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EXACT SOLITARY WAVE AND PERIODIC WAVE SOLUTIONS OF THE KAUP-KUPERSCHMIDT EQUATION*

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Abstract In this paper we investigate the exact traveling wave solutions of the fifth-order Kaup-Kuperschmidt equation. The bifurcation and exact solutions of a general first-order nonlinear equation are investigated firstly. With the help of Maple and by using the bifurcation and exact solutions of two derived subequations, we obtain two families of solitary wave solutions and two families of periodic wave solutions of the KK equation. The relationship of the two subequations and the two known first integrals are analyzed.

Keywords Traveling wave solutions, the Kaup-Kuperschmidt equation, dynamical system method, Jacobi elliptic function, subequation.

MSC(2010) 35Q51, 35Q53, 34C23, 34C37.

1. Introduction

In this paper we study the traveling wave solutions of the fifth-order Kaup-Kupersch -midt(KK) equation [10, 11, 14–16]

$$u_{xxxxx} - 15uu_{xxx} - \frac{75}{2}u_x u_{xx} + 45u^2 u_x + u_t = 0. \tag{1.1}$$

Actually, this equation is related to a general fifth-order nonlinear wave equation [8, 10]

$$u_{xxxxx} + \alpha uu_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_t = 0. \tag{1.2}$$

Obviously, Equation (1.2) reduces to the KK equation (1.1) by choosing (α, β, γ) to be $(-15, -75/2, 45)$. Actually, equation (1.2) involves many important nonlinear equations that have been studied in the literature. For example, letting $\alpha = 10, \beta = 20$ and $\gamma = 30$, equation (1.2) becomes the Lax equation. It is the Sawada-Kortera (SK) equation [5, 18, 19] if $\alpha = \beta = \gamma = 5$ in equation (1.2). When $\alpha = \beta = 30$ and $\gamma = 180$, it is the Caudrey-Dodd-Gibbon (CDG) equation [4]. If (α, β, γ) is chosen to be $(3, 6, 2)$, equation (1.1) is known as the Ito equation [9]. It has been shown in

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the literature that the properties of equation (1.2) drastically changes as α, β and γ take different values. For instance, the Lax equation and the SK equation are completely integrable and possess N-soliton solution. However, the Ito equation is not completely integrable but has a finite numbers of conservation laws [9].

In [14], Li studied the traveling wave solutions of the KK equation (1.1) by using the bifurcation approach. Under the traveling wave coordinates, the KK equation (1.1) reduces to a nonlinear ordinary differential equation (ODE) of the independent variable $\xi = x - ct$, where c is a constant representing the wave speed. Integrating the derived ODE once with respect to ξ gives

$$\frac{d^4y}{d\xi^4} - 15y\frac{d^2y}{d\xi^2} - \frac{45}{4}\left(\frac{dy}{d\xi}\right)^2 + 15y^3 - cy + g = 0, \tag{1.3}$$

where g is a constant of integration. Clearly, $y(\xi) = y(x - ct)$ is a traveling wave solution of equation (1.1) if and only if $y(\xi)$ satisfies (1.3) with the wave speed c and any constant g. Equation (1.3) corresponds to the F-III form of the higher-order Painleve equation in Cosgrove's paper [2]. By using the method of dynamical systems and the two first integrals and some solution formulas of equation (1.3) presented in [2], Li [14] obtained some explicit solitary wave and periodic wave solutions of the KK equation (1.1).

Let $x_1 = y, x_2 = x_1' = y', x_3 = x_2' = y'', x_4 = x_3' = y'''$, then equation (1.3) can be rewritten as the following four dimensional system:

$$x_1' = x_2,\ x_2' = x_3,\ x_3' = x_4,\ x_4' = 15x_1x_3 + \frac{45}{4}x_2^2 - 15x_1^3 + cx_1 - g. \tag{1.4}$$

Generally speaking, we have to study the dynamical behavior of the fourth-order ODE (1.4) in the 4-dimensional phase space, for which it is usually very difficult to obtain the orbits. However, for the case when the first integral of this equation is found, this problem possibly reduces to the one in lower dimensional space which might be easier to handle. In [2], Cosgrove obtained the two first integrals of equation (1.4) which can be rewritten as:

$$\begin{aligned}\Phi_1(x_1, x_2, x_3, x_4) =& (x_4 - 12x_1x_2)^2 - 3x_1x_3^2 + (\frac{3}{2}x_2^2 + 30x_1^3)x_3 - 93x_1^2x_2^2 - 72x_1^5 \\ &- c(2x_1x_3 - x_2^2 - 8x_1^3) + 2g(x_3 - 6x_1^2) + \frac{4}{3}cg = K_1\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}&\Phi_2(x_1, x_2, x_3, x_4) \\ =& x_1x_4^2 - (x_3 + 18x_1^2)x_2x_4 + (\frac{1}{3}x_3 - 6x_1^2)x_3^2 + (\frac{27}{2}x_1x_2^2 + 30x_1^4)x_3 \\ &- c(\frac{2}{3}x_2x_4 - \frac{1}{3}x_3^3 + 2x_1^2x_3 - \frac{15}{2}x_1x_2^2 - 2x_1^4) + \frac{1}{3}c^2x_1^2 - \frac{2}{3}cgx_1 \\ &- \frac{4}{81}c^3 - g^2 - \frac{9}{16}x_2^4 + \frac{135}{2}x_1^3x_2^2 - 45x_1^6 + g(2x_1x_3 - \frac{3}{2}x_2^2 - 6x_1^3) \\ =& K_2.\end{aligned} \tag{1.6}$$

According to the Theorem on first integrals, we know system (1.4) can be reduced to a two dimensional space provided that x_3 and x_4 can be figured out from (1.5) and (1.6) for any arbitrary constants K_1 and K_2. Unfortunately, it is intractable to solve equations (1.5) and (1.6) for x_3 and x_4 because they are higher-order polynomial equations of x_3 and x_4. However, for some special values of K_1 and K_2, (1.5) and (1.6) might be applied to derive the solutions of the higher-order equations (1.4). For instance, Li [14] obtained the solitary wave solutions approaching $\frac{\sqrt{15c}}{15}$ at infinity and some periodic wave solutions of the KK equation corresponding to system (1.4) with $g = 0$.

In this paper, we derived a lower-order subequation of the 4th-order nonlinear equation (1.3). In order to obtain the solitary wave solutions and periodic solutions, we study the bifurcation and exact solutions of a general first-order nonlinear equation in Section 2. In Section 3, by applying the bifurcation and the formulas obtained in Section 2 and with the help of computer algebra and symbolic computation, we derive two families of periodic wave solutions and two families of solitary wave solutions approaching $\frac{\sqrt{c}}{3}$ and $\frac{2\sqrt{11c}}{33}$ respectively as time ξ goes into infinity. In addition, we exploit the relationships of the subequations and the two known first integral (1.5) and (1.6). Finally we present some conclusions and discussions in Section 4.

2. Bifurcation and exact solutions of a subequation of equation (1.3)

2.1. Lower-order subequations of equation (1.3)

Notice that equation (1.3) consists of $d^4y/d\xi^4$, $d^2y/d\xi^2$, $(dy/d\xi)^2$ and polynomial of y. If y satisfies equation

$$\left(\frac{dy}{d\xi}\right)^2 = P_m(y), \tag{2.1}$$

where $P_m(y)$ is a polynomial function of degree m, then it solves

$$\frac{d^2y}{d\xi^2} = \frac{1}{2}P'_m(y) \tag{2.2}$$

and

$$\frac{d^4y}{d\xi^4} = \frac{1}{2}P'''_m(y)P_m(y) + \frac{1}{4}P''_m(y)P'_m(y). \tag{2.3}$$

Obviously, the right-hand sides of equations (2.2) and (2.3) are both polynomials of y. This observation motivates us to try to find some suitable polynomial P_m in y such that y solves the higher-order equation (1.3) if it satisfies equation (2.1).

Suppose that the function $y = y(\xi)$ satisfies equation (1.3), then $\frac{d^4y}{d\xi^4}$, $y\frac{d^2y}{d\xi^2}$ and $(\frac{dy}{d\xi})^2$ are all polynomials in y and their degrees are $2m-3$, m and m, respectively. Accordingly, we are trying to find the possible polynomial $P_3(y) = a_3y^3 + a_2y^2 + a_1y + a_0$ such that y solves equation (1.3) if it satisfies equation

$$\left(\frac{dy}{d\xi}\right)^2 = a_3y^3 + a_2y^2 + a_1y + a_0. \tag{2.4}$$

Inserting (2.4) with the higher-order derivatives $\frac{d^4y}{d\xi^4}$, $\frac{d^2y}{d\xi^2}$ obtained by (2.4) into (1.3) and comparing the coefficients of like powers of y, we have

$$\begin{aligned} &y^3: 15+\frac{15}{2}a_3^2-\frac{135}{4}a_3=0,\\ &y^2: -\frac{105}{4}a_2+\frac{15}{2}a_2a_3=0,\\ &y^1: \frac{9}{2}a_1a_3+a_2^2-\frac{75}{4}a_1-c=0,\\ &y^0: \frac{1}{2}a_1a_2-\frac{45}{4}a_0+3a_0a_3+g=0. \end{aligned} \tag{2.5}$$

Solving system (2.5) for a_is, $i = 0, 1, 2, 3$, gives two solutions of (2.5) $(a_0, a_1, a_2, a_3) = (-\frac{4}{3}g, -\frac{4}{3}c, 0, 4)$ and $(a_0, a_1, a_2, a_3) = (\frac{4}{39}g, -\frac{2}{33}c, 0, \frac{1}{2})$. Hence, we obtained two subequations of equation (1.3) and we state this result in the following theorem.

Theorem 2.1. *The function $y = y(\xi)$ solves the fourth-order differential equation (1.3) if it satisfies*

$$\left(\frac{dy}{d\xi}\right)^2 = 4y^3-\frac{4c}{3}y-\frac{4}{3}g \tag{2.6}$$

or

$$\left(\frac{dy}{d\xi}\right)^2 = \frac{1}{2}y^3-\frac{2c}{33}y+\frac{4}{39}g. \tag{2.7}$$

According to the conclusion of Theorem 2.1, we know that the fourth-order ODE (1.3) can be reduced to the first-order nonlinear ODEs (2.6) and (2.7). In what follows, we study the bifurcation and exact solutions of these two subequations.

2.2. Bifurcation and explicit solutions of equation (2.4)

Let $y' = v$, then the first-order ODE (2.1) is equivalent to the following planar dynamical system:

$$\begin{cases} y' = v,\\ v' = \frac{3}{2}a_3y^2+a_2y+\frac{a_1}{2}, \end{cases} \tag{2.8}$$

which is a Hamiltonian system with Hamiltonian

$$H(y,v)=\frac{1}{2}[v^2-(a_3y^3+a_2y^2+a_1y)]=h. \tag{2.9}$$

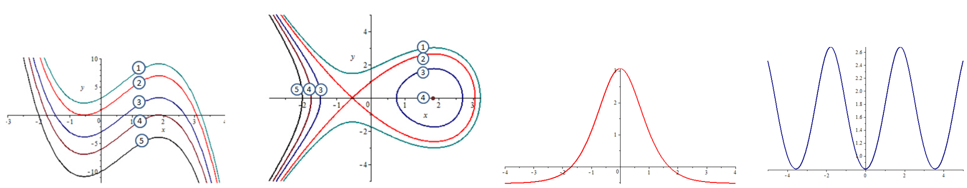
Obviously, $H(y, v) = h = a_0/2$ corresponds to equation (2.1). Consequently, we know the dynamical behaviors of ODE (2.1) from the orbits of the system (2.8) corresponding to $H(y, v) = a_0/2$. Clearly, the phase orbits defined by the vector field of system (2.8) determine all solutions of equation (2.1). Especially, the bounded solutions of equation (2.1) correspond to the bounded phase orbits of system (2.8). By investigating the bifurcation of the planar dynamical system (2.8), one gets different kinds of solutions of equation (2.1) under various coefficients conditions. Therefore, the dynamical behaviors and exact solutions of equation (2.1) are obtained.

To obtain the dynamical behaviors of system (2.8), we firstly study the equilibrium points of this system. Obviously, the roots of $P_3'(y) = 0$ are the abscissas of the equilibrium points of system (2.8). Suppose that y_e is a root of $P_3'(y) = 0$, that is to say, $(y_e, 0)$ is an equilibrium point of system (2.8). By the theory of planar dynamical systems [1, 7], we need to study the matrix

$$Df(y_0, 0) = \begin{pmatrix} 0 & 1 \\ P_3''(y_e) & 0 \end{pmatrix}$$

of the linearized system of (2.8) at $(y_e, 0)$. The equilibrium point $(y_e, 0)$ is a center, having a punctured neighborhood in which any solution is a periodic orbit, if $det(Df(y_e, 0)) = -P_3''(y_e) > 0$. It is a saddle point if $det(Df(y_e, 0)) < 0$. However, it is a cusp point if $det(Df(y_e, 0)) = 0$. To obtain the phase portraits, besides the equilibriums, we also need to investigate the boundary curves of the centers and the orbits connecting the saddle points or cusp points which are determined by the Hamiltonian $H(y, v) = h$.

Let $\Delta = a_2^2 - 3a_1a_3$. Obviously, system (2.8) has no equilibrium point or just a cusp when $\Delta \leq 0$, and thus system (2.8) has no nontrivial bounded solutions. However, system (2.8) has two equilibrium points when $\Delta > 0$. Let $y_e^{\pm} = \frac{-a_2 \pm \sqrt{\Delta}}{3a_3}$ and $H(y_e^{\pm}, 0) = h_{\pm} = \frac{2\Delta(-a_2 \pm \sqrt{\Delta}) + 3a_1a_2a_3}{54a_3^2}$, then $(y_e^+, 0)$ is a saddle, $(y_e^-, 0)$ is a center and $h_+ > h_-$. When $h_+ > h > h_-$, $H(y, v) = h$ defines a family of periodic orbits around the center $(u_e^-, 0)$ enclosed by the boundary curves defined by $H(y, v) = h_+$. However, $H(y, v) = h_+$ defines a homoclinic orbit passing through the saddle $(y_e^+, 0)$ (See Fig. 1(b) or Fig. 2(b)) which corresponds to a solution $y(\xi)$ of (2.1) approaching y_e^+ as ξ goes to infinity (See Fig. 1(c) or Fig. 2(c)). The relationship between the polynomial $P_3(y)$, the phase portraits of system (2.8) and the solutions of ODE (2.1) is shown in Figure 1 and Figure 2. In fact, the exact formulas of this family of solutions can be derived by integrating along the orbits [3]. Consequently, we have the following theorem.



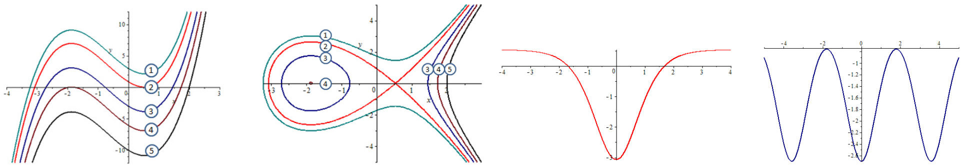
(a) (b) (c) (d)

Figure 1. For $a_3 < 0$ and $a_2^2 - a_1a_3 > 0$, (a) portrait of $P_3(y) = a_3y^3 + a_2y^2 + a_1y + a_0$ with different values of a_0,(b) phase portrait of system (2.8), corresponding to Figure 1(a), (c) portrait of solitary wave, corresponding to the homoclinic orbit ② in Figure 1(b), (d) portrait of periodic solution corresponding to the periodic orbit ③ on the right side of Figure 1(b).

Theorem 2.2. *Let* $h_{\pm} = \frac{2\Delta(-a_2 \pm \sqrt{\Delta}) + 3a_1a_2a_3}{54a_3^2}$ *and* $y_e^{\pm} = \frac{-a_2 \pm \sqrt{\Delta}}{3a_3}$, *where* $\Delta = a_2^2 - 3a_1a_3 > 0$, *then the following conclusions hold:*

(1) For $a_0 = 2h_+$, *equation (2.4) has a bounded solution approaching* y_e^+ *as* ξ *goes to infinity (See Figure 1(c) or Figure 2(c)) given by*

$$y = \frac{-a_2 + \sqrt{\Delta}}{3a_3} - \frac{\sqrt{\Delta}}{a_3} sech^2 \left[\frac{1}{2} \Delta^{\frac{1}{4}} (\xi - \xi_0) \right]. \tag{2.10}$$



(a) (b) (c) (d)

Figure 2. For $a_3 > 0$ and $a_2^2 - a_1 a_3 > 0$, (a) portrait of $P_3(y) = a_3 y^3 + a_2 y^2 + a_1 y + a_0$ with different values of a_0,(b) phase portrait of system (2.8), corresponding to Figure 2(a), (c) portrait of solitary wave, corresponding to the homoclinic orbit ② in Figure 2(b), (d) portrait of periodic solution corresponding to the periodic orbit ③ on the right side of Figure 2(b).

A constant solution

$$y = \frac{-a_2 + \sqrt{\Delta}}{3a_3} \tag{2.11}$$

and an unbounded solution

$$y = \frac{-a_2 + \sqrt{\Delta}}{3a_3} + \frac{\sqrt{\Delta}}{a_3} csch^2 \left[\frac{1}{2}\Delta^{\frac{1}{4}}(\xi - \xi_0)\right], \tag{2.12}$$

where ξ_0 is an arbitrary constant.

(2) Suppose that $a_0 \in (2h_-, 2h_+)$.

Case a) If $a_3 > 0$, then for any $y_3 \in \left(\frac{-a_2 - 2\sqrt{\Delta}}{3a_3}, \frac{-a_2 - \sqrt{\Delta}}{3a_3}\right)$,

$$y = y_3 - \frac{1}{2}\left(3y_3 + \frac{a_2}{a_3} + \sqrt{\Delta_+}\right) sn^2\left(\Omega_+(\xi - \xi_0), k_+\right), \tag{2.13}$$

is a family of smooth periodic solutions of equation (2.1) (See Figure 2(d)). Here $k_+ = \frac{2\sqrt{3y_3^2 + 2\frac{a_2}{a_3}y_3 + \frac{a_1}{a_3}}}{-3y_3 - \frac{a_2}{a_3} + \sqrt{\Delta_+}}$, $\Omega_+ = \frac{\sqrt{2}}{4}\sqrt{-3a_3 y_3 - a_2 + a_3\sqrt{\Delta_+}}$, $\Delta_+ = -3y_3^2 - 2\frac{a_2}{a_3}y_3 + \left(\frac{a_2}{a_3}\right)^2 - 4\frac{a_1}{a_3}$ and sn represents the Jacobi elliptic sine-amplitude function [6].

Case b) If $a_3 < 0$, then for any $y_1 \in \left(\frac{-a_2 - \sqrt{\Delta}}{3a_3}, \frac{-a_2 - 2\sqrt{\Delta}}{3a_3}\right)$,

$$y = y_1 - \frac{1}{2}\left(3y_1 + \frac{a_2}{a_3} - \sqrt{\Delta_-}\right) sn^2\left(\Omega_-(\xi - \xi_0), k_-\right), \tag{2.14}$$

is a family of smooth periodic solutions of equation (2.1) (See Figure 1(d)). Here $k_- = \frac{2\sqrt{3y_1^2 + 2\frac{a_2}{a_3}y_1 + \frac{a_1}{a_3}}}{3y_1 + \frac{a_2}{a_3} + \sqrt{\Delta_-}}$, $\Omega_- = \frac{\sqrt{2}}{4}\sqrt{-3a_3 y_1 - a_2 - a_3\sqrt{\Delta_-}}$ and $\Delta_- = -3y_1^2 - 2\frac{a_2}{a_3}y_1 + \left(\frac{a_2}{a_3}\right)^2 - 4\frac{a_1}{a_3}$.

(3) For $a_0 \in (-\infty, 2h_-] \cup (2h_+, +\infty)$, equation (2.1) has no non-trivial bounded solutions. When $a_0 = 2h_-$, an unbounded solution is given by

$$y = -\frac{a_2 + \sqrt{\Delta}}{3a_3} + \frac{\sqrt{\Delta}}{a_3} sec^2 \left[\frac{1}{2}\Delta^{\frac{1}{4}}(\xi - \xi_0)\right] \tag{2.15}$$

and a constant solution given by

$$y = -\frac{a_2 + \sqrt{\Delta}}{3a_3}. \tag{2.16}$$

Proof. (1) When $h = a_0 = 2h_+$, we have $a_3y^3 + a_2y^2 + a_1y + a_0 = a_3(y - \frac{-a_2+\sqrt{\Delta}}{3a_3})^2(y - \frac{-a_2-2\sqrt{\Delta}}{3a_3})$ (See orbits ② in Figures 1(a), 1(b), 2(a) and 2(b)). Solving for v from (2.9) and substituting its value into the first equation of (2.8), we have

$$\frac{dy}{d\xi} = \pm\sqrt{a_3\left(y - \frac{-a_2+\sqrt{\Delta}}{3a_3}\right)^2\left(y - \frac{-a_2-2\sqrt{\Delta}}{3a_3}\right)}. \tag{2.17}$$

Integrating (2.17) with respect to ξ gives (2.10). This completes the proof of (1).

(2) When $h = a_0 \in (2h_-, 2h_+)$, (See orbits ③ in Figure 1(a), 1(b), 2(a) and 2(b)) suppose that $a_3y^3+a_2y^2+a_1y+a_0 = a_3(y_1-y)(y_2-y)(y-y_3)$, where $y_1 > y_2 > y_3$. Then by the relationship between coefficients and roots of a polynomial, we have

$$\begin{aligned} &a_3(-y_1 - y_2 - y_3) = a_2, \\ &a_3\{y_3(y_1 + y_2) + y_1y_2\} = a_1. \end{aligned} \tag{2.18}$$

Clearly, $\frac{-a_2-2\sqrt{\Delta}}{3a_3} < y_3 < \frac{-a_2-\sqrt{\Delta}}{3a_3}$ if $a_3 > 0$ and $\frac{-a_2-\sqrt{\Delta}}{3a_3} < y_3 < \frac{-a_2-2\sqrt{\Delta}}{3a_3}$ if $a_3 < 0$.

For the case $a_3 > 0$, by letting $\Delta_+ = \frac{a_2^2}{a_3^2} - 3y_3^2 - 2\frac{a_2}{a_3}y_3 - 4\frac{a_1}{a_3}$, from (2.18), we obtain

$$y_1 = \frac{-y_3 - \frac{a_2}{a_3} + \sqrt{\Delta_+}}{2}, \quad y_2 = \frac{-y_1 - \frac{a_2}{a_3} - \sqrt{\Delta_+}}{2}. \tag{2.19}$$

Solving for v from (2.9) and substituting it into the first equation of (2.8) gives

$$\frac{dy}{d\xi} = \pm\sqrt{a_3(y_1 - y)(y_2 - y)(y - y_3)}, \tag{2.20}$$

where $y_3 \in \left(\frac{-a_2-\sqrt{\Delta}}{3a_3}, \frac{-a_2-2\sqrt{\Delta}}{3a_3}\right)$ and y_1 and y_2 are defined by (2.17). Integrating (2.20) with respect to ξ gives

$$y = y_3 + (y_2 - y_3)sn^2\left(\frac{\sqrt{a_3(y_1 - y_3)}}{2}(\xi - \xi_0), \sqrt{\frac{y_2 - y_3}{y_1 - y_3}}\right). \tag{2.21}$$

Substituting (2.19) into (2.21) gives (2.13), which corresponds to the orbits like the right-hand side ③ in Figure 2(b). The solution (2.14) can be proved in the similar way. We omit the details here.

(3) For $a_0 = 2h_-$, (See orbits ④ in Figures 1(a), 1(b), 2(a) and 2(b)) we have

$$a_3y^3 + a_2y^2 + a_1y + a_0 = a_3(y - y_0)(y - y_e^-)^2, \tag{2.22}$$

where $y_0 = \frac{-a_2+2\sqrt{\Delta}}{3a_3}$ and $y_e^- = \frac{-a_2-\sqrt{\Delta}}{3a_3}$. Solving for v from (2.9) and substituting it into the first equation of (2.8) gives

$$\frac{dy}{d\xi} = \pm(y - y_e^-)\sqrt{a_3(y - y_0)}. \tag{2.23}$$

Integrating (2.23) with respect to ξ gives (2.15), which corresponds to the orbit like the left-hand side ④ in Figures 1(b) and 2(b).

□

Remark 2.1. From Theorem 2.2, we conclude that (2.10)-(2.16) are solutions of the second-order ODE $d^2y/d\xi^2 = (1/2)P_3'(y)$, i.e. $d^2y/d\xi^2 = (3a_3y^2 + 2a_2y + a_1)/2$.

3. Exact traveling wave solutions of KK equations

In this section, we will study the traveling wave solutions of the KK equation (1.1) in terms of the two theorems obtained in Section 2. Notice that g in ODEs (2.6) and (2.7) is a constant of integration, so the function $y = y(\xi)$ is a traveling wave solution of the KK equation (1.1) if it solves the fourth-order differential equation (1.3) with arbitrary constant g. That is to say, we can obtain the traveling wave solutions of the KK equation (1.1) by solving the first order ODEs (2.6) or (2.7) with arbitrary constant g.

Inserting $(a_0, a_1, a_2, a_3) = (-\frac{4}{3}g, -\frac{4}{3}c, 0, 4)$ and $(a_0, a_1, a_2, a_3) = (\frac{4}{39}g, -\frac{2}{33}c, 0, \frac{1}{2})$ into Theorem 2.2 respectively, we derive two families of solitary wave solutions and two families of periodic wave solutions of the KK eqution (1.1), which we state in the following theorem.

Theorem 3.1. *The Kaup-Kupershmidt equation (1.1) has the following four families of bounded traveling wave solutions:*

(1) The KK equation has two families of valley-form solitary wave solutions given by

$$u(x,t) = \frac{1}{3}\sqrt{c} - \sqrt{c}\ sech^2\left[c^{\frac{1}{4}}(x - ct - \xi_0)\right] \tag{3.1}$$

and

$$u(x,t) = \frac{2}{33}\sqrt{11c} - \frac{2}{11}\sqrt{11c}\ sech^2\left[\frac{1}{2}\left(\frac{c}{11}\right)^{\frac{1}{4}}(x - ct - \xi_0)\right], \tag{3.2}$$

where the wave speed $c > 0$.

(2) For any arbitrary $c > 0$ and $u_3 \in (-2\sqrt{c}/3, -\sqrt{c}/3)$,

$$u(x,t) = u_3 + \frac{1}{2}\left(u_3 + \sqrt{\Delta_5}\right) sn^2\left(2\Omega_5(x - ct - \xi_0), k_5\right), \tag{3.3}$$

is a family of smooth periodic traveling wave solutions.

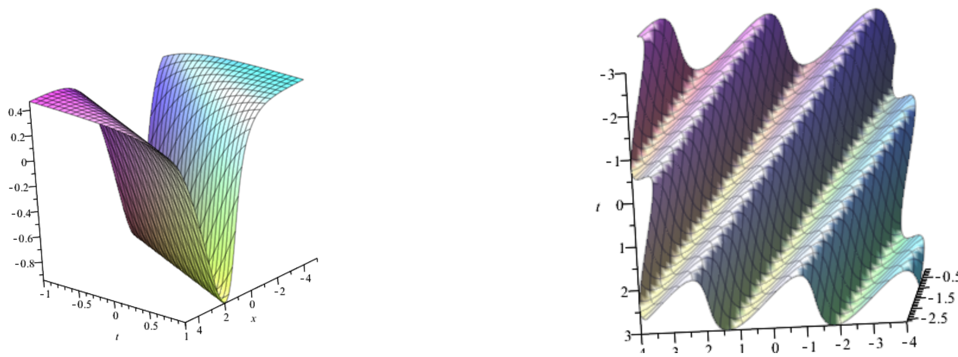
(3) For any arbitrary $c > 0$ and $u_3 \in \left(-4\sqrt{11c}/33, -2\sqrt{11c}/33\right)$,

$$u(x,t) = u_3 - \frac{1}{2}\left(3u_3 + \sqrt{\Delta_7}\right) sn^2\left(\Omega_7(x - ct - \xi_0), k_7\right), \tag{3.4}$$

is a family of smooth periodic traveling wave solutions, where $\Omega_7 = \sqrt{-3u_3 + 2\sqrt{\Delta_7}}/4$, $k_7 = \frac{2\sqrt{3u_3^2 - 4c/33}}{-3u_3 + \sqrt{\Delta_7}}$ and $\Delta_7 = 16c/33 - 3u_3^2$.

It is easy to see that the solution (3.1) $u(\xi) = u(x - ct)$ approaching $\frac{1}{3}\sqrt{c}$ as $\xi = x - ct \to \infty$ and any n-th order derivative of $u(\xi)$ with respect to ξ approaching 0 as $\xi \to \infty$, which implies that the point $P_1(\frac{1}{3}\sqrt{c}, 0, 0, 0)$ is a equilibrium point of system (1.4) with $g = -\frac{2}{9}c\sqrt{c}$. There is at least one homoclinic orbit connecting the equilibrium point P_1. For the case when $g = -\frac{2}{9}c\sqrt{c}$, we can see that P_1 is the unique equilibrium point of (1.4). The coefficient matrix of the linearized system of (1.4) at the equilibrium point P_1 can be

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4c & 0 & 5\sqrt{c} & 0 \end{pmatrix}. \tag{3.5}$$



(a) (b)

Figure 3. Portrait of the traveling wave solutions of the KK equation (1.1) with $c = 3$. (a) solitary wave solution; (b) periodic wave solution.

At P_1, one can easily calculate the eigenvalues of the matrix (3.5) $\pm 2c^{\frac{1}{4}}$ and $\pm c^{\frac{1}{4}}$, which implies that system (1.4) has a unique saddle-saddle point when $g = -\frac{2}{9}c\sqrt{c}$ for any $c > 0$. From solution (3.1), we know there is a homoclinic orbit connecting the equilibrium point P_1 and the exact formula of the projection of this homoclinic orbit onto the $(x_1, x_2)-$plane is

$$x_2^2 = 4x_1^3 - \frac{4}{3}cx_1 + \frac{8}{27}c\sqrt{c}. \tag{3.6}$$

Theoretically, we know that the fourth-order ODEs (1.4) can be reduced into second order if we can solve for x_3 and x_4 from the two first integrals (1.5) and (1.6). Unfortunately, one will realize that it is practically impossible to implement because the roots of a general sixth-degree polynomial have to be find firstly. In what follows, we will exploit the relationship of the subequations (2.6) and (2.7) obtained in Section 2 and ODEs (1.4) and the two known first integrals (1.5) and (1.6).

Note that subequation (2.6) is equivalent to

$$x_2^2 = 4x_1^3 - \frac{4}{3}cx_1 - \frac{4}{3}g. \tag{3.7}$$

Derivating subequation (3.7) with respect to ξ once and twice gives

$$x_3 = 6x_1^2 - \frac{2}{3}c \tag{3.8}$$

and

$$x_4 = 12x_1x_2, \tag{3.9}$$

respectively. With the help of Maple and by inserting (3.7)-(3.9) into (1.5) and (1.6) respectively, we get $K_1 = 0$ and $K_2 = -\frac{4}{81}c^3 - \frac{4}{9}gc\sqrt{c} - g^2$. This means that all the solutions we derived through the subequation (2.6) are defined by $\Phi_1(x_1, x_2, x_3, x_4) = 0$ and $\Phi_2(x_1, x_2, x_3, x_4) = -\frac{4}{81}c^3 - \frac{4}{9}gc\sqrt{c} - g^2$. That is to say that the solution curves of ODE (1.4) corresponding to (3.1) and (3.2) lie on the the supersurfaces $\Phi_1(x_1,x_2,x_3,x_4)=0$ and $\Phi_2(x_1,x_2,x_3,x_4)=-\frac{4}{81}c^3-\frac{4}{9}gc\sqrt{c}-g^2$. Especially, for the homoclinic solutions of (1.4) with $g=-\frac{2}{9}c\sqrt{c}$ corresponding to solitary wave solution (3.1) are defined by the two supersurfaces $\Phi_1(x_1,x_2,x_3,x_4)=0$ and $\Phi_2(x_1,x_2,x_3,x_4)=0$. Similarly, we can get the two constants $K_1=\frac{196}{143}cg$ and $K_2=-\frac{196}{169}g^2-\frac{196}{3993}c^3$ corresponding to the solutions determined by (2.7). Especially, the solitary wave solution (3.2) corresponds to the homoclinic orbit connecting the equilibrium point $(\frac{2}{33}\sqrt{11c},0,0,0)$ of system (1.4) for the case when $g=\frac{26c}{1089}\sqrt{11c}$.

4. Conclusion and discussion

The exact traveling wave solutions of the fifth-order KK equation were studied in this paper. By investigating the bifurcation and exact solutions to the subequations of the corresponding traveling wave equation of the KK equation,we derived two families of solitary wave solutions and two families of periodic wave solutions. From the discussion above, one may find that the main reasons why we can find (2.6) and (2.7) as subequations of the higher-order nonlinear equation (1.3) can be listed as follows: (1) the even-order derivatives of y are all polynomials in y provided that y satisfies ODE (2.1); (2) higher-order ODE (1.3) involves only even-order derivatives of y, $(\frac{dy}{d\xi})^2$ and polynomial in y. Consequently, for the other higher-order differential equations one may find the subequations in the form (2.1), that is, the approach we proposed in this paper might be applied to investigate other higher-order differential equations in the form $F(u^{(2k)},u^{(2k-2)},...,u'',u'^2,u)=0$, where F is a polynomial function.

There are amount of higher-order nonlinear wave equations such as the Lax equation, the Ito equation and the general fifth-order wave equation (1.2) can be reduced to the form $F(u^{(2k)},u^{(2k-2)},...,u'',u'^2,u)=0$. However, some nonlinear wave equations, especially those equations with nonlinear dispersion, usually reduce to singular dynamical systems. It has been proved that this class of nonlinear wave equations possess non-smooth singular wave solutions, such as compacton, peakon, etc by using the dynamical system method [12, 13, 17, 20] and other methods. The singular dynamical systems possess abundant varieties of interesting wave solutions reflecting some phenomena in the real world. To the best of our knowledge, this method has only been used to investigate the lower-order wave equations. The question now is whether the higher-order wave equations, such as the sixth-order KdV equations, possess non-smooth singular wave solutions? How can we use the planar singular dynamical systems to find the singular wave solutions to higher-order singular differential equations? These questions will be investigated in our future work.

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