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SOME NOVEL APPLICATIONS OF CERTAIN HIGHER ORDER ORDINARY COMPLEX DIFFERENTIAL EQUATIONS TO NORMALIZED ANALYTIC FUNCTIONS

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Abstract The aim of this work is first to reveal some nonlinear connections between normalized analytic functions and certain higher order ordinary (complex) differential equations and then to point some of their geometric implications appertaining to normalized analytic functions out.

Keywords Normalized analytic functions, univalent functions, domains in the complex plane, starlikeness and convexity, inequalities in the complex plane, ordinary complex differential equation.

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1. Introduction and Definitions

In the literature, frequently, we encounter several scientific researches relating to both univalent function theory (UFT) and ordinary differential equations (ODE). As we know, ODE, which is an equation containing a function of one independent variable and its derivatives, is an important scientific field for mathematics, engineering and also many other sciences. A great number of its basic concept and related scientific researches can be easily obtained by a basic research. In the same time, UFT is a fascinating interplay of geometry and analysis, directed primarily toward extremal problems. A branch of complex analysis with classical roots, it is an active field of modern research. For its detail, one may refer to [1,2,11]. Especially, an ordinary complex differential equation (OCDE), which is ODE consisting of complex variables, plays an important role for our main purpose. Furthermore, in the light of a different and novel idea asserted by this investigate, the essential objective of this work is first to reveal some comprehensive results between normalized analytic functions and certain types of OCDE, and then to determine some of their consequences relating to UFT. For this reason, this investigation is both important and interesting in making a major contribution concerning the related results between different scientific fields of mathematics, which are UFT and also OCDE. For some results between complex functions analytic in $\mathbb{U}$ and first (or second) order complex differential equations, as example, see the papers in [3,4]. Since the main result includes some comprehensive results between certain analytic functions and higher order complex differential equations, this work makes a significant

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contribution to the related fields. In that case, we can begin by presenting and by recalling the following information which will be needed for our investigation.

First of all, let us denote by $\mathbb{N}$, $\mathbb{C}$, $\mathbb{U}$ and $\mathcal{H}$ the set of natural numbers, the set of complex numbers, the unit open disk and the class of all analytic functions in $\mathbb{U}$, respectively, and let $\mathcal{A}_n$ denote the family of the functions $f(z) \in \mathcal{H}$ normalized by the following Taylor-Maclaurin series:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \quad (n \in \mathbb{N}),$$

which are univalent in $\mathbb{U}$.

Specially, in the sense of the geometric properties of UFT, as is known, the well-known subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of the general class $\mathcal{H}$ are called the classes of all starlike functions and convex functions of order α $(0 \leq \alpha < 1)$ in $\mathbb{U}$, and their special subclasses $\mathcal{K} := \mathcal{K}(0)$ and $\mathcal{S}^* := \mathcal{S}^*(0)$ are also known as the classes of convex functions and starlike functions with respect to the origin in $\mathbb{U}$, respectively. For their details, may be checked the works in [1,2] and also [11].

The following assertions, which are Lemma 1.1, Lemma 1.2 and Lemma 1.3 below, will be required in our present investigation.

Lemma 1.1 ([12]). *If a function $f(z) \in \mathcal{A}_n$ satisfies the following inequality:*

$$|f''(z)| \leq \frac{2(1-\alpha)}{2-\alpha} \quad (0 \leq \alpha < 1;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Lemma 1.2 ([12]). *If a function $f(z) \in \mathcal{A}_n$ satisfies the following inequality:*

$$|f''(z)| \leq \frac{1-\alpha}{2-\alpha} \quad (0 \leq \alpha < 1;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{K}(\alpha)$.

Lemma 1.3 ([7]). *Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ satisfies $\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all $K \geq mM\frac{M-|a|}{M+|a|}$, $\theta \in \mathbb{R}$, and $z \in \mathbb{U}$, and also let the function $p(z)$ be in the class $\mathcal{H}[a,m] \equiv \{p(z) \in \mathcal{H} \ :\ p(z) = a + a_m z^m + a_{m+1}z^{m+1} + \ldots (z \in \mathbb{U})\}$ and $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$. Then $|p(z)| < M$, where $0 \leq |a| < M$ and $m \in \mathbb{N}$.*

As well, literature presentes us that there are several investigations including some analytic and geometric results between certain complex inequalities constituted by functions belonging to the general class $\mathcal{A}_n$. As certain basic information relating to our main results, we also need to recall extra information. Especially, the problem of finding λ satisfying the following proposition:

$$|f''(z)| \leq \lambda \ \ (f(z) \in \mathcal{A}_n; z \in \mathbb{U}) \quad \Rightarrow \quad f(z) \in \mathcal{S}^*$$

was first considered by Mocanu [8]. Later, Ponnusamy and Singh [10] derived a better value of the parameter parameter λ Afterwards, Obradovic [9] focused on same problems for the value of parameter $\lambda = 2/3$ and proved that his result is sharp. In [12], by using the methods used by Obradovic [9], Tuneski also obtained certain results dealing with the same problems above.

By means of the assertions (Lemmas 1.1 and 1.2) and also the assertion (Lemma 1.3), which is one of the novel forms produced by the more general result obtained by Miller and Mocanu in [7, p. 33-35], as a different technique for the proof of main result, an interesting result (Theorem 2.1 below) dealing with certain inequalities consisting of functions in the class $\mathcal{A}_n$ and the function which is the solving of a differential equation is first produced and its certain applications related to geometric properties of analytic and univalent functions are also pointed out. For the other certain novel usages of the more general result in [7], see the results given by [5] and also [6].

2. Main result and Consequences

We begin first by setting and then by proving the following result which includes several interesting relations between normalized analytic functions and functions satisfying initial value problems for higher-order linear equations.

Theorem 2.1. *Let $\phi(z)$ be an analytic function in $\mathbb{U}$ and satisfy the inequality:*

$$\big|z\phi(z)\big| < \frac{M(M-|a|)}{(M+1)(M+|a|)} \quad \big(0 \le |a| < M;\ z \in \mathbb{U}\big), \tag{2.1}$$

and also let $W := W(z)$ be the (unique) solution of the initial value problem for higher-order linear differential equation given by

$$\left\{\begin{array}{l} W^{(n+1)} \pm \phi(z)W^{(n)} \ = \ \phi(z) \quad (z \in \mathbb{U}; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \\ W(0) \ = \ 0 \\ W'(0) \ = \ 1 \\ W^{(k)}(0) \ = \ 0 \quad \big(k = 2, 3, ..., n-1\big) \\ W^{(n)}(0) \ = \ a \end{array}\right\},$$

where $W^{(n)}(z) = \frac{d^nW}{dz^n}$ $(n \in \mathbb{N}_0)$. Then the inequality $\big|W^{(n)}(z)\big| < M$ $(0 \le |a| < M;\ z \in \mathbb{U})$ holds.

Proof. Define $p(z)$ by

$$p(z) = W^{(n)}(z) \quad (z \in \mathbb{U}).$$

Clearly, the function $p(z)$ is in the class $\mathcal{H}[a, 1]$, and then by the related implicit function it immediately follows that

$$\frac{zp'(z)}{1 \pm p(z)} = \frac{zW^{(n+1)}(z)}{1 \pm W^{(n)}(z)} = z\phi(z) \quad \Big(z \in \mathbb{U};\ W^{(n)}(z) \neq \pm 1\Big),$$

and also denote $\psi(r, s; z)$ and Ω by

$$\psi(r, s; z) := \frac{s}{1 \pm r} \quad (r \neq \pm 1)$$

and

$$\Omega := \left\{ w \in \mathbb{C} \ :\ |w| < \frac{M(M-|a|)}{(M+1)(M+|a|)} \quad \big(0 \le |a| < M\big) \right\},$$

respectively.

We then obtain

$$\psi\big(p(z), zp'(z); z\big)\left(= \frac{zp'(z)}{1 \pm p(z)}\right) = \frac{zW^{(n+1)}(z)}{1 \pm W^{(n)}(z)}$$

belonging to the complex domain Ω for all z in $\mathbb{U}$.

Further, for any

$$\theta \in \mathbb{R}\ ,\ K \geq mM\frac{M-|a|}{M+|a|} \geq M\frac{M-|a|}{M+|a|} \quad \text{and} \quad z \in \mathbb{U},$$

we also obtain that

$$\left|\psi(Me^{i\theta}, Ke^{i\theta}; z)\right| = \left|\frac{Ke^{i\theta}}{1 \pm Me^{i\theta}}\right| \geq \frac{M}{M+1} \cdot \frac{M-|a|}{M+|a|} \quad (\text{since } m \geq 1),$$

which immediately yields that

$$\psi\big(Me^{i\theta}, Ke^{i\theta}; z\big) \notin \Omega.$$

Therefore, in view of Lemma 1.3, the definition of the function $p(z)$ easily follows that

$$\big|p(z)\big| = \big|W^{(n)}(z)\big| < M \quad \big(z \in \mathbb{U};\ M > |a| \geq 0\big),$$

which is the proof of Theorem 2.1. □

Clearly, the theorem above includes several interesting or important results. By selecting suitable values of the parameters in that theorem (and also its consequences), some of them can be easily generated. It is not easy to list of them. But, as its certain applications, we want to indicate only some of them, which have an important role for both analytic and geometric function theory.

By letting $n := 2$ in the related theorem, the following corollary, i.e., Corollary 2.1, can be easily derived.

Corollary 2.1. *Let an analytic function $\phi(z)$ in $\mathbb{U}$ satisfy the inequality in (2.1), and also let the function $w := w(z)$ be the (unique) solution of the initial value problem for third-order linear differential equation:*

$$\begin{cases} w''' \pm \phi(z)w'' &=& \phi(z) \quad (z \in \mathbb{U}), \\ w(0) &=& 0, \\ w'(0) &=& 1, \\ w''(0) &=& a. \end{cases} \tag{2.2}$$

Then $\big|w''(z)\big| < M$, where $0 \leq |a| < M$ and $z \in \mathbb{U}$.

Corollary 2.1 (or Theorem 2.1) gives us the following interesting results concerning geometric properties of analytic and univalent functions, which are Propositions 2.3 and 2.4 below.

Proposition 2.1. *Let $\phi(z)$ be an analytic function in $\mathbb{U}$ and satisfy:*

$$|z\phi\,(z)| < \frac{2\,(1-\alpha)\,[2\,(1-\alpha) - |a|\,(2-\alpha)]}{(4-3\alpha)[2\,(1-\alpha) + |a|\,(2-\alpha)]} \quad \left(0 \leq |a| < \frac{2(1-\alpha)}{2-\alpha}\right),$$

and also let $w(z)$ be the (unique) solution of the initial value problem for third-order linear differential equation given by (2.2). Then $w(z) \in \mathcal{S}^*(\alpha)$, where $0 \leq \alpha < 1$.

Proof. By taking

$$M := \frac{2(1-\alpha)}{2-\alpha} \quad (0 \leq \alpha < 1),$$

in Corollary 2.1 (or in Theorem 2.1, of course, with $n := 2$) and just then by making use of Lemma 1.1, the proof of Proposition 2.1 can be easily obtained. □

Proposition 2.2. *Let $\phi(z)$ be an analytic function in $\mathbb{U}$ and satisfy:*

$$|z\phi(z)| < \frac{(1-\alpha)[1-\alpha-|a|(2-\alpha)]}{(3-2\alpha)[1-\alpha+|a|(2-\alpha)]} \quad \left(0 \leq |a| < \frac{1-\alpha}{2-\alpha}\right),$$

and also let $w(z)$ be the (unique) solution of the initial value problem for third-order linear differential equation given by (2.2). Then $w(z) \in \mathcal{K}^*(\alpha)$, where $0 \leq \alpha < 1$.

Proof. By letting

$$M := \frac{1-\alpha}{2-\alpha} \quad (0 \leq \alpha < 1),$$

in Corollary 2.1 (or in Theorem 2.1, of course, with $n := 2$) and just then by using Lemma 1.2, the proof of Proposition 2.2 can be easily stated. □

By taking $\alpha := 0$ in both propositions, respectively, the following two corollaries are then obtained.

Corollary 2.2. *If an analytic function $\phi(z)$ in $\mathbb{U}$ satisfies the inequality:*

$$|z\phi(z)| < \frac{1-|a|}{2(1+|a|)} \quad (0 \leq |a| < 1)$$

and the function $w(z)$ is the (unique) solution of the initial value problem for third-order linear differential equation given by (2.2)*, then $w(z) \in \mathcal{S}^*$.*

Corollary 2.3. *If an analytic function $\phi(z)$ in $\mathbb{U}$ satisfies the inequality:*

$$\Big|z\phi(z)\Big| < \frac{1-2|a|}{3(1+2|a|)} \quad \left(0 \leq |a| < \frac{1}{2}\right)$$

and also $w(z)$ be the (unique) solution of the initial value problem for third-order linear equation in (2.2)*, then $w(z) \in \mathcal{K}$.*

In view of Corollaries 2.2 and 2.3, the following special examples dealing with geometric properties of analytic functions can be given.

Example 2.1.

(i) One of the solution of the initial value problem for third-order linear differential equation given by (2.2) is $w(z) = ze^{\frac{z}{8}}$ and it is a starlike function in the disk $\mathbb{U}$, where, of course, $a := \frac{1}{4}$ and $\phi(z) := \frac{z+24}{64(8e^{-z}+z+16)}$.

(ii) One of the solution of the initial value problem for third-order linear differential equation given by (2.2) is $w(z) = \frac{z}{1-\alpha z}$ and also it is a convex function in the disk $\mathbb{U}$, where $a := 2\alpha$ and $\phi(z) := \frac{6\alpha^2}{2\alpha(1-\alpha z)+(1-\alpha z)^4}$ $\left(-\frac{1}{32} \leq \alpha \leq \frac{1}{32}\right)$.

(iii) One of the solution of the initial value problem for third-order linear differential equation given by (2.2) is $w(z) = -64e^{-\frac{z}{8}} + \frac{z^2}{2} - 7z + 64$ and this function is convex in the disk $\mathbb{U}$, where $a := 0$ and $\phi(z) := \frac{1}{8}$.

(iv) One of the solution of the initial value problem for linear third-order linear differential equation given by (2.2) is $w(z) = -16e^{-\frac{z}{4}} + \frac{z^2}{2} - 3z + 16$ and also this function is starlike in the disk $\mathbb{U}$, where $a := 0$ and $\phi(z) := \frac{1}{4}$.

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