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OPTIMAL SELLING STRATEGY WITH A LARGE BLOCK OF STOCK*

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Abstract In this paper, we develop an optimal stock selling strategy with the stochastic upper bound of selling rate over an infinite time horizon. Moreover, the temporary and permanent price impact are considered. We treat the problem by using a fluid model. In the model that the number of shares is treated as fluid (continuous) and the overall liquidation is dictated by the rates of selling over time. The goal is to maximize the overall return under state constraints. The corresponding value function with the selling strategies is shown to be continuous and the unique viscosity solution to the associated HJB equation. Finally, a numerical example is given to illustrate the result.

Keywords Optimal selling strategy, temporary price impact, HJB, viscosity solution.

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1. Introduction

A large institutional investor, when selling a large block of shares, is faced with the following crucial problem. On one hand, selling a large position in a market place normally depresses the market if sold in a short period of time, which would result in poor filling prices. An advisable strategy for selling stock of large size is to sell much smaller number of shares over the time. On the other hand, upper bound of the quantity of shares which can be sold at some price is not only determined by seller, but also depend on the liquidity of market. Therefore, the stochastic upper bound of selling rate has to be considered in the optimal selling strategy.

There is an extensive literature devoted to the contra-trend strategy. For instance, Bertsimas and Lo [4] derived dynamic optimal trading strategies that minimize the expected cost of trading a large block of equity over a fixed time horizon. This model had been extended by Almgren and Chriss [1]. They considered the execution of portfolio transactions with the aim of minimizing a combination of volatility risk and transaction costs arising from permanent and temporary market impact, and constructed the efficient frontier in the space of time-dependent liquidation strategies.

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The following improvement of this work was finished by Almgren [2,3] . A. Schied and T. SchÄoneborn [15,16] studied the infinite and finite horizon optimal portfolio liquidation problem for a von Neumann-Morgenstern investor, and characterized the value function and the optimal strategy as classical solutions of nonlinear parabolic partial differential equations by stochastic control approach.

M. Pemy, Q. Zhang and G. Yin [11] studied the liquidation strategy for selling a large block of stock. In particular, they treated the selling rule problem by using a fluid model, and the corresponding liquidation was dictated by the rate of selling over time. This model with regime switching has been extended by M. Pemy, Q. Zhang and G. Yin [12]. In M. Pemy [13], the stock price movements are modeled by a Markov switching Levy process. B. Bian, M. Dai, L. Jiang, J. Zhang and Y.F. Zhong [5] considered the liquidation strategy for selling an illiquid stock by combining selling with occasional buying over a period of time. In their model, the buying activities helped to stabilize the stock price when heavy selling was in progress.

Temporary price impact function of stock was studied by F. Lillo, J. Farmer, and R. Manttegna [10]. They studied the short-term response to a single trade by using huge amounts of data. The similar work was done by M. Potters and J.-P. Bouchard [14]. They determined the price impact function using French and British stocks, and found a logarithmic dependence of the price response on the volume.

This paper differs from the aforementioned papers in the following significant ways: (1) The stochastic upper bound of selling rate is considered in our model to better reflect market conditions. In fact, as a result of stochastic liquidity, the selling strategies are not always satisfied. (2) Our model is analyzed in the viscosity framework. In general, HJB equation method is the ideal choice for continuous selling strategy model, because the target (i.e. value function) of the model always can be described as a supremum or infimum of expectation. However, the HJB equation in our article is a fully nonlinear degenerate parabolic equation and is not well-posed in the classical sense, and the value function is not smooth enough to satisfy the HJB equation in the classical sense. Therefore, it is natural to ask for a weak solution such that the value function is unique even though it is not smooth. One such weak solution called viscosity solution was introduced by Crandall and Lions [6,7]. More properties was discussed by H M. Soner [17,18].

The rest of this article is organized as follows. In section 2, we rigorously develope the mathematical model for the optimal selling strategy with stochastic upper bound of selling rate, in which the temporary and permanent price impact are considered. We prove that the value function is continuous, and show that it is the viscosity solutions of the HJB equation. In section 3, we prove the comparison principle of the viscosity solutions and then the uniqueness follows the comparison principle. In section 4, we present the numerical examples to demonstrate how to apply the method to find the optimal selling rules.

2. Problem Formulation

Let X_t denote the stock price at time t, which satisfies the stochastic differential equation

$$dX_t = (\mu - al_t(Z_t \wedge z_{up}))X_t dt + \sigma_1 X_t dB_t^1, \ \ t \in [0, +\infty) \quad X(0) = x \ , \qquad (2.1)$$
$$x \in (0, +\infty)$$

with positive μ, a, z_{up} and σ_1, where a is the permanent price impact parameter and B_t^1 is a standard Brownian motion.

Here, Z_t denote the liquidity of the stock in the market(i.e. market's capacity to absorb the stock), which is assumed to satisfy the following equation

$$dZ_t = \sigma_2 Z_t dB_t^2,\ \ t \in [0, +\infty) \quad Z(0) = z\ \ ,\ z \in (0, +\infty), \tag{2.2}$$

where ${B_t}^2$ is a standard Brownian motion. σ_2 is positive constants and volatility. Therefore, $Z_t \wedge z_{up}$ can describe stochastic upper bound of selling rate in the real market, which is stochastic and finite, where z_{up} is positive constant. In fact, the upper bound of selling rate is not only determined by seller, but also influenced by liquidity of the market (i.e., Z_t), so it is stochastic. Moreover, because the upper bound of selling rate is constrained by the actual trade conditions (i.e., z_{up}) (i.e. positions, quantity of buyers, firm trade policy and so on), it is finite.

$l_t \in [0, 1]$ is the control variable, which denote the ratio of the stochastic upper bound. Therefore, $l_t(Z_t \wedge z_{up})$ means the selling rate at time t with strategy l_t

The number of shares of a stock yet hold is denoted by $Y(t)$ satisfying the following first-order differential equation

$$dY_t = -l_t(Z_t \wedge z_{up})dt,\ t \in [0, +\infty),\ Y(0) = y,\ y \in [0, +\infty). \tag{2.3}$$

Thus, the variables at any time t consists of the (X_t, Y_t, Z_t) , and the space is $\bar{Q} = R^+ \times \bar{R}^+ \times R^+$, where $\bar{R}^+ = [0, +\infty)$.

Definition 2.1. We say that a control l_t is admissible with respect to the initial values $(x, y, z) \in \bar{Q}$, if (i) l_t is an $\mathscr{F}_t = \sigma\{X_s : s < t\}$ adapted; (ii) $l_t \in [0, 1]$ for all $t \geq 0$. (iii) For arbitrary l_t, $(X(t), Y(t), Z(t)) \in \bar{Q}$, when $t \geq 0$. We use $\mathfrak{A} = \mathfrak{A}(x, y, z)$ to denote the set of all admissible controls.

The optimal control model can be expressed as the following form that we need to maximize

$$J(x, y, z; l) = E[\int_0^\tau e^{-\rho t} l_t(Z_t \wedge z_{up}) h(l_t, Z_t) X_t dt], \tag{2.4}$$

where ρ is the discount rate, and

$$\tau = \inf_{0 \leq t \leq \infty} \{s\,|Y_t \leq 0\}.$$

$h(l_t, Z_t)$ is the temporary price impact function, which is assumed to be

$$h(l_t, Z_t) = (1 + g') \exp[g|l_t(Z_t \wedge z_{up})|^\beta],$$

where g' is the bid-ask spread parameter, g is the temporary price impact factor, β is the price impact exponent [10, 14].

In the following, assuming $\beta = 1$, $gl_t(Z_t \wedge z_{up}) \ll 1$ and $g' = 0$. Then, the temporary impact function is approximately

$$h(l_t, Z_t) \approx 1 - gl_t(Z_t \wedge z_{up}).$$

Define the value function as follows:

$$W(x,y,z) = \sup_{l\in\mathfrak{A}(x,y,z)} J(x,y,z,l),\ (x,y,z)\in\bar{Q} \tag{2.5}$$
$$= \sup_{l\in\mathfrak{A}(x,y,z)} E[\int_0^{\tau} e^{-\rho t} l_t(Z_t\wedge z_{up})(1-gl_t(Z_t\wedge z_{up}))X_t dt].$$

Lemma 2.1. *Assume $\rho > \mu$, the following assertion hold.*

(a) *For each (x,z), $W(x,y,z)$ is nondecreasing in y.*

(b) *$W(x,y,z)$ is continuous in $(x,y,z)\in\bar{Q}$.*

Proof. (a) Note that for $0\le y_1\le y_2$, $\mathfrak{A}(x,y_1,z)\subset\mathfrak{A}(x,y_2,z)$. Given $l\in\mathfrak{A}(x,y_1,z)$, then $l\in\mathfrak{A}(x,y_2,z)$. We have

$$\begin{aligned} W(x,y_2,z) &\ge J(x,y_2,z;l)\\ &\ge J(x,y_1,z;l)\end{aligned}$$

for any (x,z). This implies $W(x,y_2,z)\ge W(x,y_1,z)$.

Next we process to prove (b). Note that

$$X_{i,t} = x_i\exp(\int_0^t \mu-\frac{1}{2}\sigma_1^2 - al(Z_s\wedge z_{up})ds+\sigma_1 B_t^1),\quad (i=1,2).$$

Note also that for any $x_1>0, x_2>0$, $\mathfrak{A}(x_1,y,z)=\mathfrak{A}(x_2,y,z)$, where $(y,z)\in\bar{R}^+\times R^+$.

For any $l\in\mathfrak{A}(x_1,y,z)=\mathfrak{A}(x_2,y,z)$, we have

$$\begin{aligned} &|J(x_1,y,z;l)-J(x_2,y,z;l)| \qquad (2.6)\\ \le& E[\int_0^\infty e^{-\rho t} l(Z_t\wedge z_{up})\left|X_{1,t}-X_{2,t}\right| dt]\\ \le& \left|x_1-x_2\right| z_{up} E[\int_0^\infty e^{-(\rho-\mu)t-\frac{1}{2}\sigma_1^2 t+\sigma_1 B_t^1}dt]\\ \le& \frac{\left|x_1-x_2\right| z_{up}}{\rho-\mu}.\end{aligned}$$

The inequality (2.6) implies the continuity of $W(x,y,z)$ with respect to x.

It remains to show that $W(x,y,z)$ is continuous in y. In view of (a), it suffices to show that for $0\le y_1\le y_2<+\infty$, $W(x,y_1,z)\le W(x,y_2,z)$. Let $l_{y,2}\in\mathfrak{A}(x,y_2,z)$, such that

$$y_2=\int_0^\infty l_{y,2}(s)(Z_s\wedge z_{up})ds,\quad \textit{and}\quad W(x,y_2,z)\le J(x,y_2,z;l_{y,2})+\left|y_2-y_1\right|.$$

Let

$$\tau_1=\inf\{t>0:\int_0^t l_{y,2}(s)(Z_s\wedge z_{up})ds=y_2-y_1\}.$$

It follows that $\int_0^{\tau_1} l_{y,2}(s)(Z_s\wedge z_{up})ds=y_2-y_1$. Define

$$l_{y,1}=\begin{cases} 0, & \text{if } 0\le t\le\tau_1,\\ l_{y,2}, & \text{if } \qquad t>\tau_1.\end{cases}$$

It is obviously, $l_{y,1} \in \mathfrak{A}(x, y_1, z)$. We have

$$
\begin{aligned}
&|J(x, y_2, z; l_{y,2}) - J(x, y_1, z; l_{y,1})| \\
\leq& E\bigg[\int_0^\infty e^{-\rho t}(Z_t \wedge z_{up})\Big|l_{y,2}(1 - gl_{y,2}(Z_t \wedge z_{up}))X_t(l_{y,2}) \\
&- l_{y,1}(1 - gl_{y,1}(Z_t \wedge z_{up}))X_t(l_{y,1})\Big|dt\bigg] \\
\leq& E\bigg[\int_0^\infty e^{-\rho t}(Z_t \wedge z_{up})\Big|l_{y,2}(1 - gl_{y,2}(Z_t \wedge z_{up}))X_t(l_{y,2}) \\
&- l_{y,2}(1 - gl_{y,2}(Z_t \wedge z_{up}))X_t(l_{y,1})\Big|dt\bigg] \\
&+ E\bigg[\int_0^\infty e^{-\rho t}(Z_t \wedge z_{up})\Big|l_{y,2}(1 - gl_{y,2}(Z_t \wedge z_{up}))X_t(l_{y,1}) \\
&- l_{y,1}(1 - gl_{y,1}(Z_t \wedge z_{up}))X_t(l_{y,1})\Big|)dt\bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
&E\bigg[\int_0^\infty e^{-\rho t}(Z_t \wedge z_{up})\Big|l_{y,2}(1 - gl_{y,2}(Z_t \wedge z_{up}))X_t(l_{y,2}) \\
&- l_{y,2}(1 - gl_{y,2}(Z_t \wedge z_{up}))X_t(l_{y,1})\Big|dt\bigg] \\
\leq& z_{up}xE[\int_0^\infty e^{-(\rho-\mu+\frac{1}{2}\sigma_1^2)t+\sigma_1 B_t^1}\left|e^{-\int_0^t al_{y,2}(Z_s \wedge z_{up})ds} - e^{-\int_0^t al_{y,1}(Z_s \wedge z_{up})ds}\right|dt] \\
\leq& z_{up}xaE[\int_0^\infty e^{-(\rho-\mu+\frac{1}{2}\sigma_1^2)t+\sigma_1 B_t^1}\left|\int_0^t l_{y,2}(Z_s \wedge z_{up})ds - \int_0^t l_{y,1}(Z_s \wedge z_{up})ds\right|dt] \\
=& z_{up}xaE[\int_0^\infty e^{-(\rho-\mu+\frac{1}{2}\sigma_1^2)t+\sigma_1 B_t^1}\int_0^\infty |l_{y,2} - l_{y,1}|\,(Z_s \wedge z_{up})dsdt] \\
=& z_{up}xaE[\int_0^\infty e^{-(\rho-\mu+\frac{1}{2}\sigma_1^2)t+\sigma_1 B_t^1}\int_0^{\tau_1} l_{y,2}(Z_s \wedge z_{up})dsdt] \\
\leq& \frac{z_{up}xa\,|y_2 - y_1|}{\rho - \mu},
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&E\bigg[\int_0^\infty e^{-\rho t}(Z_t \wedge z_{up})\Big|l_{y,2}(1 - gl_{y,2}(Z_t \wedge z_{up}))X_t(l_{y,1}) \\
&- l_{y,1}(1 - gl_{y,1}(Z_t \wedge z_{up}))X_t(l_{y,1})\Big|)dt\bigg] \\
\leq& E[\int_0^{\tau_1} e^{-\rho t}(Z_t \wedge z_{up})X_t(0)l_{y,2}dt] \\
=& E[\int_0^{\tau_1} e^{-\rho t}X_t(0)d\int_0^t l_{y,2}(Z_S \wedge z_{up})ds] \\
=& E[e^{-\rho t}X_t(0)\int_0^t l_{y,2}(Z_S \wedge z_{up})ds\Big|_0^{\tau_1}] \\
&+ E[\int_0^{\tau_1}\int_0^t l_{y,2}(Z_s \wedge z_{up})ds(\rho - \mu)e^{-\rho t}X_t(0)dt]
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
=&E[e^{-\rho\tau_1}X_{\tau_1}(0)\int_0^{\tau_1} l_{y,2}(Z_s\wedge z_{up})ds]\\
&+E[\int_0^\infty\int_0^{\tau_1} l_{y,2}(Z_s\wedge z_{up})ds(\rho-\mu)e^{-\rho t}X_t(0)dt]\\
\le&x\,|y_2-y_1|+\frac{|y_2-y_1|\,(\rho-\mu)x}{(\rho-\mu)}=2x\,|y_2-y_1|\,.
\end{aligned}
$$

Therefore, using (2.7), (2.8), we obtain $|J(x,y_2,z;l_{y,2})-J(x,y_1,z;l_{y,1})|\le(\frac{z_{up}xa}{\rho-\mu}+2x)\,|y_2-y_1|$. It follows that

$$
\begin{aligned}
W(x,y_1,z)&\ge J(x,y_1,z;l_{y,1})\\
&\ge J(x,y_2,z;l_{y,2})-(\frac{z_{up}xa}{\rho-\mu}+2x)\,|y_2-y_1|\\
&\ge W(x,y_2,z)-(\frac{z_{up}xa}{\rho-\mu}+2x+1)\,|y_2-y_1|\,.
\end{aligned}
$$

Combining with (a), we have the continuity of $W(x,y,z)$ with respect to y.

It remains to show that $W(x,y,z)$ is continuous in z. For $0\le z_1\le z_2<\infty$, $\mathfrak{A}(x,y,z_2)\subset\mathfrak{A}(x,y,z_1)$. We have

$$
\begin{aligned}
&\left|\sup_{\mathfrak{A}(x,y,z_2)}J(x,y,z_2)-\sup_{\mathfrak{A}(x,y,z_1)}J(x,y,z_1)\right|\\
\le&\left|\sup_{\mathfrak{A}(x,y,z_2)}J(x,y,z_2)-\sup_{\mathfrak{A}(x,y,z_2)}J(x,y,z_1)\right|\\
&+\left|\sup_{\mathfrak{A}(x,y,z_2)}J(x,y,z_1)-\sup_{\mathfrak{A}(x,y,z_1)}J(x,y,z_1)\right|.
\end{aligned}
\tag{2.9}
$$

Above all, we estimate the first part of the right side in (2.9). In fact

$$
\begin{aligned}
&|J(x,y,z_2)-J(x,y,z_1)|\\
\le&E\left[\int_0^\infty e^{-\rho t}l\Big|X_t(z_2)(1-gl(Z_{2,t}\wedge z_{up}))(Z_{2,t}\wedge z_{up})\right.\\
&\left.-X_t(z_1)(1-gl(Z_{1,t}\wedge z_{up}))(Z_{1,t}\wedge z_{up})\Big|dt\right]\\
\le&E\left[\int_0^\infty e^{-\rho t}l\Big|X_t(z_2)(1-gl(Z_{2,t}\wedge z_{up}))(Z_{2,t}\wedge z_{up})\right.\\
&\left.-X_t(z_2)(1-gl(Z_{1,t}\wedge z_{up}))(Z_{1,t}\wedge z_{up})\Big|dt\right]\\
&+E\left[\int_0^\infty e^{-\rho t}l\Big|X_t(z_2)(1-gl(Z_{1,t}\wedge z_{up}))(Z_{1,t}\wedge z_{up})\right.\\
&\left.-X_t(z_1)(1-gl(Z_{1,t}\wedge z_{up}))(Z_{1,t}\wedge z_{up})\Big|dt\right],
\end{aligned}
\tag{2.10}
$$

where $Z_{i,t}=z_ie^{\frac12\sigma_2^2t+\sigma_2B_t^2},\ X_t(z_i)=xe^{\int_0^t\mu-\frac12\sigma_1^2-al(Z_{i,s}\wedge z_{up})ds+\sigma_1B_t^1}$.

In (2.10), we have following estimate:

$$\begin{aligned}
&E\Big[\int_0^\infty e^{-\rho t} l\Big|X_t(z_2)(1-gl(Z_{2,t}\wedge z_{up}))(Z_{2,t}\wedge z_{up})\\
&-X_t(z_2)(1-gl(Z_{1,t}\wedge z_{up}))(Z_{1,t}\wedge z_{up})\Big|dt\Big]\\
\le&(1+2gz_{up})E[\int_0^\infty e^{-\rho t}X_t(z_2)\left|Z_{2,t}-Z_{1,t}\right|dt]\\
\le&(1+2gz_{up})x\left|z_2-z_1\right|E[\int_0^\infty e^{-(\rho-\mu+\frac{1}{2}\sigma_1^2)t+\sigma_1 B_t^1-\frac{1}{2}\sigma_2^2 t+\sigma_2 B_t^2}dt]\\
\le&\frac{x(1+2gz_{up})\left|z_2-z_1\right|}{\rho-\mu}
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&E\Big[\int_0^\infty e^{-\rho t} l\Big|X_t(z_2)(1-gl(Z_{1,t}\wedge z_{up}))(Z_{1,t}\wedge z_{up})\\
&-X_t(z_1)(1-gl(Z_{1,t}\wedge z_{up}))(Z_{1,t}\wedge z_{up})\Big|dt\Big]\\
\le&z_{up}E[\int_0^\infty e^{-\rho t}l\left|X_t(z_2)-X_t(z_1)\right|dt]\\
\le&z_{up}xE[\int_0^\infty e^{-(\rho-\mu+\frac{1}{2}\sigma_1^2)t+\sigma_1 B_t^1}\left|e^{-\int_0^t al(Z_{1,s}\wedge z_{up})ds}-e^{-\int_0^t al(Z_{2,s}\wedge z_{up})ds}\right|dt]\\
\le&z_{up}xE[\int_0^\infty e^{-(\rho-\mu+\frac{1}{2}\sigma_1^2)t+\sigma_1 B_t^1}\int_0^t al\left|(Z_{1,s}\wedge z_{up})-(Z_{2,s}\wedge z_{up})\right|dsdt]\\
\le&z_{up}gxE[\int_0^\infty e^{-(\rho-\mu+\frac{1}{2}\sigma_1^2)t+\sigma_1 B_t^1}\int_0^t \left|Z_{1,s}-Z_{2,s}\right|dsdt]\\
\le&z_{up}ax\left|z_2-z_1\right|E[\int_0^\infty e^{-(\rho-\mu+\frac{1}{2}\sigma_1^2)t+\sigma_1 B_t^1}\int_0^t e^{\frac{1}{2}\sigma_2^2 t+\sigma_2 B_t^2}dsdt]\\
\le&z_{up}ax\left|z_2-z_1\right|\int_0^\infty e^{-(\rho-\mu)t}tdt=\frac{z_{up}ax\left|z_2-z_1\right|}{(\rho-\mu)^2}.
\end{aligned} \tag{2.12}$$

Combining with (2.10)-(2.12), we obtain

$$\begin{aligned}
&\left|\sup_{\mathfrak{A}(x,y,z_2)}J(x,y,z_2)-\sup_{\mathfrak{A}(x,y,z_2)}J(x,y,z_1)\right|\\
\le&\sup_{\mathfrak{A}(x,y,z_2)}\left|J(x,y,z_2)-J(x,y,z_1)\right|\\
\le&\frac{z_{up}gx\left|z_2-z_1\right|}{(\rho-\mu)^2}+\frac{x\left|z_2-z_1\right|}{\rho-\mu}=(\frac{z_{up}a}{(\rho-\mu)^2}+\frac{1+2gz_{up}}{\rho-\mu})x\left|z_2-z_1\right|.
\end{aligned} \tag{2.13}$$

Next, let us estimate the second part of (2.9), i.e.,

$$\left|\sup_{\mathfrak{A}(x,y,z_2)}J(x,y,z_1)-\sup_{\mathfrak{A}(x,y,z_1)}J(x,y,z_1)\right|.$$

Because $\mathfrak{A}(x,y,z_2)\subset\mathfrak{A}(x,y,z_1)$, it is obviously that

$$\sup_{\mathfrak{A}(x,y,z_2)} J(x,y,z_1) \le \sup_{\mathfrak{A}(x,y,z_1)} J(x,y,z_1). \tag{2.14}$$

Let $l_{z,1}\in\mathfrak{A}(x,y,z_1)$, such that $y=\int_0^\infty l_{z,1}(s)(Z_{1,s}\wedge z_{up})ds$, and $W(x,y,z_1)\le J(x,y,z_1;l_{z,1})+|z_2-z_1|$. Define $\tau_2=inf\{t>0:\int_0^t l_{z,1}(s)(Z_{2,s}\wedge z_{up})ds\ge y\}$, which implies $\int_0^{\tau_2} l_{z,1}(s)(Z_{2,s}\wedge z_{up})ds=y$.

Let $l_{z,2}=\begin{cases} l_{z,1}, & \text{if } 0\le t\le\tau_2,\\ 0, & \text{if } t>\tau_2,\end{cases}$ and such that $l_{z,2}\in\mathfrak{A}(x,y,z_2)$.

In fact, using integration by parts, we have

$$\begin{aligned}
&|J(x,y,z_1;l_{z,2})-J(x,y,z_1;l_{z,1})|\\
\le& E[\int_{\tau_2}^\infty e^{-\rho t}l_{z,1}X_t(l_{z,1},z_1)(1-gl_{z,1}(Z_{1,t}\wedge z_{up}))(Z_{1,t}\wedge z_{up})dt]\\
\le& E[\int_{\tau_2}^\infty e^{-\rho t}l_{z,1}X_t(0,z_1)(Z_{1,t}\wedge z_{up})dt]\\
\le& E[\int_{\tau_2}^\infty e^{-\rho t}X_t(0,z_1)d\int_{\tau_2}^t l_{z,1}(Z_{1,s}\wedge z_{up})ds]\\
=&E[e^{-\rho t}X_t(0,z_1)\int_{\tau_2}^t l_{z,1}(Z_{1,s}\wedge z_{up})ds\Big|_{\tau_2}^\infty]\\
&-E[\int_{\tau_2}^\infty\int_{\tau_2}^t l_{z,1}(Z_{1,s}\wedge z_{up})dsd(e^{-\rho t}X_t(0,z_1))]\\
\le& E[\int_{\tau_2}^\infty\int_{\tau_2}^\infty l_{z,1}(Z_{1,s}\wedge z_{up})ds(\rho-\mu)e^{-\rho t}X_t(0,z_1)dt)]\\
=&E[\int_{\tau_2}^\infty\int_0^{\tau_2} l_{z,1}\left|(Z_{2,s}\wedge z_{up})-(Z_{1,s}\wedge z_{up})\right|ds(\rho-\mu)e^{-\rho t}X_t(0,z_1)dt)]\\
\le&(\rho-\mu)E[\int_{\tau_2}^\infty e^{-\rho t}X_t(0,z_1)\int_0^{\tau_2} l_{z,1}\left|Z_{2,s}-Z_{1,s}\right|dsdt)]\\
\le&(\rho-\mu)\left|z_2-z_1\right|xE[\int_{\tau_2}^\infty e^{-(\rho-\mu+\frac12\sigma_1^2)t+\sigma_1W_t^1}\int_0^{\tau_2} e^{-\frac12\sigma_2^2t+\sigma_2W_t^2}dsdt)]\\
\le&(\rho-\mu)\left|z_2-z_1\right|xE[\int_{\tau_2}^\infty e^{-(\frac{\rho-\mu}{2}+\frac12\sigma_1^2)t+\sigma_1W_t^1}\int_0^{\tau_2} e^{-(\frac{\rho-\mu}{2}+\frac12\sigma_2^2)t+\sigma_2W_t^2}dsdt)]\\
\le&(\rho-\mu)\left|z_2-z_1\right|xE[\int_0^\infty e^{-(\frac{\rho-\mu}{2}+\frac12\sigma_1^2)t+\sigma_1W_t^1}\int_0^\infty e^{-(\frac{\rho-\mu}{2}+\frac12\sigma_2^2)t+\sigma_2W_t^2}dsdt)]\\
\le&\frac{4x\left|z_2-z_1\right|}{\rho-\mu},
\end{aligned}$$

which leads to

$$\begin{aligned}
\sup_{\mathfrak{A}(x,y,z_2)} J(x,y,z_1;l_{z,2}) &\ge J(x,y,z_1;l_{z,2})\\
&\ge J(x,y,z_1;l_{z,1})-\frac{4x\left|z_2-z_1\right|}{\rho-\mu}\\
&\ge \sup_{\mathfrak{A}(x,y,z_1)} J(x,y,z_1;l_{z,1})-(\frac{4x}{\rho-\mu}+1)\left|z_2-z_1\right|.
\end{aligned}$$

Combining (2.14), we obtain

$$\begin{aligned}
&\left|\sup_{\mathfrak{A}(x,y,z_2)} J(x,y,z_2) - \sup_{\mathfrak{A}(x,y,z_1)} J(x,y,z_1)\right| \qquad (2.15)\\
\leq&\left(\frac{z_{up}gx}{(\rho-\mu)^2} + \frac{x+2gz_{up}x}{\rho-\mu}\right)|z_2-z_1| + \left(\frac{4x}{\rho-\mu}+1\right)|z_2-z_1|\\
=&\left(\frac{z_{up}gx}{(\rho-\mu)^2} + \frac{5x+2gz_{up}x}{\rho-\mu}+1\right)|z_2-z_1|.
\end{aligned}$$

□

According to general hypothesis and dynamical programming principle(DPP), we derive formally the HJB equation for the value function (2.5) as following

$$\begin{aligned}
&\mu x\frac{\partial W}{\partial x} + \frac{1}{2}{\sigma_1}^2x^2\frac{\partial^2 W}{\partial x^2} + \frac{1}{2}{\sigma_2}^2z^2\frac{\partial^2 W}{\partial z^2} - \rho W\\
&+ (z\wedge z_{up})\sup_{l\in\mathfrak{A}}[l(1-gl(z\wedge z_{up}))x - alx\frac{\partial W}{\partial x} - l\frac{\partial W}{\partial y}] = 0,\ (x,y,z)\in Q, \qquad (2.16)\\
&W(x,0,z)=0,\ (x,z)\in R^{+2}, \qquad (2.17)
\end{aligned}$$

where $Q = R^{+3}$. In the next, we will show that the value function defined in (2.5) is the viscosity solution of the HJB equation (2.16)-(2.17).

Above all, we introduce the following notion of a viscosity solution ([18]).

Definition 2.2. Let $W : \bar{Q} \to R$ be locally bounded

$$\begin{aligned}
&F(x,y,z,W,\frac{\partial W}{\partial x},\frac{\partial W}{\partial y},\frac{\partial W}{\partial z},\frac{\partial^2 W}{\partial x^2},\frac{\partial^2 W}{\partial z^2})\\
=&\mu x\frac{\partial W}{\partial x} + \frac{1}{2}{\sigma_1}^2x^2\frac{\partial^2 W}{\partial x^2} + \frac{1}{2}{\sigma_2}^2z^2\frac{\partial^2 W}{\partial z^2} - \rho W\\
&(z\wedge z_{up})\sup_{l\in\mathfrak{A}}[l(1-gl(z\wedge z_{up}))x - alx\frac{\partial W}{\partial x} - l\frac{\partial W}{\partial y}].
\end{aligned}$$

(1) $W \in USC(\bar{Q})$ is a viscosity subsolution of (2.16)-(2.17) if

$$F\left(\bar{x},\bar{y},\bar{z},W,\frac{\partial \phi}{\partial x}(\bar{x},\bar{y},\bar{z}),\frac{\partial \phi}{\partial y}(\bar{x},\bar{y},\bar{z}),\frac{\partial \phi}{\partial z}(\bar{x},\bar{y},\bar{z}),\frac{\partial^2 \phi}{\partial x^2}(\bar{x},\bar{y},\bar{z}),\frac{\partial^2 \phi}{\partial z^2}(\bar{x},\bar{y},\bar{z})\right) \geq 0$$

for all $(\bar{x},\bar{y},\bar{z}) \in Q$ and for all $\phi \in C^2(Q)$ such that $(\bar{x},\bar{y},\bar{z})$ is a maximum point of $W-\phi$.

(2) $W \in LSC(\bar{Q})$ is a viscosity supersolution of (2.16)-(2.17) if

$$F\left(\bar{x},\bar{y},\bar{z},W,\frac{\partial \phi}{\partial x}(\bar{x},\bar{y},\bar{z}),\frac{\partial \phi}{\partial y}(\bar{x},\bar{y},\bar{z}),\frac{\partial \phi}{\partial z}(\bar{x},\bar{y},\bar{z}),\frac{\partial^2 \phi}{\partial x^2}(\bar{x},\bar{y},\bar{z}),\frac{\partial^2 \phi}{\partial z^2}(\bar{x},\bar{y},\bar{z})\right) \leq 0$$

for all $(\bar{x},\bar{y},\bar{z}) \in Q$ and for all $\phi \in C^2(Q)$ such that $(\bar{x},\bar{y},\bar{z})$ is a minimum point of $W-\phi$.

(3) We say that W is a viscosity solution of (2.16)-(2.17) if it is both a subsolution and supersolution of (2.16)-(2.17) .

Lemma 2.2. *The value function (2.5) satisfies a linear growth in $\bar{Q}$. This means that there exists a finite positive constant C_p such that for any (x, y, z) in $\bar{Q}$,*

$$|W(x,y,z)| \leq C_p(1+|x|+|y|+|z|). \tag{2.18}$$

Proof. Indeed, let the optimal control l^* replace l.

$$|W(x,y,z)| \leq E[\int_0^\infty e^{-\rho s} l^*(z \wedge z_{up})(1 - gl^*(z \wedge z_{up}))X_s ds] \leq \frac{z_{up}}{\rho - \mu}x. \tag{2.19}$$

□

Theorem 2.1. *The value function (2.5) is a viscosity solution of the HJB equation (2.16)-(2.17) .*

Proof. The theorem can be proved easily from DPP. □

For discussing the uniqueness of the viscosity solution, we let

$$\xi = \ln x,\ \ \eta = \ln z,\ \ u(\xi, y, \eta) = W(x, y, z).$$

Then, HJB equation (2.16)-(2.17) is written as following.

$$\begin{aligned}&(\mu x - \frac{1}{2}{\sigma_1}^2)\frac{\partial u}{\partial \xi} + \frac{1}{2}{\sigma_1}^2\frac{\partial^2 u}{\partial \xi^2} - \frac{1}{2}{\sigma_2}^2\frac{\partial u}{\partial \eta} + \frac{1}{2}{\sigma_2}^2\frac{\partial^2 u}{\partial \eta^2} - \rho u \\ &+ (e^\eta \wedge z_{up}) \sup_{l \in \mathscr{A}(\xi,y,\eta)}[l(1 - gl(e^\eta \wedge z_{up}))e^\xi - al\frac{\partial u}{\partial \xi} - l\frac{\partial u}{\partial y}] = 0,\ \ (\xi, y, \eta) \in Q^1\end{aligned} \tag{2.20}$$

$$u(\xi, 0, \eta) = 0,\ \ (\xi, \eta) \in R^2, \tag{2.21}$$

where $Q^1 = R \times R^+ \times R$. and $\bar{Q}^1 = R \times \bar{R}^+ \times R$.

Lemma 2.3. *For all $(\xi, y, \eta) \in \bar{Q}^1$, $u(\xi, y, \eta)$ is the viscosity solution of (2.20) (2.21), and the value function $u(\xi, y, \eta)$ satisfies*

$$|u(\xi,y,\eta)| \leq C_p(e^\xi + |y| + |\eta| + 1),$$

where C_p is a positive constant.

3. Comparison Principle and Uniqueness

Theorems 2.1 show the value function (2.5) is a viscosity solution of (2.16)-(2.17), which provides the existence result for the problem . In this section, we develop the comparison principle and uniqueness of the viscosity solutions. Firstly, we shall need a formulation of viscosity sub-and super-solutions based on the so called semijets.

Definition 3.1. (1) The set of second-order superjet of a USC function U at point $\bar{v} \in Q^1$ is

$$\begin{aligned}P^{2,+}U(\bar{v}) =&\{(\bar{q}, \bar{M}) \in R^3 \times S^3\,|U(v) \leq U(\bar{v}) + \bar{q}(v - \bar{v}) \\ &+ \frac{1}{2}\bar{M}(v-\bar{v})(v-\bar{v}) + o(|v-\bar{v}|^2), Q^1 \ni v \to \bar{v}\},\end{aligned}$$

where S^3 is 3×3 symmetric matrix, and $v = (\xi, y, \eta)$.

(2) The set of second-order subjet of a LSC function V at point at point $\bar{v} \in Q^1$ is

$$P^{2,-}V(\bar{v}) = \{ (\bar{q}, \bar{M}) \in R^3 \times S^3 \,|V(v,t) \geq V\ (\bar{v}) + \bar{q}(v-\bar{v}) + \frac{1}{2}\bar{M}(v-\bar{v})(v-\bar{v}) + o(|v-\bar{v}|^2), Q^1 \ni v \to \bar{v}\}.$$

The closure $\bar{P}^{2,+}U(\bar{v})(\bar{P}^{2,-}V(\bar{v}))$ is defined as the set of $(\bar{q}, \bar{M}) \in R^3 \times S^3$ for which there exists a sequence $(u^k, q^k, M^k) \in Q^1 \times R^3 \times S^3$ such that $(v^k, U(v^k), q^k, M^k) \to (\bar{v}, U(\bar{v}), \bar{q}, \bar{M})$,$((v^k, V(v^k), q^k, M^k) \to (\bar{v}, V(\bar{v}), \bar{q}, \bar{M}))$ as $k \to \infty$, and $(q^k, M^k) \in P^{2,+}U(\bar{v}) \left(P^{2,-}V(\bar{v})\right)$ for all k.

Secondly, we need the maximum principle for semicontinuous functions for elliptic equation.

Lemma 3.1 (Ishii's lemma, [6]). *Let $U_i \in USC\,(O_i)$, for $i = 1, \cdots, k$, where Q_i is a locally compact subset of R^N. Let ϕ be defined on an open neighborhood of $O_1 \times \cdots \times O_k$, and such that $\phi(x_1, \cdots, x_k)$ is twice continuously differentiable in $(x_1, \cdots, x_k) \in O_1 \times \cdots \times O_k$, suppose*

$$(\widehat{x}_1, \cdots, \widehat{x}_k) \in O_1 \times \cdots \times O_k,$$
$$F(x_1, \cdots, x_k) = U_1(x_1) + \cdots + U_k(x_k) - \phi(x_1, \cdots, x_k) \leq F(\widehat{x}_1, \cdots, \widehat{x}_k),$$

for $(x_1, \cdots, x_k) \in O_1 \times \cdots \times O_k$. Then for each $\eta > 0$, there are $M_i \in S^{N_i}$ such that

$$\left\{ \begin{array}{l} (D_{x_i}\phi(\widehat{x}_1, \cdots, \widehat{x}_k), M_i) \in \bar{P}^{2,+}U_i(\bar{x}_i) \;\; for \;\; i = 1, \cdots, k, \\ -\left(\frac{1}{\eta} + \|A\|\right) I \leq \begin{pmatrix} M_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_K \end{pmatrix} \leq A + \eta A^2, \end{array} \right. \tag{3.1}$$

where $A = (D_x^2\phi)(\widehat{x}_1, \cdots, \widehat{x}_k)$. The norm of symmetric matrix A is defined as $\|A\| = \sup\{|\langle A\xi, \xi\rangle| : |\xi| \leq 1\}$.

Finally, the next theorem is a comparison principle for the viscosity sub- and super-solutions that satisfies a exponential growth (lemma 2.3).

Theorem 3.1. *Suppose $\rho > \hat{\rho}$, and $U \in USC\,(\bar{Q}^1)$ is a viscosity subsolution of (2.20),$V \in LSC\,(\bar{Q}^1)$ is a viscosity supersolution of (2.20), satisfying $U(\xi, y, \eta) \leq V(\xi, y, \eta)$, for $(\xi, y, \eta) \in \partial Q^1$ then we have $U(\xi, y, \eta) \leq V(\xi, y, \eta)$, for $(\xi, y, z) \in \bar{Q}^1$, where $\partial Q^1 = \bar{Q}^1 \setminus Q^1$, $\hat{\rho} = \max\{2\mu + \sigma_1^2, \mu + az_{up} + \frac{1}{2}\sigma_1^2, \frac{1}{2}\sigma_2^2\}$.*

Proof. Before the proof, we define some notations and operators as following for simplifying the theorem proving procedure.

a. (ξ, y, η) is denoted by P, and (ξ^*, y^*, η^*) is denoted by P^*. P_1, P_2, $P_{1,\alpha}$, $P_{2,\alpha}$, $P_{1,\varepsilon}$, $P_{2,\varepsilon}$, P_ε are similarly defined.

b. $|P_1P_2| = \sqrt{(\xi_1 - \xi_2)^2 + (y_1 - y_2)^2 + (\eta_1 - \eta_2)^2}$, and $\|P\|^n = e^{n\xi} + |\xi|^n + |y|^n + |\eta|^n$, $n \in N$.

We argue by contradiction, which yields:

$$U(P^*) \geq V(P^*) + 2\delta, \text{ for some } P^* \in \bar{Q}^1 \tag{3.2}$$

with $\delta > 0$.

We now use the dedoubling variable technique by considering for any $1 > \varepsilon > 0$ *and* α, the functions are defined as follows:

$$\Phi\left(P_1, P_2\right) = U(P_1) - V(P_2) - \phi(P_1, P_2),$$
$$\phi(P_1, P_2) = \frac{\alpha}{2}|P_1 P_2|^2 + \frac{\varepsilon}{2}(\|P_1\|^2 + \|P_2\|^2).$$

Let $F_\alpha = \sup\limits_{R\times\bar{R}^+\times R\times R\times\bar{R}^+\times R} \Phi\left(P_1, P_2\right)$. Because of growth condition and upper semicontinuity of $\Phi\left(P_1, P_2\right)$, we get that $F_\alpha < +\infty$, and there exists $(P_{1,\alpha}, P_{2,\alpha})$, such that

$$F_\alpha = \Phi\left(P_{1,\alpha}, P_{2,\alpha}\right). \tag{3.3}$$

Hence,

$$F_\alpha \geq U(P^*) - V(P^*) - \varepsilon\|P^*\|^2 \geq \delta \tag{3.4}$$

for each ε small enough. This implies that $U(P_{1,\alpha}) \geq V(P_{2,\alpha}) + \delta$, for each $\alpha > 0$ and ε small enough.

Because of $\Phi\left(0, 0\right) \leq \Phi\left(P_{1,\alpha}, P_{2,\alpha}\right)$ and linear growth condition,

$$U(0) - V(0) - \varepsilon \leq U(P_{1,\alpha}) - V(P_{2,\alpha}) - \frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 - \frac{\varepsilon}{2}\left(\|P_{1,\alpha}\|^2 + \|P_{2,\alpha}\|^2\right).$$

Then we have

$$\begin{aligned}\frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 + \frac{\varepsilon}{2}\left(\|P_{1,\alpha}\|^2 + \|P_{2,\alpha}\|^2\right) - \varepsilon &\leq U(P_{1,\alpha}) - V(P_{2,\alpha}) - U(0) + V(0)\\ &\leq C_p\left(\|P_{1,\alpha}\| + \|P_{2,\alpha}\| + 1\right).\end{aligned}$$

It means that there are positive constants $C_{1,\varepsilon}$, that $\|P_{1,\alpha}\|^2 + \|P_{2,\alpha}\|^2 \leq C_{1,\varepsilon}$, where $C_{1,\varepsilon}$ is determined by ε.

From these, the conclusion can be obtained that there exists a subsequence, still denoted by $(P_{1,\alpha}, P_{2,\alpha})$, which converges some $(P_{1,\varepsilon}, P_{2,\varepsilon}) \in R\times\bar{R}^+\times R\times R\times\bar{R}^+\times R$, when $\alpha \to \infty$ (for each fixed ε).

Furthermore, we can get $\frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 \leq C_{2,\varepsilon}$ is a positive constant for fixed ε. So $\xi_{1,\alpha} - \xi_{2,\alpha} \to 0$, $y_{1,\alpha} - y_{2,\alpha} \to 0$, $\eta_{1,\alpha} - \eta_{2,\alpha} \to 0$, as $\alpha \to \infty$, and $P_{1,\varepsilon} = P_{2,\varepsilon}$. Let $P_\varepsilon = P_{1,\varepsilon} = P_{2,\varepsilon}$ in the following.

Because of $\Phi\left(P_{1,\varepsilon}, P_{2,\varepsilon}\right) \leq \Phi\left(P_{1,\alpha}, P_{2,\alpha}\right)$, we get

$$\begin{aligned}\frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 \leq& U(P_{1,\alpha}) - U(P_{1,\varepsilon}) - V(P_{2,\alpha}) + V(P_{2,\varepsilon})\\ &+ \frac{\varepsilon}{2}\left(\|P_{1,\varepsilon}\|^2 + \|P_{2,\varepsilon}\|^2\right) - \frac{\varepsilon}{2}\left(\|P_{1,\alpha}\|^2 + \|P_{2,\alpha}\|^2\right).\end{aligned}$$

The semicontinuity of U and V help us yield $\frac{\alpha}{2}|P_{1,\alpha}P_{2,\alpha}|^2 \to 0$ as $\alpha \to \infty$ (for each fixed ε).

Because of $\Phi\left(P^*, P^*\right) \leq \Phi\left(P_{1,\alpha}, P_{2,\alpha}\right)$, we have

$$\begin{aligned}\frac{\varepsilon}{2}\left(\|P_{1,\alpha}\|^2 + \|P_{2,\alpha}\|^2\right) &\leq U(P_{1,\alpha}) - V(P_{2,\alpha}) - U(P^*) + V(P^*) + \varepsilon\|P^*\|^2\\ &\leq U(P_{1,\alpha}) - V(P_{2,\alpha}) + \varepsilon\|P^*\|^2.\end{aligned}$$

If $P_\varepsilon \in \partial Q^1$, it is obvious that $U(P_{1,\alpha}) - V(P_{2,\alpha}) \geq F_\alpha \geq U(P^*) - V(P^*) - \varepsilon\|P^*\|^2$. For the upper semicontinuity of $U - V$ and $U(P_\varepsilon) \leq V(P_\varepsilon)$, as $\alpha \to \infty$ and $\varepsilon \to 0$, we get $U(P^*) \leq V(P^*)$, this contradices (3.2). Therefore $P_\varepsilon \in Q^1$.

In what follows, we assume $P_\varepsilon \in R \times R^+ \times R$, so that $P_{1,\alpha} \in R \times R^+ \times R$, $P_{2,\alpha} \in R \times R^+ \times R$ for any α large enough. An application of Ishii lemma yields

$$(q_{1,\alpha}, M_\alpha) \in \bar{P}^{2,+}U(P_{1,\alpha}), \quad (q_{2,\alpha}, M_\alpha) \in \bar{P}^{2,-}V(P_{2,\alpha}),$$

where $q_{1,\alpha} = D_{P_{1,\alpha}}\phi, q_{2,\alpha} = -D_{P_{2,\alpha}}\phi$.

Since U and V are viscosity subsolution and viscosity supersolution of (2.20), there exists a constant l^*, such that

$$\begin{aligned} &\begin{pmatrix} (\mu - al^*(e^{\eta_{1,\alpha}} \wedge z_{up})) - \frac{1}{2}{\sigma_1}^2 \\ -l^*(e^{\eta_{1,\alpha}} \wedge z_{up}) \\ -\frac{1}{2}{\sigma_2}^2 \end{pmatrix}^T \begin{pmatrix} \alpha\,(\xi_{1,\alpha} - \xi_{2,\alpha}) + \varepsilon(e^{2\xi_{1,\alpha}} + \xi_{1,\alpha}) \\ \alpha\,(y_{1,\alpha} - y_{2,\alpha}) + \varepsilon y_{1,\alpha} \\ \alpha\,(\eta_{1,\alpha} - \eta_{2,\alpha}) + \varepsilon \eta_{1,\alpha} \end{pmatrix} \\ &+ \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}^T M_\alpha\right) \\ &+ l^*(e^{\eta_{1,\alpha}} \wedge z_{up})(1 - gl^*(e^{\eta_{1,\alpha}} \wedge z_{up}))e^{\xi_{1,\alpha}} - \rho U \geq 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\begin{pmatrix} (\mu - al^*(e^{\eta_{2,\alpha}} \wedge z_{up})) - \frac{1}{2}{\sigma_1}^2 \\ -l^*(e^{\eta_{2,\alpha}} \wedge z_{up}) \\ -\frac{1}{2}{\sigma_2}^2 \end{pmatrix}^T \begin{pmatrix} \alpha\,(\xi_{1,\alpha} - \xi_{2,\alpha}) - \varepsilon(e^{2\xi_{2,\alpha}} + \xi_{2,\alpha}) \\ \alpha\,(y_{1,\alpha} - y_{2,\alpha}) - \varepsilon y_{2,\alpha} \\ \alpha\,(\eta_{1,\alpha} - \eta_{2,\alpha}) - \varepsilon \eta_{2,\alpha} \end{pmatrix} \\ &+ \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}^T M_\alpha\right) \\ &+ l^*(e^{\eta_{2,\alpha}} \wedge z_{up})(1 - gl^*(e^{\eta_{2,\alpha}} \wedge z_{up}))e^{\xi_{2,\alpha}} - \rho V \leq 0, \end{aligned} \tag{3.6}$$

We subtract (3.6) from (3.5), and get

$$\begin{aligned} (3.6) - (3.5) =& I + II - \rho\,(U - V) + l^*(e^{\eta_{1,\alpha}} \wedge z_{up})(1 - gl^*(e^{\eta_{1,\alpha}} \wedge z_{up}))e^{\xi_{1,\alpha}} \\ & - l^*(e^{\eta_{2,\alpha}} \wedge z_{up})(1 - gl^*(e^{\eta_{2,\alpha}} \wedge z_{up}))e^{\xi_{2,\alpha}} \geq 0. \end{aligned} \tag{3.7}$$

Notice,

$$\begin{aligned} I =& \begin{pmatrix} \mu x_{1,\alpha} - al^*(z_{1,\alpha} \wedge z_{up}) \\ -l^*(z_{1,\alpha} \wedge z_{up}) \\ 0 \end{pmatrix}^T \begin{pmatrix} \alpha\,(x_{1,\alpha} - x_{2,\alpha}) + \varepsilon x_{1,\alpha} \\ \alpha\,(y_{1,\alpha} - y_{2,\alpha}) + \varepsilon y_{1,\alpha} \\ \alpha\,(z_{1,\alpha} - z_{2,\alpha}) + \varepsilon z_{1,\alpha} \end{pmatrix} \\ &- \begin{pmatrix} \mu x_{2,\alpha} - al^*(z_{2,\alpha} \wedge z_{up}) \\ -l^*(z_{2,\alpha} \wedge z_{up}) \\ 0 \end{pmatrix}^T \begin{pmatrix} \alpha\,(x_{1,\alpha} - x_{2,\alpha}) - \varepsilon x_{1,\alpha} \\ \alpha\,(y_{1,\alpha} - y_{2,\alpha}) - \varepsilon y_{1,\alpha} \\ \alpha\,(z_{1,\alpha} - z_{2,\alpha}) - \varepsilon z_{1,\alpha} \end{pmatrix}, \end{aligned} \tag{3.8}$$

$$II = \frac{1}{2}tr\left(\begin{pmatrix} CC^T & CD^T \\ DC^T & DD^T \end{pmatrix}\begin{pmatrix} M_\alpha & 0 \\ 0 & -N_\alpha \end{pmatrix}\right), \tag{3.9}$$

where $C = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix} = D.$

From ishii lemma, we conclude

$$\begin{pmatrix} M_\alpha & 0 \\ 0 & -N_\alpha \end{pmatrix} \leq A + \frac{1}{\alpha}A^2,$$

where $A = \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \varepsilon \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$, and $R_i = \begin{pmatrix} 2e^{2\xi_{i,\alpha}}+1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $i = 1, 2$.

Then, we have

$$\begin{aligned} A^2 =& 2\alpha^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \varepsilon\alpha \begin{pmatrix} R_1 & -R_2 \\ -R_1 & R_2 \end{pmatrix} + \varepsilon\alpha \begin{pmatrix} R_1 & -R_1 \\ -R_2 & R_2 \end{pmatrix} \\ &+ \varepsilon^2 \begin{pmatrix} {R_1}^2 & 0 \\ 0 & {R_2}^2 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} II =& \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} CC^T & CD^T \\ DC^T & DD^T \end{pmatrix} \begin{pmatrix} M_\alpha & 0 \\ 0 & -N_\alpha \end{pmatrix} \right) \\ \leq& \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} CC^T & CD^T \\ DC^T & DD^T \end{pmatrix} (A + \frac{1}{\alpha}A^2) \right). \end{aligned} \tag{3.10}$$

Let $\alpha \to +\infty$. The limit of (3.8) and (3.9) can be obtained as the following

$$\lim_{\alpha\to\infty} I = 2\varepsilon(\mu - al^*(e^{\eta_\varepsilon} \wedge z_{up}) - \frac{1}{2}{\sigma_1}^2)(e^{2\xi_\varepsilon} + \xi_\varepsilon) - 2l^*\varepsilon y_\varepsilon(e^{\eta_\varepsilon} \wedge z_{up}) - \varepsilon{\sigma_2}^2\eta_\varepsilon, \tag{3.11a}$$

$$\lim_{\alpha\to\infty} II \leq 2\varepsilon{\sigma_1}^2 e^{2\xi_\varepsilon} + \varepsilon{\sigma_1}^2 + \varepsilon{\sigma_2}^2. \tag{3.11b}$$

Finally, from (3.8)-(3.11), the (3.8) can be written as

$$\begin{aligned} 0 \leq& 2\varepsilon(\mu - al^*(e^{\eta_\varepsilon} \wedge z_{up}) - \frac{1}{2}{\sigma_1}^2)(e^{2\xi_\varepsilon} + \xi_\varepsilon) - \varepsilon{\sigma_2}^2\eta_\varepsilon + 2\varepsilon{\sigma_1}^2 e^{2\xi_\varepsilon} + \varepsilon{\sigma_1}^2 + \varepsilon{\sigma_2}^2 \\ &- \rho(U(P_\varepsilon) - V(P_\varepsilon)) \\ \leq& \varepsilon(2\mu - 2al^*(e^{\eta_\varepsilon} \wedge z_{up}) + \sigma_1^2)e^{2\xi_\varepsilon} + \varepsilon(2\mu - 2al^*(e^{\eta_\varepsilon} \wedge z_{up}) - \sigma_1^2)\xi_\varepsilon \\ &- \varepsilon\sigma_2^2\eta_\varepsilon + \varepsilon(\sigma_1^2 + \sigma_2^2) - \rho(U(P_\varepsilon) - V(P_\varepsilon)) \\ \leq& \varepsilon(\mu + az_{up} + \frac{3}{2}\sigma_1^2 + \frac{3}{2}\sigma_2^2) + \varepsilon(2\mu + \sigma_1^2)e^{2\xi_\varepsilon} + \varepsilon(\mu + az_{up} + \frac{1}{2}\sigma_1^2)\xi_\varepsilon^2 + \varepsilon\frac{1}{2}\sigma_2^2\eta_\varepsilon^2 \\ &- \rho(U(P_\varepsilon) - V(P_\varepsilon)) \\ \leq& \varepsilon(\mu + az_{up} + \frac{3}{2}\sigma_1^2 + \frac{3}{2}\sigma_2^2) + \hat{\rho}\varepsilon\|P_\varepsilon\|^2 - \rho(U(P_\varepsilon) - V(P_\varepsilon)) \\ \leq& \varepsilon(\mu + az_{up} + \frac{3}{2}\sigma_1^2 + \frac{3}{2}\sigma_2^2) + \hat{\rho}\varepsilon\|P^*\|^2 - (\rho - \hat{\rho})(U(P_\varepsilon) - V(P_\varepsilon)) \\ \leq& \varepsilon(\mu + az_{up} + \frac{3}{2}\sigma_1^2 + \frac{3}{2}\sigma_2^2) + \hat{\rho}\varepsilon\|P^*\|^2 - (\rho - \hat{\rho})\delta. \end{aligned}$$

If ε is chosen sufficiently small, this is the contradiction and the proof is completed. □

Corollary 3.1. *The value function $u(\xi, y, \eta)$ is the unique viscosity solution of (2.20)-(2.21).*

4. Numerical Simulation

In this section, we discuss the numerical illustrative examples, and give the concrete analysis. Firstly, we will show the optimal strategy, before the finite difference calculation.

4.1. Optimal Strategy

From the HJB equation (2.16), the strategy should have the following form:

$$l^* = \begin{cases} 0, & \frac{x-ax\frac{\partial W}{\partial x}-\frac{\partial W}{\partial y}}{2g(z\wedge z_{up})x} \le 0, \\ \frac{x-ax\frac{\partial W}{\partial x}-\frac{\partial W}{\partial y}}{2g(z\wedge z_{up})x}, & 0 < \frac{x-ax\frac{\partial W}{\partial x}-\frac{\partial W}{\partial y}}{2g(z\wedge z_{up})x} < 1, \\ 1, & \frac{x-ax\frac{\partial W}{\partial x}-\frac{\partial W}{\partial y}}{2g(z\wedge z_{up})x} \ge 1. \end{cases}$$

The above strategy can be obtained easily, after simplifying the HJB equation, because it is the stochastic quadratic control problem. However, the close-form solution of (2.16) always can't be obtained, we seek numerical solutions for the problem under consideration.

4.2. Boundary Conditions

In equation (2.16), there are three state variables. If we hope to solve this equation by implicit scheme or C-N scheme, we need five boundary conditions. Except (2.17), there are another two boundary conditions.
(1) $x \to +\infty$, $\frac{\partial^2 W}{\partial X^2} \approx 0$ [9].
(2) $z \to +\infty$, $(z \wedge z_{up}) \to z_{up}$. It means the state variable z can be ignored, we have

$$W(x,y,z) \to \sup_{l\in\mathfrak{A}(x,y)} E[\int_0^\tau e^{-\rho t} l z_{up}(1-glz_{up})X_t dt].$$

4.3. Finite Difference Scheme

The finite difference method, which was discussed by [9], is used in the following simulation to get a positive coefficient discretization of equation (2.16). The main steps can be summarized as following.

(Step 1)Consider the step size Δx, Δy and Δz for x, y, z. Define the infinite lattice

$$\Sigma^h_{Inf} = \{(x,y,z) = (i\Delta x, j\Delta y, k\Delta z) | i,j,k = 0,1,2,\cdots\}.$$

For actual numerical calculations, Σ^h_{Inf} must be replaced by some finite lattice Σ^h_F as the subset of Σ^h_{Inf}. Denote

$$\Sigma^h_F = \left\{(x,y,z) \in \Sigma^h_{Inf} | 0 \le x \le M, 0 \le y \le N, 0 \le z \le L\right\},$$

where $M > 0, N > 0, L > 0$ is large enough and $M = m\Delta x, N = n\Delta y, L = l\Delta z$.

Let $i = 0,\cdots,m$; $j = 0,\cdots,n$; $k = 0,\cdots,l$. Then we have $x_i = i\Delta x$, $y_k = k\Delta y$, $z_j = j\Delta z$. The equation (2.6) can be discretized using forward, backward or central differencing to give

$$
\begin{aligned}
&(\mu - al(z_j \wedge z_{up}))i(W^{k+1}_{i+1,j} - W^{k+1}_{i,j})\mathbf{1}_{(\mu > al(z_j \wedge z_{up}))} \\
&- (\mu - al(z_j \wedge z_{up}))i(W^{k+1}_{i,j} - W^{k+1}_{i-1,j})\mathbf{1}_{(\mu \leq al(z_j \wedge z_{up}))} \\
&+ \frac{1}{2}\sigma_1^2 i^2 (W^{k+1}_{i+1,j} - 2W^{k+1}_{i,j} + W^{k+1}_{i-1,j}) \\
&+ \frac{1}{2}\sigma_2^2 j^2 (W^{k+1}_{i,j+1} - 2W^{k+1}_{i,j} + W^{k+1}_{i,j-1}) - \rho W^{k+1}_{i,j} + - \frac{l(z_j \wedge z_{up})}{\Delta y}(W^{k+1}_{i,j} - W^{k}_{i,j}) \\
&+ l(z_j \wedge z_{up})(1 - gl(z_j \wedge z_{up}))x_i = 0,
\end{aligned}
\tag{4.1}
$$

where l can be assigned any initial value.

(Step 2)Calculate the matrix $(W^{k+1})^1$ from W^k. If $\left|(W^{k+1})^1 - (W^{k+1})^0\right| < er$, the iteration stop, where er is the allowed error, and $(W^{k+1})^0 = W^k$. Otherwise, computer l^1 by $(W^{k+1})^1$ and replace the l in (4.1).

(Step 3)Repeat the step 3, until $\left|(W^{k+1})^n - (W^{k+1})^{n-1}\right| < er$.

4.4. Computational Examples

The parameters for this section are shown in Table 1.

Table 1. Parameters for this numerical simulation

Parameter	Value	Parameter	Value
μ	0.02	a	0.01
σ_1	0.2	g	0.02
σ_2	0.1	M	30
ρ	0.5	N	20
z_{up}	2	L	8

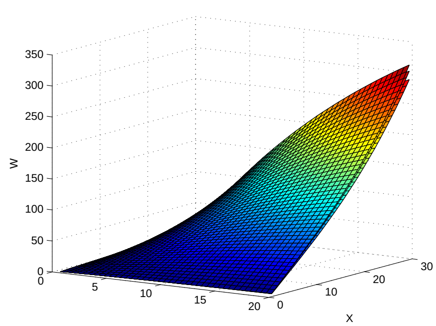

Figure 1. The parametric is shown in Table 1. The Values of cash flow, as defined in (2.5), and X, Y, as defined in (2.1, 2.3) are reported at z=1.6, z=3.2 for various position level. The case that Z wasn't considered in the model was also shown in the Fig. 1.

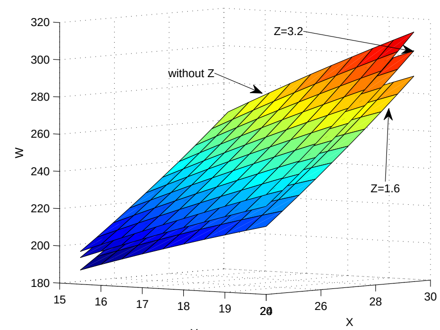

Figure 2. is the part of Fig. 1, which was magnified. This means that the income perhaps is overestimated, if the stochastic upper bound of selling rate is ignored.

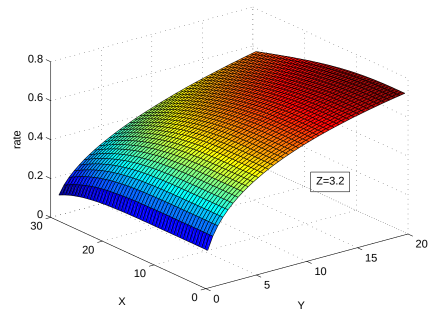

Figure 3. On one hand, the bigger Y correspond the bigger l, because $\rho > \mu$, i.e., more positions means more discounted cost. On the other hand, l declined as X increases. In fact, the seller will be unwise , if he choose bigger l, when the X is bigger at the same time. The reason is that the cost on account of the temporary and permanent price impact may be more than the discounted cost.

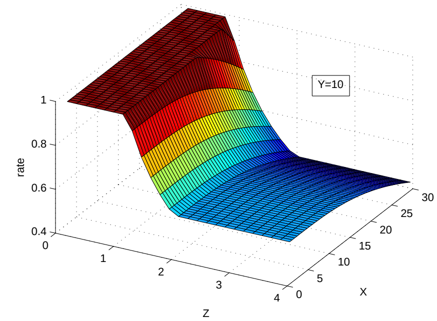

Figure 4. Be similar to Figure.3, we show the strategy with X and Z.

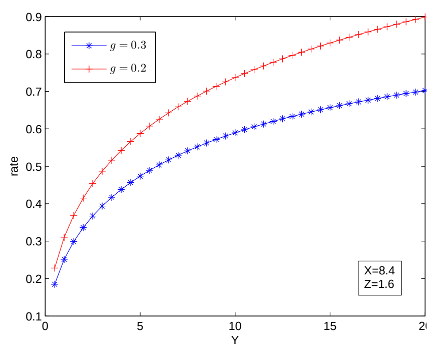

Figure 5. g is the temporary price impact factor. The smaller g means smaller losses in each deal. Therefore the seller can liquidate more shares with the smaller temporary price impact.

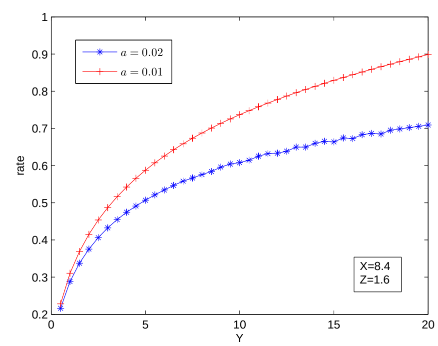

Figure 6. a is the permanent price impact factor. The bigger a means lower price of the stock will be. Consequently, the seller will liquidate less shares with bigger permanent price impact.

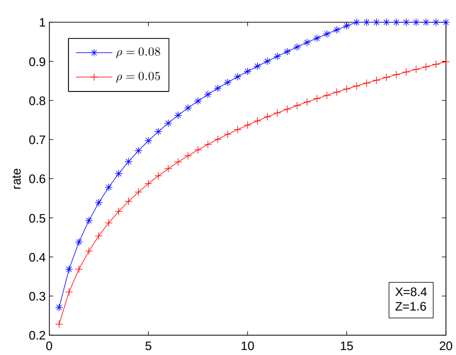

Figure 7. ρ means the market rate of return. The bigger ρ implies there are better investment chance in the market, so the seller will liquidate shares more quickly.

5. Conclusion

In this paper, we developed an optimal selling model with the temporary and permanent price impact. We show that the value function is continuous and the viscosity solutions of the HJB equation and prove the comparison principle of the viscosity solutions. Finally, we give numerical illustrative examples and numerical solution of optimal selling strategies with finite difference method.

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