = n$, $\deg g = m$. System (1.1) is called a polynomial Liénard system of type (n, m). Its unperturbed system is

$$\dot{x} = y, \quad \dot{y} = -g(x). \tag{1.2}$$

Let $H(n, m)$ denote the upper bound on the number of limit cycles that the system (1.1) can have on the plane for ε sufficiently small.

For $m = 1$ and 2, Blows and Lloyd [2] obtained $H(n, 1) \ge [\frac{n}{2}]$ and Han [4] gave $H(n, 2) \ge [\frac{2n+1}{3}], n \ge 2$ in 1999.

For $m = 3$, Christopher and Lynch [3] obtained $H(n, 3) \ge 2\left[\frac{3n+6}{8}\right], 2 \le n \le 50$. Then, Yang and Han [19] gave a new estimation of the lower bound of $H(n, 3)$ with $H(n, 3) \ge [\frac{3n+4}{4}]$ for $3 \le n \le 8$. From [12] and [20] we know that

$$H(n, 3) \ge n + 2 - [\frac{n+1}{4}],\ 9 \le n \le 22.$$

[†]the corresponding author: jmyzhw@sina.com(J. Yang)
[1]College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050016, China
[2]Hebei Key Laboratory of Computational Mathematics and Applications, Shijiazhuang 050016, China
[3]College of Mathematics and Computer Science, Anhui Normal University, Wuhu 241000, China
*The project was supported by National Natural Science Foundation of China (11101118), Natural Science Foundation of Hebei Province(A2012205074) and Science Foundation of Hebei Normal University (L2011B01).

For $m = 4$, Christopher and Lynch [3] proved $H(9,4) \geq 9$. Then, Yu and Han [24] obtained $H(n,4) \geq n$ for $10 \leq n \leq 14$ and Han et al. [9] gave

$$H(n,4) \geq n+3,\ n = 2,3,5,6,7,8,\ H(4,4) \geq 6.$$

In 2011, Yang and Han [21] obtained

$$H(n,4) \geq n+1-\left[\frac{n+1}{5}\right],\ \text{ for }\ 3 \leq n \leq 18. \tag{1.3}$$

Han and Romanovski [7] gave a new method to find a lower bound of $H(n,m)$ for many integers m and n. One of their results is

$$H(n,4) \geq H(n,3) \geq 2\left[\frac{n-1}{4}\right] + \left[\frac{n-1}{2}\right],\ n \geq 3. \tag{1.4}$$

More results on the bifurcation of limit cycles of system (1.1) can be found in [1, 6, 14, 15, 18, 24] etc.

In this paper we will study the limit cycles of system (1.1) in the case $m = 4$. Suppose system (1.2) has four singular points taking into account their multiplicity. Then according to [17], one may assume

$$g(x) = x(x-1)(x-\alpha)(x-\beta),\ 0 \leq \alpha \leq \beta \leq 1. \tag{1.5}$$

We are interested in the cases that system (1.2) has at least a family of periodic orbits. Due to Xiao [17], system (1.2) has a nilpotent singular point and at least a family of periodic orbits if and only if one of the following conditions holds:

(a) $\alpha = \beta = 0$, (b) $\alpha = 0, 0 < \beta < 1$, (c) $0 < \alpha = \beta < 1$, (d) $0 < \alpha < \frac{2}{5}, \beta = 1$,
(e) $\alpha = \frac{2}{5}, \beta = 1$, (f) $\frac{2}{5} < \alpha < 1$, $\beta = 1$, (g) $0 < \alpha < \frac{2}{5}, \beta = 1$.

Yang and Han [22] considered the case (g) and proved $H(n,4) \geq n - \left[\frac{n+1}{5}\right]$ for $1 \leq n \leq 12$. The authors considered cases (c) and (f) in [21] and gave (1.3). Recently, cases (a) and (b) were studied in [23]. Now only the cases (e) and (d) are not considered.

In this paper, we will give the lower bound of limit cycles of system (1.1) where $g(x)$ satisfies (1.5) and the condition (e). This will make the study of limit cycles of system (1.1) with $g(x)$ satisfies (1.5) be more complete. One of our main result is as follows.

Theorem 1.1. *Consider system* (1.1) *with* $\deg g(x) = 4$ *and* $g(x)$ *satisfies* (1.5) *and case (g). Let* $\widetilde{H}(n)$ *denote the maximal number of limit cycles of system* (1.1) *for all possible polynomial* f *with degree* n. *Then we have*

$$\widetilde{H}(n) \geq n+1-\left[\frac{n+1}{5}\right],\ 3 \leq n \leq 27. \tag{1.6}$$

2. Preliminaries and main results

Consider a near-Hamiltonian system in the form of

$$\begin{aligned} \dot{x} &= H_y + \varepsilon p(x,y,\delta), \\ \dot{y} &= -H_x + \varepsilon q(x,y,\delta), \end{aligned} \tag{2.1}$$

where p, q and H are C^ω functions, ε is a small perturbation parameter $(0 \le \varepsilon \ll 1)$, $\delta \in D \subset \mathbf{R}^m$ and D is compact. When $\varepsilon = 0$, system (2.1) becomes the Hamiltonian system

$$\dot{x} = H_y, \quad \dot{y} = -H_x. \tag{2.2}$$

We suppose that system (2.2) has an elementary center $C(x_c, y_c)$, a nilpotent cusp $A(x_a, y_a)$ and a hyperbolic saddle $S(x_s, y_s)$. Introduce

$$h_s = H(x_s, y_s) = H(x_a, y_a), \; h_c = H(x_c, y_c).$$

Without loss of generality we suppose $h_s = 0$ and the equation $H(x, y) = 0$ defines a heteroclinic loop L_0 containing S, A and two heteroclinic orbits L_1, L_2. For definiteness, we assume the periodic orbits are oriented clockwise. Then $h_c < 0$. The equation $H(x, y) = h, h \in (h_c, 0)$ defines a family of periodic orbits denoted by L_h as shown in Fig 1.

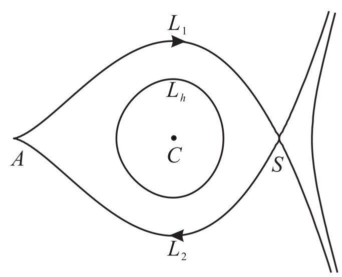

Figure 1. The phrase portraits of system (2.2)

Correspondingly we have a Melnikov function as follows

$$M(h, \delta) = \oint_{L_h} q dx - p dy, \quad h \in (h_c, 0).$$

Sun, Han and Yang [16] studied the number of limit cycles of system (2.1) in a neighborhood of the heteroclinic loop L_0. To introduce their main result in [16], we need make some preparations.

There exist three transformations of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = T_i \begin{pmatrix} u \\ v \end{pmatrix} + S_i, \; i = 1, 2, 3, \; S_1 = \begin{pmatrix} x_a \\ y_a \end{pmatrix}, \; S_2 = \begin{pmatrix} x_s \\ y_s \end{pmatrix}, \; S_3 = \begin{pmatrix} x_c \\ y_c \end{pmatrix},$$

where T_i is a 2×2 matrix satisfying $\det T_i = 1$, such that system (2.1) becomes

$$\dot{u} = \frac{\partial H_i}{\partial v} + \varepsilon p_i(u, v, \delta), \quad \dot{v} = -\frac{\partial H_i}{\partial u} + \varepsilon q_i(u, v, \delta),$$

where

$$\begin{aligned} H_1(u, v) &= \frac{v^2}{2} + \sum_{i+j \ge 3} \widetilde{h}_{ij} u^i v^j, \quad \widetilde{h}_{30} < 0, \\ p_1(u, v, \delta) &= \sum_{i+j \ge 0} \widetilde{a}_{ij} u^i v^j, \quad q_1(u, v, \delta) = \sum_{i+j \ge 0} \widetilde{b}_{ij} u^i v^j, \end{aligned} \tag{2.3}$$

$$\begin{aligned} H_2(u,v) &= \frac{1}{2}(v^2-u^2)+\sum_{i+j\geq 3}\bar{h}_{ij}u^iv^j, \\ p_2(u,v,\delta) &= \sum_{i+j\geq 0}\bar{a}_{ij}u^iv^j, \quad q_2(u,v,\delta)=\sum_{i+j\geq 0}\bar{b}_{ij}u^iv^j, \end{aligned} \tag{2.4}$$

$$\begin{aligned} H_3(u,v) &= \frac{1}{2}(v^2+u^2)+\sum_{i+j\geq 3}\hat{h}_{ij}u^iv^j, \\ p_3(u,v,\delta) &= \sum_{i+j\geq 0}\hat{a}_{ij}u^iv^j, \quad q_3(u,v,\delta)=\sum_{i+j\geq 0}\hat{b}_{ij}u^iv^j. \end{aligned} \tag{2.5}$$

By use of some results in [10] and [12] the authors gave their main result as shown in the following Lemma.

Lemma 2.1. ([16]) *Consider system* (2.1)*. Then we have*

$$\begin{aligned} M(h,\delta)=&c_0(\delta)+B_{00}c_1(\delta)|h|^{\frac{5}{6}}+c_2(\delta)h\ln|h|+[c_3(\delta)+b_1c_1(\delta)+b_2c_2(\delta)]h \\ &+B_{10}c_4(\delta)|h|^{\frac{7}{6}}+c_5(\delta)h^2\ln|h|-\frac{1}{11}B_{00}c_6(\delta)|h|^{\frac{11}{6}}+O(h^2) \end{aligned}$$

for $0\leq -h\leq 1$*, where* $B_{00}(>0), B_{10}(>0), b_1$ *and* b_2 *are some constants, and*

$$\begin{aligned} c_0(\delta) &= M(0,\delta)=\oint_{L_0} qdx+pdy=\sum_{i=1}^{2}\int_{L_i} qdx+pdy, \\ c_1(\delta) &= 2\sqrt{2}h_{30}^{-\frac{1}{3}}(\widetilde{a}_{10}+\widetilde{b}_{01}), \quad c_2(\delta)=-(\bar{a}_{10}+\bar{b}_{01}), \\ c_3(\delta)|_{c_1(\delta)=c_2(\delta)=0} &= \oint_{L_0}(p_x+q_y)dt, \\ c_4(\delta) &= 2\sqrt{2}\widetilde{h}_{30}^{-\frac{5}{3}}\left[\widetilde{h}_{30}(2\widetilde{a}_{20}+\widetilde{b}_{11}-\widetilde{h}_{12}(\widetilde{a}_{10}+\widetilde{b}_{01}))+\frac{1}{3}(\widetilde{h}_{21}^2-2\widetilde{h}_{40})(\widetilde{a}_{10}+\widetilde{b}_{01})\right], \\ c_5(\delta) &= -\frac{1}{2}\{(-3\bar{a}_{30}-\bar{b}_{21}+\bar{a}_{12}+3\bar{b}_{03}) \\ &\quad -\ [(2\bar{b}_{02}+\bar{a}_{11})(3\bar{h}_{03}-\bar{h}_{21})+(2\bar{a}_{20}+\bar{b}_{11})(3\bar{h}_{30}-\bar{h}_{12})]\}+\bar{b}c_2(\delta), \\ c_6(\delta) &= 9\mu_1^{-1}\alpha_{01}-2\mu_1^{-7}[(20\mu_2^3-20\mu_1\mu_2\mu_3+4\mu_1^2\mu_4)\alpha_{00}+(4\mu_1^2\mu_3-10\mu_1\mu_2^2)\alpha_{10} \\ &\quad +4\mu_1^2\mu_2\alpha_{20}-\mu_1^3\alpha_{30}], \end{aligned} \tag{2.6}$$

with

$$\begin{aligned} \mu_1 =& \sqrt[3]{\widetilde{h}_{30}}, \\ \mu_2 =& -\frac{1}{6}\widetilde{h}_{30}^{-\frac{2}{3}}(-2\widetilde{h}_{40}+\widetilde{h}_{21}^2), \\ \mu_3 =& \frac{1}{36}\widetilde{h}_{30}^{-\frac{5}{3}}[12\widetilde{h}_{30}(\widetilde{h}_{50}-\widetilde{h}_{31}\widetilde{h}_{21}+\widetilde{h}_{12}\widetilde{h}_{21}^2)-4\widetilde{h}_{40}^2+4\widetilde{h}_{40}\widetilde{h}_{21}^2-\widetilde{h}_{21}^4], \\ \mu_4 =& -\frac{1}{648}\widetilde{h}_{30}^{-\frac{8}{3}}[5\widetilde{h}_{21}^6-40\widetilde{h}_{40}^3+30\widetilde{h}_{40}\widetilde{h}_{21}^2(2\widetilde{h}_{40}-\widetilde{h}_{21}^2) \\ &+144\widetilde{h}_{30}\widetilde{h}_{40}(\widetilde{h}_{50}-\widetilde{h}_{31}\widetilde{h}_{21}+\widetilde{h}_{12}\widetilde{h}_{21}^2)+72\widetilde{h}_{30}\widetilde{h}_{21}^2(-\widetilde{h}_{50}+\widetilde{h}_{21}h_{31}-\widetilde{h}_{21}^2\widetilde{h}_{12}) \\ &+216\widetilde{h}_{30}^2(-\widetilde{h}_{60}-\widetilde{h}_{22}\widetilde{h}_{21}^2+\widetilde{h}_{41}\widetilde{h}_{21}+\widetilde{h}_{03}\widetilde{h}_{21}^3)+108\widetilde{h}_{30}^2\widetilde{h}_{31}^2 \\ &+432\widetilde{h}_{30}^2\widetilde{h}_{12}\widetilde{h}_{21}(\widetilde{h}_{12}\widetilde{h}_{21}-\widetilde{h}_{31})], \end{aligned}$$

$$\begin{aligned}
\alpha_{00} =& 2\sqrt{2}\,(\widetilde{a}_{10}+\widetilde{b}_{01}),\\
\alpha_{10} =& 2\,\sqrt{2}(-\widetilde{h}_{12}(\widetilde{a}_{10}+\widetilde{b}_{01})+2\widetilde{a}_{20}+\widetilde{b}_{11}),\\
\alpha_{20} =& 2\sqrt{2}[(\widetilde{a}_{10}+\widetilde{b}_{01})(3\widetilde{h}_{03}\widetilde{h}_{21}-\widetilde{h}_{22}+\frac{3}{2}\widetilde{h}_{12}^2)-2\widetilde{h}_{12}\widetilde{a}_{20}-\widetilde{h}_{12}\widetilde{b}_{11}\\
&+3\widetilde{a}_{30}+\widetilde{b}_{21}-\widetilde{a}_{11}\widetilde{h}_{21}-2\widetilde{b}_{02}\widetilde{h}_{21}],\\
\alpha_{30} =& 2\sqrt{2}\,[(\widetilde{a}_{10}+\widetilde{b}_{01})(3\,\widetilde{h}_{13}\widetilde{h}_{21}+3\,\widetilde{h}_{03}\widetilde{h}_{31}+3\,\widetilde{h}_{12}\widetilde{h}_{22}-15\,\widetilde{h}_{12}\widetilde{h}_{03}\widetilde{h}_{21}-\frac{5}{2}\,\widetilde{h}_{12}^3-\widetilde{h}_{32})\\
&+(3\,\widetilde{a}_{11}+6\,\widetilde{b}_{02})\,\widetilde{h}_{12}\widetilde{h}_{21}-2\,(\widetilde{b}_{12}+\widetilde{a}_{21})\,\widetilde{h}_{21}-(\widetilde{a}_{11}+2\,\widetilde{b}_{02})\,\widetilde{h}_{31}+4\,\widetilde{a}_{40}+\widetilde{b}_{31}\\
&+(3\,\widetilde{b}_{11}+6\,\widetilde{a}_{20})\widetilde{h}_{03}\widetilde{h}_{21}-(2\,\widetilde{a}_{20}+\widetilde{b}_{11})\,h_{22}+(3\,\widetilde{a}_{20}+\frac{3}{2}\,\widetilde{b}_{11})\,\widetilde{h}_{12}^2\\
&-(3\,\widetilde{a}_{30}+\widetilde{b}_{21})\widetilde{h}_{12}],\\
\alpha_{01} =& 2\sqrt{2}\,[\frac{2}{3}\,\widetilde{a}_{12}+2\,\widetilde{b}_{03}-2\,\widetilde{h}_{03}\widetilde{a}_{11}-4\,\widetilde{h}_{03}\widetilde{b}_{02}+(\widetilde{a}_{10}+\widetilde{b}_{01})(5\,\widetilde{h}_{03}^2-2\,\widetilde{h}_{04})].
\end{aligned}$$

Introduce

$$\begin{aligned}
&\bar{c}_i(\delta)=c_i(\delta), i=0,1,2,4,6,\\
&\bar{c}_3(\delta)=c_3(\delta)|_{c_1(\delta)=c_2(\delta)=0},\ \ \bar{c}_5(\delta)=c_5(\delta)|_{c_2(\delta)=0}.
\end{aligned}\tag{2.7}$$

If there exists a parameter $\delta_0\in\mathbf{R}^m$ *such that*

$$\bar{c}_j(\delta_0)=0, j=0,1,\cdots,l-1,\quad \bar{c}_l(\delta_0)\neq 0,\quad 1\le l\le 6$$

and

$$\text{rank}\frac{\partial(\bar{c}_0,\bar{c}_1,\cdots,\bar{c}_{l-1})}{\partial(\delta_1,\cdots,\delta_m)}=l,\quad (m\ge l).$$

Then system (2.1) *can have* l *limit cycles near* L_0 *for some* (ε,δ) *near* $(0,\delta_0)$.

In the above Lemma, the results of $c_1(\delta), c_4(\delta), c_6(\delta)$ can be found in [12]. The results of $c_2(\delta), c_5(\delta)$ can be found in [10].

The expansion of $M(h,\delta)$ near $C(x_c,y_c)$ can be written as (see [5])

$$M(h,\delta)=\sum_{j\ge 0}b_j(\delta)(h-h_c)^{j+1},\ 0<h-h_c\ll 1.\tag{2.8}$$

The formulas of $b_j, j=0,1,2,3$ were given in [13]. Let (2.5) hold. Han, Yang and Yu [11] established a computationally efficient algorithm to compute $b_j(\delta)(j=0,1,2,3,\cdots)$ systematically.

By perturbing the elementary center and the heteroclinic loop we can get more limit cycles as showed in the following theorem which is another main result of this paper.

Theorem 2.1. *Let* (2.8) *hold and system* (2.1) *satisfy the condition of Lemma 2.1. If there exist* $k\ge 1$ *and* $\delta_0\in\mathbf{R}^m$ *such that*

$$\begin{aligned}
&\bar{c}_j(\delta_0)=0,\ j=0,1,\cdots,l-1,\ \bar{c}_l(\delta_0)\neq 0,\ 1\le l\le 6,\\
&b_i(\delta_0)=0,\ i=0,1,\cdots,k-1,\ b_k(\delta_0)\neq 0,
\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}
&\bar{c}_l(\delta_0)b_k(\delta_0)<0(>0),\ l=0,1,2,4,\quad \text{or}\quad \bar{c}_l(\delta_0)b_k(\delta_0)>0(<0),\ l=3,5,6,\\
&\text{rank}\ \frac{\partial(\bar{c}_0,\cdots,\bar{c}_{l-1},b_0,\cdots,b_{k-1})}{\partial(\delta_1,\delta_2,\cdots,\delta_m)}\bigg|_{\delta_0}=l+k.
\end{aligned}\tag{2.10}$$

Then system (2.1) *can have* $l+k+1(resp., l+k)$ *limit cycles for some* (ε,δ) *near* (ε,δ_0).

The idea of proof on the above theorem is similar to the idea of Theorem 3.1 in [8].

3. Proof of Theorem 1.1

Consider

$$\dot{x}=y, \quad \dot{y}=-x(x-\frac{2}{5})(x-1)^2-\varepsilon y f(x), \tag{3.1}$$

where $f(x)=\sum_{i=0}^{n} a_i x^i$.

If $\varepsilon=0$, system (3.1) becomes a Hamiltonian system

$$\dot{x}=y, \quad \dot{y}=-x(x-\frac{2}{5})(x-1)^2. \tag{3.2}$$

It has a Hamiltonian function

$$H(x,y)=\frac{1}{2}y^2+\frac{1}{5}x^5-\frac{3}{5}x^4+\frac{3}{5}x^3-\frac{1}{5}x^2. \tag{3.3}$$

System (3.2) has an elementary center $C(\frac{2}{5},0)$, a nilpotent cusp $A(1,0)$, a hyperbolic saddle $O(0,0)$ and a heteroclinic loop $L_0=L_1\cup L_2\cup\{A,O\}$ defined by $H(x,y)=0$. It is easy to get $H(\frac{2}{5},0)=-\frac{108}{15625}$. For simplicity, we write $h_c=-\frac{108}{15625}$. The equation $H(x,y)=h, h\in(h_c,0)$ defines a family of periodic orbits denoted by L_h. See Fig 2.

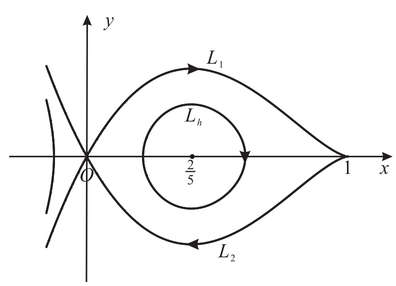

Figure 2. The phrase portraits of system (3.2)

In the following we will make some suitable transformations to compute $\bar{c}_j$ in (2.7) and $\bar{b}_j$ in (2.8).

We introduce the following change of variables

$$u=1-x, \ v=-y.$$

Then system (3.1) becomes

$$\begin{aligned}\dot{u}&=v,\\ \dot{v}&=\tfrac{3}{5}u^2-\tfrac{8}{5}u^3+u^4-\varepsilon v f(1-u).\end{aligned} \tag{3.4}$$

The Hamiltonian function of system $(3.4)|_{\varepsilon=0}$ is

$$H_1(u,v) = \frac{1}{2}\,v^2 - \frac{1}{5}\,u^5 + \frac{2}{5}\,u^4 - \frac{1}{5}\,u^3. \tag{3.5}$$

Let

$$q_1(u,v) = -vf(1-u) \equiv \sum_{i=0}^{n} \widetilde{b}_{ij} u^i v. \tag{3.6}$$

One can obtain

$$\widetilde{b}_{01} = -\sum_{i=0}^{n} a_i,\quad \widetilde{b}_{11} = \sum_{i=1}^{n} i a_i,\quad \widetilde{b}_{21} = -\sum_{i=2}^{n} \tfrac{i(i-1)}{2} a_i,$$
$$\widetilde{b}_{31} = \sum_{i=3}^{n} \tfrac{i(i-1)(i-2)}{6} a_i,\ \widetilde{b}_{03} = 0.$$

By (2.6) and (2.7), we obtain

$$\bar{c}_1 = 2\sqrt{3}\,\widetilde{b}_{01} = -2\sqrt{3}\sum_{i=0}^{n} a_i,$$
$$\bar{c}_4 = \frac{2}{3}\sqrt{2}\cdot 5^{\frac{2}{3}}(4\widetilde{b}_{01} + 3\widetilde{b}_{11}),$$
$$\bar{c}_6 = \frac{10}{81}\,\sqrt{2}\cdot\sqrt[3]{5}\left(1232\,\widetilde{b}_{01} + 792\,\widetilde{b}_{11} + 432\,\widetilde{b}_{21} - \frac{1458}{5}\,\widetilde{b}_{03} + 162\,\widetilde{b}_{31}\right).$$

We introduce another change of variables

$$u = \frac{1}{5}\,\sqrt{10}\,x,\ \ v = y$$

and a time scaling $t \to \frac{\sqrt{10}}{5}t$. Then system (3.1) becomes

$$\begin{aligned} \dot{u} &= v,\\ \dot{v} &= u - \tfrac{9}{4}\,\sqrt{10}\,u^2 + 15\,u^3 - \tfrac{25}{8}\,\sqrt{10}u^4 + \varepsilon q_2(u,v), \end{aligned} \tag{3.7}$$

where

$$q_2(u,v) = -\frac{\sqrt{10}}{2}\sum_{i=0}^{n} a_i \left(\frac{1}{2}\,\sqrt{10}u\right)^i v. \tag{3.8}$$

Obviously, $q_1(u,v)$ can be rewritten as

$$q_2(u,v) = \sum_{i=0}^{n} \bar{b}_{ij} u^i v^j,$$

where

$$\bar{b}_{01} = -\frac{1}{2}\,\sqrt{10}\,a_0,\quad \bar{b}_{11} = -\frac{5}{2}\,a_1,\quad \bar{b}_{21} = -\frac{5}{4}\sqrt{10}\,a_2,\quad \bar{b}_{ij} = 0, j \neq 1.$$

The Hamiltonian function of system $(3.7)|_{\varepsilon=0}$ is

$$H_2(u,v) = \frac{1}{2}\,(v^2 - u^2) + \frac{3}{4}\,\sqrt{10}\,u^3 + \frac{5}{8}\,\sqrt{10}\,u^5 - \frac{15}{4}\,u^4. \tag{3.9}$$

By (2.6) and (2.7), we get

$$\bar{c}_2 = \frac{1}{2}\sqrt{10}\,a_0, \quad \bar{c}_5 = -\frac{5}{16}\sqrt{10}\,(2\,a_2 + 9\,a_1)\,.$$

Next, we will give the formulas of $\bar{c}_0$ and $\bar{c}_3$ in (2.7).

From $H(x,y)=0$, we obtain

$$y = \pm\frac{1}{5}\,x\,(1-x)\sqrt{10\,(1-x)}.$$

Let $q(x,y) = -yf(x)$. By (2.6) and (2.7) we have

$$\begin{aligned} \bar{c}_0 &= \oint_{L_0} q(x,y)dx = -\oint_{L_0} yf(x)dx = -\sum_{i=0}^{n} a_i I_i, \\ \bar{c}_3|_{\bar{c}_1=\bar{c}_2=0} &= \oint_{L_0} q_y dt = -\oint_{L_0}\left(\sum_{i=2}^{n} a_i(x^i - x)\right)dt = \sum_{i=0}^{n} a_i J_i, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} I_i &= \oint_{L_0} x^i y dx = \frac{2}{5}\int_0^1 x^{i+1}(1-x)\sqrt{10\,(1-x)}\,dx, \\ J_i &= -\oint_{L_0}(x^i - x)dt = \oint_{L_0}\frac{(x-x^i)}{y}dx = 10\int_0^1 \frac{1-x^{i-1}}{(1-x)\sqrt{10\,(1-x)}}dx. \end{aligned} \tag{3.11}$$

By Maple, we get

$I_0 = \frac{8}{175}\sqrt{10}$, $I_1 = \frac{32}{1575}\sqrt{10}$, $I_2 = \frac{64}{5775}\sqrt{10}$, $I_3 = \frac{512}{75075}\sqrt{10}$, $I_4 = \frac{1024}{225225}\sqrt{10}$,
$I_5 = \frac{4096}{1276275}\sqrt{10}$, $I_6 = \frac{8192}{3464175}\sqrt{10}$, $I_7 = \frac{131072}{72747675}\sqrt{10}$, $I_8 = \frac{262144}{185910725}\sqrt{10}$,
$I_9 = \frac{1048576}{929553625}\sqrt{10}$, $I_{10} = \frac{2097152}{2281631625}\sqrt{10}$, $I_{11} = \frac{33554432}{52594534125}\sqrt{10}$,
$I_{12} = \frac{33554432}{52594534125}\sqrt{10}$, $I_{13} = \frac{134217728}{247945660875}\sqrt{10}$, $I_{14} = \frac{268435456}{578539875375}\sqrt{10}$,
$I_{15} = \frac{8589934592}{21405975388875}\sqrt{10}$, $I_{16} = \frac{17179869184}{49107825892125}\sqrt{10}$, $I_{17} = \frac{68719476736}{223713429064125}\sqrt{10}$,
$I_{18} = \frac{137438953472}{506298813145125}\sqrt{10}$, $I_{19} = \frac{1099511627776}{4556689318306125}\sqrt{10}$, $I_{20} = \frac{2199023255552}{10198304664780375}\sqrt{10}$,
$I_{21} = \frac{8796093022208}{45428811688567125}\sqrt{10}$, $I_{22} = \frac{17592186044416}{100733452005083625}\sqrt{10}$,
$I_{23} = \frac{281474976710656}{1779624318756477375}\sqrt{10}$, $I_{24} = \frac{562949953421312}{3915173501264250225}\sqrt{10}$,
$I_{25} = \frac{2251799813685248}{17166529967081712525}\sqrt{10}$, $I_{26} = \frac{4503599627370496}{37512046965104482925}\sqrt{10}$,
$I_{27} = \frac{36028797018963968}{326890694981624779775}\sqrt{10}$, $J_2 = 2\sqrt{10}$, $J_3 = \frac{10}{3}\sqrt{10}$, $J_4 = \frac{22}{5}\sqrt{10}$,
$J_5 = \frac{186}{35}\sqrt{10}$, $J_6 = \frac{386}{63}\sqrt{10}$, $J_7 = \frac{1586}{231}\sqrt{10}$, $J_8 = \frac{3238}{429}\sqrt{10}$, $J_9 = \frac{52666}{6435}\sqrt{10}$,
$J_{10} = \frac{106762}{12155}\sqrt{10}$, $J_{11} = \frac{431910}{46189}\sqrt{10}$, $J_{12} = \frac{872218}{88179}\sqrt{10}$, $J_{13} = \frac{7036530}{676039}\sqrt{10}$,
$J_{14} = \frac{14177066}{1300075}\sqrt{10}$, $J_{15} = \frac{57079714}{5014575}\sqrt{10}$, $J_{16} = \frac{114828038}{9694845}\sqrt{10}$, $J_{17} = \frac{3693886906}{300540195}\sqrt{10}$,
$J_{18} = \frac{7423131482}{583401555}\sqrt{10}$, $J_{19} = \frac{29822170718}{2268783825}\sqrt{10}$, $J_{20} = \frac{59883160786}{4418157975}\sqrt{10}$,
$J_{21} = \frac{480832549478}{34461632205}\sqrt{10}$, $J_{22} = \frac{964947159166}{67282234305}\sqrt{10}$, $J_{23} = \frac{3872021770174}{263012370465}\sqrt{10}$,
$J_{24} = \frac{7766914181258}{514589420475}\sqrt{10}$, $J_{25} = \frac{124613686513778}{8061900920775}\sqrt{10}$, $J_{26} = \frac{249872325101218}{15801325804719}\sqrt{10}$,
$J_{27} = \frac{1001920273605598}{61989816618513}\sqrt{10}$.

Finally, we calculate the coefficients $b_j (j = 0, 1, \cdots)$ in (2.8). There is a transformation of the form

$$u = \frac{3}{25}\sqrt{10}(x - \frac{2}{5}),\ v = y$$

and a time scaling $t \to \frac{3}{25}\sqrt{10}\, t$ such that system (3.1) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + \tfrac{25}{36}\sqrt{10}\, u^2 + \tfrac{3125}{81}\, u^3 - \tfrac{78125}{1944}\, \sqrt{10} u^4 + \varepsilon\, q_3(u, v), \end{aligned} \tag{3.12}$$

where

$$q_3(u, v) = -\frac{5}{6}\sqrt{10}\, f\left(\frac{5}{6}\sqrt{10}\, u + \frac{2}{5}\right) v = -\frac{5}{6}\sqrt{10} \sum_{i=0}^{n} a_i \left(\frac{5}{6}\sqrt{10}\, u + \frac{2}{5}\right)^i v. \tag{3.13}$$

System $(3.12)|_{\varepsilon=0}$ has the following Hamiltonian function

$$H_3(u, v) = \frac{1}{2}(v^2 + u^2) - \frac{25}{108}\sqrt{10} u^3 - \frac{3125}{324} u^4 + \frac{15625}{1944}\sqrt{10} u^5. \tag{3.14}$$

Now we are ready to use the programs in [11] to compute the coefficients b_j in (2.8) for system (3.1). Since the formulas of $b_j, j = 0, 1, 2, \cdots, 17$ are so long, we only present b_0, b_1, and b_2 here and omit others.

We obtain (for $n = 27$)

$$\begin{aligned} b_0 = \; & -\sqrt{10}\pi(5/3\, a_0 + 2/3\, a_1 + \tfrac{4}{15}\, a_2 + \tfrac{8}{75}\, a_3 + \tfrac{16}{375}\, a_4 + \tfrac{32}{1875}\, a_5 + \tfrac{64}{9375}\, a_6 \\ & + \tfrac{128}{46875}\, a_7 + \tfrac{256}{234375}\, a_8 + \tfrac{512}{1171875}\, a_9 + \tfrac{1024}{5859375}\, a_{10} + \tfrac{2048}{29296875}\, a_{11} \\ & + \tfrac{4096}{146484375}\, a_{12} + \tfrac{8192}{732421875}\, a_{13} + \tfrac{16384}{3662109375}\, a_{14} + \tfrac{32768}{18310546875}\, a_{15} \\ & + \tfrac{65536}{91552734375}\, a_{16} + \tfrac{131072}{457763671875}\, a_{17} + \tfrac{262144}{2288818359375}\, a_{18} + \tfrac{524288}{11444091796875}\, a_{19} \\ & + \tfrac{1048576}{57220458984375}\, a_{20} + \tfrac{2097152}{286102294921875}\, a_{21} + \tfrac{4194304}{1430511474609375}\, a_{22} \\ & + \tfrac{8388608}{7152557373046875}\, a_{23} + \tfrac{16777216}{35762786865234375}\, a_{24} + \tfrac{33554432}{178813934326171875}\, a_{25} \\ & + \tfrac{67108864}{894069671630859375}\, a_{26} + \tfrac{134217728}{4470348358154296875}\, a_{27}), \\ b_1 = \; & -\sqrt{10}\pi(\tfrac{640625}{23328}\, a_0 + \tfrac{184375}{11664}\, a_1 + \tfrac{81875}{5832}\, a_2 + \tfrac{32125}{2916}\, a_3 + \tfrac{10925}{1458}\, a_4 + \tfrac{3355}{729}\, a_5 \\ & + \tfrac{1918}{729}\, a_6 + \tfrac{5204}{3645}\, a_7 + \tfrac{13576}{18225}\, a_8 + \tfrac{34352}{91125}\, a_9 + \tfrac{84832}{455625}\, a_{10} + \tfrac{205376}{2278125}\, a_{11} \\ & + \tfrac{489088}{11390625}\, a_{12} + \tfrac{1148672}{56953125}\, a_{13} + \tfrac{2665984}{284765625}\, a_{14} + \tfrac{6124544}{1423828125}\, a_{15} + \tfrac{13944832}{7119140625}\, a_{16} \\ & + \tfrac{31502336}{35595703125}\, a_{17} + \tfrac{70672384}{177978515625}\, a_{18} + \tfrac{157564928}{889892578125}\, a_{19} + \tfrac{349339648}{4449462890625}\, a_{20} \\ & + \tfrac{770637824}{22247314453125}\, a_{21} + \tfrac{1692270592}{111236572265625}\, a_{22} + \tfrac{3700686848}{556182861328125}\, a_{23} \\ & + \tfrac{8061976576}{2780914306640625}\, a_{24} + \tfrac{17501782016}{13904571533203125}\, a_{25} + \tfrac{37872467968}{69522857666015625}\, a_{26} \\ & + \tfrac{81709236224}{347614288330078125}\, a_{27}), \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
b_2 = \quad & -\sqrt{10}\pi\Big(\tfrac{830908203125}{544195584}\,a_0 + \tfrac{250205078125}{272097792}\,a_1 + \tfrac{112197265625}{136048896}\,a_2 + \tfrac{47737890625}{68024448}\,a_3 \\
& + \tfrac{19306953125}{34012224}\,a_4 + \tfrac{7451828125}{17006112}\,a_5 + \tfrac{2730565625}{8503056}\,a_6 + \tfrac{948390625}{4251528}\,\pi\,a_7 \\
& + \tfrac{313131125}{2125764}\,a_8 + \tfrac{98783725}{1062882}\,a_9 + \tfrac{29942585}{531441}\,a_{10} + \tfrac{17532482}{531441}\,a_{11} + \tfrac{9961364}{531441}\,a_{12} \\
& + \tfrac{137836712}{13286025}\,\pi\,a_{13} + \tfrac{372833104}{66430125}\,a_{14} + \tfrac{988442144}{332150625}\,a_{15} + \tfrac{2574528064}{1660753125}\,a_{16} \\
& + \tfrac{1320224896}{1660753125}\,a_{17} + \tfrac{16689620224}{41518828125}\,a_{18} + \tfrac{41668755968}{207594140625}\,a_{19} + \tfrac{102860612608}{1037970703125}\,a_{20} \\
& + \tfrac{251320444928}{5189853515625}\,a_{21} + \tfrac{121669357568}{5189853515625}\,a_{22} + \tfrac{1460066459648}{129746337890625}\,a_{23} \\
& + \tfrac{3476979859456}{648731689453125}\,a_{24} + \tfrac{8220787245056}{3243658447265625}\,a_{25} + \tfrac{19308506054656}{16218292236328125}\,a_{26} \\
& + \tfrac{9014747987968}{16218292236328125}\,a_{27}\Big), \\
\cdots\,.
\end{aligned}
$$

In the following we first give the proof of theorem 1.1 for $n = 27$.

By solving the equations $b_0 = b_1 = \cdots = b_{16} = \bar{c}_0 = \bar{c}_1 = \cdots = \bar{c}_4 = 0$, we obtain

$$
\begin{aligned}
a_0 = \quad & 0, \\
a_1 = \quad & -\tfrac{2}{5}\,a_4 - \tfrac{6}{5}\,a_9 - \tfrac{504}{5}\,a_{14} - \tfrac{299178}{5}\,a_{19} - \tfrac{3589185612}{25}\,a_{24} \\
& - \tfrac{121669546266205091921932140953533563642 8}{777026707344293594360351562 5}\,a_{27}, \\
a_2 = \quad & \tfrac{9}{5}\,a_4 + \tfrac{27}{5}\,a_9 + \tfrac{2268}{5}\,a_{14} + \tfrac{1346301}{5}\,a_{19} + \tfrac{16151335254}{25}\,a_{24} \\
& + \tfrac{4927616623781306181299835649288275728557 4}{699324036609864234924316406 25}\,a_{27}, \\
a_3 = \quad & -\tfrac{12}{5}\,a_4 - \tfrac{34}{5}\,a_9 - \tfrac{2856}{5}\,a_{14} - \tfrac{1695342}{5}\,a_{19} - \tfrac{20338718468}{25}\,a_{24} \\
& - \tfrac{689460762175169319505850871100881835302 8}{777026707344293594360351562 5}\,a_{27}, \\
a_5 = \quad & 9\,a_9 + \tfrac{3778}{5}\,a_{14} + \tfrac{2242647}{5}\,a_{19} + \tfrac{26904639852}{25}\,a_{24} \\
& + \tfrac{547223076851981098018296632460147219428 8}{466216024406576156616210937 5}\,a_{27}, \\
a_6 = \quad & -14\,a_9 - \tfrac{5859}{5}\,a_{14} - \tfrac{3477936}{5}\,a_{19} - \tfrac{41724183641}{25}\,a_{24} \\
& - \tfrac{282881023245067586650599236502650801096 7}{155405341468858718872070312 5}\,a_{27}, \\
a_7 = \quad & 12\,a_9 + \tfrac{4944}{5}\,a_{14} + \tfrac{2934758}{5}\,a_{19} + \tfrac{7041554544}{5}\,a_{24} \\
& + \tfrac{477402307755877285587389314341565582136}{31081068293771743774414062 5}\,a_{27}, \\
a_8 = \quad & -\tfrac{27}{5}\,a_9 - \tfrac{2016}{5}\,a_{14} - 239328\,a_{19} - \tfrac{2871175518}{5}\,a_{24} \\
& - \tfrac{2335914572638593645345194502261505454 2}{372972819525260925292968 75}\,a_{27}, \\
a_{10} = \quad & \tfrac{462}{5}\,a_{14} + \tfrac{273823}{5}\,a_{19} + \tfrac{3284999883}{25}\,a_{24} \\
& + \tfrac{712691119973369554938591001893872 97}{497297092700347900390625}\,a_{27}, \\
a_{11} = \quad & -\tfrac{336}{5}\,a_{14} - \tfrac{198099}{5}\,a_{19} - \tfrac{2376539424}{25}\,a_{24} \\
& - \tfrac{46403811837196654428795702307933752 4}{447567383430313110351562 5}\,a_{27}, \\
a_{12} = \quad & \tfrac{156}{5}\,a_{14} + 18018\,a_{19} + \tfrac{1080761591}{25}\,a_{24} \\
& + \tfrac{10821920498508687408363736826854 43}{22952173509246826171875}\,a_{27}, \\
a_{13} = \quad & -\tfrac{42}{5}\,a_{14} - \tfrac{21714}{5}\,a_{19} - 10418772\,a_{24} \\
& - \tfrac{28856109787654095397546976380 36}{2539389409533691406 25}\,a_{27}, \\
a_{15} = \quad & 396\,a_{19} + \tfrac{4740736}{5}\,a_{24} + \tfrac{49370909620851218416465707802 4}{477405208992333984375}\,a_{27}, \\
a_{16} = \quad & -187\,a_{19} - 444873\,a_{24} - \tfrac{2233952512828480789350270 77}{460371464794921875}\,a_{27},
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
a_{17} &= \tfrac{297}{5}\, a_{19} + \tfrac{3451734}{25}\, a_{24} + \tfrac{1437971594129908679572045146\,2}{954810417984667968\,75}\, a_{27}, \\
a_{18} &= -\tfrac{57}{5}\, a_{19} - \tfrac{589589}{25}\, a_{24} - \tfrac{982248157627817778673704 29}{3819241671938671875}\, a_{27}, \\
a_{20} &= \tfrac{5733}{5}\, a_{24} + \tfrac{113682589280637833219887}{91661800126528125}\, a_{27}, \\
a_{21} &= -\tfrac{2002}{5}\, a_{24} - \tfrac{1659212497816434996138}{3917170945578125}\, a_{27}, \\
a_{22} &= \tfrac{483}{5}\, a_{24} + \tfrac{2949132933187687886171}{30553933375509375}\, a_{27}, \\
a_{23} &= -\tfrac{72}{5}\, a_{24} - \tfrac{1454019848924182720 6}{1222157335020375}\, a_{27}, \\
a_{25} &= \tfrac{16122924049949}{80010300165}\, a_{27}, \\
a_{26} &= -\tfrac{266691911}{12335115}\, a_{27}.
\end{aligned}
$$

Let $\delta_0 = (a_0, a_1, \cdots, a_{26})$ which satisfies (3.16). Then we get

$$
\begin{aligned}
b_{17}(\delta_0) &= \tfrac{71086315553831047608923654479440301656723022460937 5}{49959879203495999031480298380460032} \sqrt{10}\, a_{27}\pi, \\
\bar{c}_5(\delta_0) &= \tfrac{1589344057950208}{428120899200439453125} \sqrt{10}\, a_{27}
\end{aligned}
$$

and

$$
\operatorname{rank} \frac{\partial(b_0, b_1, \cdots, b_{16}, \bar{c}_0, \bar{c}_1, \cdots, \bar{c}_4)}{\partial(a_0, a_1, \cdots, a_{26})} = 22.
$$

It can be seen that $b_{17}(\delta_0)\bar{c}_5(\delta_0) > 0$ if $a_{27} \neq 0$. Then by Theorem 2.1, system (3.1) has 23 limit cycles for some (ε, δ) near $(0, \delta_0)$.

For each $n, n = 3, 4, \cdots, 26$, we can similarly prove that there exists a corresponding parameter δ_{0_n} such that

$$
\begin{aligned}
&\bar{c}_0(\delta_{0_n}) = \bar{c}_1(\delta_{0_n}) = \cdots = \bar{c}_4(\delta_{0_n}) = 0,\ \bar{c}_5(\delta_{0_n}) \neq 0, \\
&b_0(\delta_{0_n}) = b_1(\delta_{0_n}) = \cdots = b_{n-6-[\frac{n+1}{5}]}(\delta_{0_n}) = 0,\ b_{n-5-[\frac{n+1}{5}]}(\delta_{0_n}) \neq 0, \\
&b_{n-5-[\frac{n+1}{5}]}(\delta_{0_n})\bar{c}_5(\delta_{0_n}) > 0,\ \operatorname{rank} \tfrac{\partial(b_0, b_1, \cdots, b_{n-6-[\frac{n+1}{5}]}, \bar{c}_0, \bar{c}_1, \cdots, \bar{c}_4)}{\partial(a_0, a_1, \cdots, a_{26})} = n - [\tfrac{n+1}{5}].
\end{aligned}
$$

Then by Theorem 2.1, system (3.1) has $n + 1 - [\frac{n+1}{5}]$ limit cycles for some (ε, δ) near $(0, \delta_0)$.

The proof of Theorem 1.1 is completed.

References

[1] A. Atabaigi and H. R. Z. Zangeneha, *Bifurcation of limit cycles in smll pertrrbations of a class of Hyper-elliptic Hamiltonian systems of degree 5 with a cusp*, J. of Appl. Anal. and Compu., 1:3(2011), 299–313.

[2] T. R. Blows and N. G. Lloyd, *The number of small-amplitude limit cycles of Li6nard equations*, Math. Proc. Camb. Phil. Soc., 95(1984), 359–366.

[3] C. J. Christopher and S. Lynch, *Small-amplitude limit cycle bifurcations for Liénard systems with quadratic or cubic damping or restoring forces*, Nonlinearity, 12(1999), 1099–1112.

[4] M. Han, *Liapunov constants and Hopf cyclicity of Lineard systems*, Ann. of Diff. Eqs., 15:2 (1999), 113–126.

[5] M. Han, *On Hopf cyclicity of planar systems,* J. Math. Anal. Appl., 245(2000), 404–422.

[6] M. Han, *Bifurcation Theory of Limit Cycles, Science Press*, Beijing, 2013.

[7] M. Han and V. Romanovski, *On the number of limit cycles of polynomial Lienard systems*, Nonlinear Analysis: Real World Applications, 14(2013), 1655–1668.

[8] M. Han, Z. Wang and H. Zang, *Limit cycles by Hopf and homoclinic bifurcations for near Hanmil- tonian systems*, (Chinese). Chinese Ann. Math. Ser. A 28:5(2007), 679–690; translation in Chinese J. Contemp. Math. 28:4(2007), 423–434.

[9] M. Han, H. Yan, J. Yang and C. Lhotka, *On the Number of Limit Cycles of Some Liénard Systems*, Can. Appl. Math. Q., 17:1(2009), 61–83.

[10] M. Han, J. Yang, T. Alina and Y. Gao, *Limit cycles near Homoclinic and Heteroclinic loops*, J. Dyn. Diff. Equat. 2(2008), 923–944.

[11] M. Han, J. Yang and P. Yu, *Hopf Bifurcations for near-Hamiltonian Systems*, Int. J. Bifurcation and Chaos, 19:12(2009), 4117–4130.

[12] M. Han, H. Zang and J. Yang, *Limit cycle bifurcations by perturbing a cuspidal loop in a Hamiltonian system*, J. Diff. Equat., 245(2008), 1086–1111.

[13] Y. Hou and M. Han, *Melnikov functions for planar near-Hamiltonian systems and Hopf bifurcations*, J.Shanghai Normal University (Natural Sciences), 35(2006), 1–10.

[14] R. Kazemi and H. R. Z. Zangeneha, *Bifurcation of limit cycles in smll pertrrbations of a Hyper-elliptic Hamiltonian system with two nilpotent saddles*, J. of Appl. Anal. and Compu., 2:4(2012), 395–413.

[15] N. Lloyd and J. Pearson, *Symmetry in planar dynamical systems*, J. Symbolic Comput, 33(2002), 357-366.

[16] X. Sun, M. Han and J. Yang, *Bifurcation of limit cycles from a heteroclinic loop with a cusp*, Nonlinear Analysis, 74 (2011), 2948–2965.

[17] D. Xiao, *Bifurcations on a Five-Parameter Family of planar Vector Field*, J. Dyn. Diff. Equat., 20:4(2008), 961–980.

[18] J. Yang, *On the limit cycles of a kind of Lienard system with a nilpotent center under perturbations*, J. of Appl. Anal. and Compu., 2:3(2012), 325–339.

[19] J. Yang, M. Han and V.G. Romanovski, *Limit cycle bifurcations of some Liénard systems*, J. Math. Anal. Appl., 366(2010), 242–255.

[20] J. Yang and M. Han, *Limit cycle bifurcations of some Linard systems with a nilpotent cusp*, Internat. J. Bifur. Chaos, 11 (2010) 3829–3839.

[21] J. Yang and M. Han, *Limit cycle bifurcations of some Lienard systems with a cuspidal loop and a homoclinic loop*, Chaos, Solitons and Fractals, 44(2011), 269–289.

[22] J. Yang and M. Han, *Computation of expansion coefficients of Melnikov functions near a nilpotent center*, Computers and Mathematics with Applications, 64(2012), 1957–1974.

[23] J. Yang and X. Sun, *Bifurcation of limit cycles for some Liénard systems with a nilpotent singular point*, to appear in IJBC.

[24] P. Yu and M. Han , *Limit cycles in generalized Liénard systems*, Chaos, Solitons and Fractals, 30:5(2006), 1048–1068.