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LOEWNER CHAINS AND UNIVALENCE CRITERIA RELATED WITH RUSCHEWEYH AND SĂLĂGEAN DERIVATIVES

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Abstract In this paper we obtain, by the method of Loewner chains, some sufficient conditions for the analyticity and the univalence of the functions defined by an integral operator. These conditions involves Ruscheweyh and Sălăgean derivative operator in the open unit disk. In particular cases, we find the well-known conditions for univalency established by Becker [3], Ahlfors [2], Kanas and Srivastava [8] and others for analytic mappings $f : \mathcal{U} \to \mathbb{C}$. Also, we obtain the corresponding new, useful and simpler conditions for this integral operator.

Keywords Univalent function, Ruscheweyh and Sălăgean derivative, univalence condition, integral operator, Loewner chain.

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1. Introduction

Denote by $\mathcal{U}_r = \{z \in \mathbb{C} : \ |z| < r\} \quad (0 < r \leqslant 1)$ the disk of radius r and let $\mathcal{U} = \mathcal{U}_1$. Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathcal{U}$ which satisfy the usual normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

By $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathcal{U}$. These classes have been one of the important subjects of research in Geometric Function Theory for a long time (see [19]). For the functions f_p $(p = 1, 2)$ given by

$$f_p(z) = z + \sum_{k=2}^{\infty} a_{k,p} z^k \quad (p = 1, 2),$$

let $f_1 * f_2$ denote the Hadamard product (or convolution) of f_1 and f_2, defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{1.1}$$

Two of the most important and well-known univalence criteria for analytic functions defined the open unit disk were obtained by Becker [3] and Ahlfors [2]. Becker

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and Ahlfors's works depends upon a clever use of the Theory of Loewner chains and the generalized Loewner differential equation. Extensions of these two criteria were given by Ruscheweyh [17], Kanas and Srivastava [8]. Recently, Ovesea [11], Deniz and Orhan [4] and Deniz *et al.* [5] obtained some generalization of these univalence criterions. Furthermore, Pascu [12] and Raducanu *et al.* [15] obtained some extensions of Becker's univalence criterion.

In the present paper, we will study a number of new criteria for univalence based upon the Ruscheweyh and Sălăgean derivative and for the functions defined by the integral operator $\mathcal{F}_\beta(z)$. The paper improve the work of Kanas and Srivastava [8] of extending univalence criteria for analytic mappings. In special cases our univalence conditions contain the results obtained by some of the authors are cited in references. Our considerations are based on the Theory of Loewner chains.

2. Loewner chains and related theorem

Before proving our main theorem we need a brief summary of the method of Loewner chains.

Let $\mathcal{L}(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$, $a_1(t) \neq 0$, be a function defined on $\mathcal{U} \times I$, where $I := [0, \infty)$ and $a_1(t)$ is a complex-valued, locally absolutely continuous function on I. $\mathcal{L}(z,t)$ is called a Loewner chain if $\mathcal{L}(z,t)$ satisfies the following conditions:

(i) $\mathcal{L}(z,t)$ is analytic and univalent in $\mathcal{U}$ for all $t \in I$,

(ii) $\mathcal{L}(z,t) \prec \mathcal{L}(z,s)$ for all $0 \leq t \leq s < \infty$,

where the symbol "$\prec$" stands for subordination. If $a_1(t) = e^t$ then we say that $\mathcal{L}(z,t)$ is a *standard Loewner chain.*

In order to prove our main results we need the following theorem due to Pommerenke [13] (also see [14]). This theorem is often used to find out univalency for an analytic function, apart from the theory of Loewner chains.

Theorem 2.1. *(see Pommerenke [14]) Let $\mathcal{L}(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$ be analytic in $\mathcal{U}_r$ for all $t \in I$. Suppose that:*

(i) $\mathcal{L}(z,t)$ is a locally absolutely continuous function in the interval I, and locally uniformly with respect to $\mathcal{U}_r$.

(ii) $a_1(t)$ is a complex valued continuous function on I such that $a_1(t) \neq 0$, $|a_1(t)| \to \infty$ for $t \to \infty$ and

$$\left\{ \frac{\mathcal{L}(z,t)}{a_1(t)} \right\}_{t \in I}$$

forms a normal family of functions in $\mathcal{U}_r$.

(iii) There exists an analytic function $p : \mathcal{U} \times I \to \mathbb{C}$ satisfying $\Re(p(z,t)) > 0$ for all $z \in \mathcal{U},\ t \in I$ and

$$z\frac{\partial \mathcal{L}(z,t)}{\partial z} = p(z,t)\frac{\partial \mathcal{L}(z,t)}{\partial t}, \quad z \in \mathcal{U}_r,\ t \in I. \tag{2.1}$$

Then, for each $t \in I$, the function $\mathcal{L}(z,t)$ has an analytic and univalent extension to the whole disk $\mathcal{U}$ or the function $\mathcal{L}(z,t)$ is a Loewner chain.

The equation (2.1) is called the generalized Loewner differential equation. Detailed information about Loewner chains and Pommerenke's theorem can be found in Hotta [6] and [7].

3. Univalence criteria and the Ruscheweyh derivative

For a function $f \in \mathcal{A}$ the Ruscheweyh derivative operator [16], $\mathcal{R}^{\lambda} : \mathcal{A} \to \mathcal{A}$ is defined by

$$\mathcal{R}^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \quad (\lambda > -1;\ z \in \mathcal{U})\,. \tag{3.1}$$

In particular, when $\lambda = n$ ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$), definition (3.1) implies that

$$\mathcal{R}^{n} f(z) = \frac{z}{n!}\frac{d^n}{dz^n}\{z^{n-1}f(z)\}. \tag{3.2}$$

It is obvious that

$$\begin{aligned} \mathcal{R}^0 f(z) &= f(z), \\ \mathcal{R}^1 f(z) &= zf'(z), \\ \mathcal{R}^2 f(z) &= \frac{z}{2}\{2f'(z) + zf''(z)\}, \end{aligned} \tag{3.3}$$

and so on. Also we can write recurrence relationship as follows:

$$z[\mathcal{R}^n f(z)]' = (n+1)\mathcal{R}^{n+1} f(z) - n\mathcal{R}^n f(z).$$

In this section, making use of the Theorem 2.1, we obtain some univalence criterions connected with the Ruscheweyh derivative operator for an integral operator. The proofs are based on the theory of Loewner chains (or Theorem 2.1), essence of which is the construction of a suitable Loewner chain.

Theorem 3.1. *Let $f, g, h \in \mathcal{A}$. Also let $m \in \mathbb{R}_+$ and α, β, c be complex numbers such that*

$$\alpha \neq 1, \quad c \neq -1, \quad \left|\frac{1+c}{1-\alpha} - \frac{m+1}{2}\right| \leqslant \frac{m+1}{2}, \qquad \left|\beta - \frac{m+1}{2}\right| < \frac{m+1}{2}. \tag{3.4}$$

If the inequalities

$$\left|\left(\frac{(1+c)f'(z)}{[\mathcal{R}^n h(z)]' - \alpha} - 1\right) - \frac{m-1}{2}\right| < \frac{m+1}{2} \tag{3.5}$$

and

$$\begin{aligned} &\Bigg| |z|^{m+1}\left(\frac{(1+c)f'(z)}{[\mathcal{R}^n h(z)]' - \alpha} - 1\right) \\ &+(1-|z|^{m+1})\left[(\beta-1)\frac{zg'(z)}{g(z)} + \frac{z[\mathcal{R}^n h(z)]''}{[\mathcal{R}^n h(z)]' - \alpha}\right] - \frac{m-1}{2}\Bigg| \leqslant \frac{m+1}{2} \end{aligned} \tag{3.6}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by

$$\mathcal{F}_\beta(z) = \left[\beta \int_0^z g^{\beta-1}(u) f'(u) du\right]^{1/\beta} \tag{3.7}$$

is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. Let a and b be two positive real numbers such that $m = \frac{b}{a}$. We will prove that there exists a real number $r \in (0,1]$ such that the function $\mathcal{L} : \mathcal{U}_r \times I \to \mathbb{C}$, defined formally by

$$\mathcal{L}(z,t) \tag{3.8}$$
$$= \left\{\beta \int_0^{e^{-at}z} g^{\beta-1}(s) f'(s) ds + \frac{\beta(e^{bt} - e^{-at})}{1+c} z g^{\beta-1}(e^{-at}z)\left([\mathcal{R}^n h(e^{-at}z)]' - \alpha\right)\right\}^{\frac{1}{\beta}}$$

is analytic in $\mathcal{U}_r$ for all $t \in I$.

Because $g \in \mathcal{A}$ the function

$$\psi(z) = \frac{g(z)}{z}$$

is analytic in $\mathcal{U}$ and $\psi(0) = 1$. Then there exist a disk $\mathcal{U}_{r_1}$, $0 < r_1 \leqslant 1$, in which $\psi(z) \neq 0$ for all $z \in \mathcal{U}_{r_1}$. We denote by ψ_1 the uniform branch of $(\psi(z))^{\beta-1}$ equal to 1 at origin.

Consider the function

$$\psi_2(z,t) = \beta \int_0^{e^{-at}z} s^{\beta-1} \psi_1(s) f'(s) ds,$$

then we have

$$\psi_2(z,t) = z^\beta \psi_3(z,t),$$

where ψ_3 is also analytic in $\mathcal{U}_{r_1}$. Hence, the function

$$\psi_4(z,t) = \psi_3(z,t) + \frac{\beta}{1+c}\left(e^{bt} - e^{-at}\right) e^{-a(\beta-1)t} \psi_1(e^{-at}z)\left([R^n h(e^{-at}z)]' - \alpha\right)$$

is analytic in $\mathcal{U}_{r_1}$ and

$$\psi_4(0,t) = e^{-a\beta t}\left[\frac{1+c-(1-\alpha)\beta}{1+c} + \frac{(1-\alpha)\beta}{1+c}(e^{(a+b)t})\right].$$

We will prove that $\psi_4(0,t) \neq 0$ for all $t \in I$. It is easy to see that $\psi_4(0,0) = 1$. Suppose that there exists $t_0 > 0$ such that $\psi_4(0,t_0) = 0$. Then we obtain the equality $e^{(a+b)t_0} = \frac{1+c+(\alpha-1)\beta}{(\alpha-1)\beta}$. This equality implies that $c < -1$ (or $c > -1$), which contradicts $|c| \leqslant 1$. We conclude that $\psi_4(0,t) \neq 0$ for all $t \in I$. Therefore, there is a disk $\mathcal{U}_{r_2}$, $r_2 \in (0, r_1]$, in which $\psi_4(z,t) \neq 0$ for all $t \in I$. Then we can choose an uniform branch of $[\psi_4(z)]^{1/\beta}$ analytic in $\mathcal{U}_{r_2}$, denoted by $\psi_5(z,t)$.

It follows from (3.8) that the

$$\mathcal{L}(z,t) = z\psi_5(z,t) = a_1(t)z + a_2(t)z^2 + ...$$

and thus, the function $\mathcal{L}(z,t)$ is analytic in $\mathcal{U}_{r_2}$.

We have

$$a_1(t) = e^{(\frac{-a\beta+a+b}{\beta})t}\left[\frac{1+c-(1-\alpha)\beta}{1+c}e^{-(a+b)t}+\frac{(1-\alpha)\beta}{1+c}\right]^{\frac{1}{\beta}}.$$

for which we consider the uniform branch equal to 1 at the origin. Since $\left|\beta - \frac{m+1}{2}\right| < \frac{m+1}{2}$ is equivalent with $\Re(\frac{1}{\beta}) > \frac{1}{m+1}$ we have that

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

Moreover, $a_1(t) \neq 0$ for all $t \in I$.

After simple calculation we obtain, for each $z \in \mathcal{U}$

$$\begin{aligned} & \lim_{t\to\infty}\frac{\mathcal{L}(z,t)}{a_1(t)} \\ = & \lim_{t\to\infty} z\left[\frac{1+c-(1-\alpha)\beta}{1+c}e^{-(a+b)t}+\frac{(1-\alpha)\beta}{1+c}\right]^{-\frac{1}{\beta}} \\ & \times\left(1+O(e^{-at}z)\right)^{1/\beta}\left\{e^{-(a+b)t}+(1-e^{-(a+b)t})\frac{\beta}{1+c}\left[1-\alpha+O(e^{-at}z)\right]\right\}^{\frac{1}{\beta}} \\ = & z. \end{aligned}$$

The limit function $\varphi(z) = z$ belongs to the family $\{\mathcal{L}(z,t)/a_1(t)\}$; then in every closed disk $\mathcal{U}_{r_3}$, $0 < r_3 < r_2$, there exists a constant $K = K(r_3)$ such that

$$\left|\frac{\mathcal{L}(z,t)}{a_1(t)}\right| < K, \quad \forall z \in \mathcal{U}_{r_3},\ t \in I$$

uniformly in this disk, provided that t is sufficiently large. Then, by Montel's Theorem, $\left\{\frac{\mathcal{L}(z,t)}{a_1(t)}\right\}_{t\in I}$ is a normal family in $\mathcal{U}_{r_3}$. From the analyticity of $\frac{\partial\mathcal{L}(z,t)}{\partial t}$, we obtain that for all fixed numbers $T > 0$ and $r_4,\ 0 < r_4 < r_3$, there exists a constant $K_1 > 0$ (that depends on T and r_4) such that

$$\left|\frac{\partial\mathcal{L}(z,t)}{\partial t}\right| < K_1, \quad \forall z \in \mathcal{U}_{r_4},\ t \in [0,T].$$

Therefore, the function $\mathcal{L}(z,t)$ is locally absolutely continuous in I, locally uniformly with respect to $\mathcal{U}_{r_4}$.

Consider the function $p : \mathcal{U}_r \times I \to \mathbb{C}$ for $0 < r < r_4$ and $t \in I$, defined by

$$p(z,t) = z\frac{\partial\mathcal{L}(z,t)}{\partial z}\Big/\frac{\partial\mathcal{L}(z,t)}{\partial t}.$$

The function $p(z,t)$ is analytic in $\mathcal{U}_r$, $0 < r < r_4$. If the function

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1} = \frac{\frac{z\partial\mathcal{L}(z,t)}{\partial z} - \frac{\partial\mathcal{L}(z,t)}{\partial t}}{\frac{z\partial\mathcal{L}(z,t)}{\partial z} + \frac{\partial\mathcal{L}(z,t)}{\partial t}} \tag{3.9}$$

is analytic in $\mathcal{U} \times I$ and $|w(z,t)| < 1$, for all $z \in \mathcal{U}$ and $t \in I$, then $p(z,t)$ has an analytic extension with positive real part in $\mathcal{U}$, for all $t \in I$. From equality (3.9) we have

$$w(z,t) = \frac{(1+a)\mathcal{G}(z,t) + 1 - b}{(1-a)\mathcal{G}(z,t) + 1 + b}, \tag{3.10}$$

where

$$\begin{aligned} \mathcal{G}(z,t) \quad = \quad & e^{-(a+b)t} \left\{ \left(\frac{(1+c)f'(e^{-at}z)}{[\mathcal{R}^n h(e^{-at}z)]' - \alpha} - 1 \right) \right. \\ & \left. +(e^{(a+b)t} - 1) \left[(\beta - 1) \frac{e^{-at}zg'(e^{-at}z)}{g(e^{-at}z)} + \frac{e^{-at}z[\mathcal{R}^n h(e^{-at}z)]''}{[\mathcal{R}^n h(e^{-at}z)]' - \alpha} \right] \right\} \end{aligned} \tag{3.11}$$

for $z \in \mathcal{U}$ and $t \in I$.

The inequality $|w(z,t)| < 1$ for all $z \in \mathcal{U}$ and $t \in I$, where $w(z,t)$ is defined by (3.10), is equivalent to

$$\left| \mathcal{G}(z,t) - \frac{m-1}{2} \right| < \frac{m+1}{2}, \quad \forall z \in \mathcal{U},\ t \in I. \tag{3.12}$$

Let us denote

$$\mathcal{H}(z,t) = \mathcal{G}(z,t) - \frac{m-1}{2}, \quad \forall z \in \mathcal{U},\ t \in I. \tag{3.13}$$

In view of (3.5), (3.6), from (3.11) and (3.13) we have

$$|\mathcal{H}(z,0)| = \left| \left(\frac{(1+c)f'(z)}{[\mathcal{R}^n h(z)]' - \alpha} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2}, \tag{3.14}$$

and for $z = 0$, $t > 0$, since (3.4)

$$\begin{aligned} |\mathcal{H}(0,t)| \quad &= \quad \left| e^{-(a+b)t} \left(\frac{c+\alpha}{1-\alpha} \right) + (1 - e^{-(a+b)t})(\beta - 1) - \frac{m-1}{2} \right| \\ &= \quad \left| e^{-(a+b)t} \left(\frac{1+c}{1-\alpha} - \frac{m+1}{2} \right) + (1 - e^{-(a+b)t}) \left(\beta - \frac{m+1}{2} \right) \right| \\ &< \quad e^{-(a+b)t} \frac{m+1}{2} + (1 - e^{-(a+b)t}) \frac{m+1}{2} = \frac{m+1}{2}. \end{aligned}$$

Since $|e^{-at}z| \leqslant |e^{-at}| = e^{-at} < 1$ for all $z \in \overline{\mathcal{U}} = \{z \in \mathbb{C} : \ |z| \leqslant 1\}$ and $t > 0$, we find that $\mathcal{H}(z,t)$ is an analytic function in $\overline{\mathcal{U}}$. Making use of the maximum modulus principle we obtain that for each $t > 0$ arbitrarily fixed there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|\mathcal{H}(z,t)| < \max_{|\zeta|=1} |\mathcal{H}(\zeta,t)| = \left| \mathcal{H}(e^{i\theta},t) \right|, \tag{3.15}$$

for all $z \in \mathcal{U}$.

Let us denote $u = e^{-at}e^{i\theta}$. Then $|u| = e^{-at}$, $e^{-(a+b)t} = (e^{-at})^{m+1} = |u|^{m+1}$ and from (3.11) we have

$$\begin{aligned} \left| \mathcal{H}(e^{i\theta},t) \right| \quad = \quad & \left| |u|^{m+1} \left(\frac{(1+c)f'(u)}{[\mathcal{R}^n h(u)]' - \alpha} - 1 \right) \right. \\ & \left. + (1 - |u|^{m+1}) \left[(\beta - 1) \frac{ug'(u)}{g(u)} + \frac{u[\mathcal{R}^n h(u)]''}{[\mathcal{R}^n h(u)]' - \alpha} \right] - \frac{m-1}{2} \right|. \end{aligned}$$

Since $u \in \mathcal{U}$, the inequality (3.6) implies that

$$\left|\mathcal{H}(e^{i\theta}, t)\right| \leqslant \frac{m+1}{2}, \tag{3.16}$$

and from (3.14) and (3.16) it follows that the inequality (3.12)

$$|\mathcal{H}(z,t)| = \left|\mathcal{G}(z,t) - \frac{m-1}{2}\right| < \frac{m+1}{2}$$

is satisfied for all $z \in \mathcal{U}$ and $t \in I$. Therefore $|w(z,t)| < 1$ for all $z \in \mathcal{U}$ and $t \in I$.

Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $\mathcal{L}(z,t)$ has an analytic and univalent extension to the whole unit disk $\mathcal{U}$, for all $t \in I$. For $t = 0$ we have $\mathcal{L}(z,0) = \mathcal{F}_\beta(z)$, for $z \in \mathcal{U}$ and therefore, the function $\mathcal{F}_\beta(z)$ is analytic and univalent in $\mathcal{U}$. □

For $n = 0$ in Theorem 3.1 we obtain another univalence criterion as follows.

Corollary 3.1. *Let $f, g, h \in \mathcal{A}$. Also let $m \in \mathbb{R}_+$ and complex numbers α, β, c be as in Theorem 3.1. If the inequalities*

$$\left|\left(\frac{(1+c)f'(z)}{h'(z)-\alpha} - 1\right) - \frac{m-1}{2}\right| < \frac{m+1}{2} \tag{3.17}$$

and

$$\left| |z|^{m+1}\left(\frac{(1+c)f'(z)}{h'(z)-\alpha} - 1\right) + (1-|z|^{m+1})\left[(\beta-1)\frac{zg'(z)}{g(z)} + \frac{zh''(z)}{h'(z)-\alpha}\right] - \frac{m-1}{2}\right| \leqslant \frac{m+1}{2} \tag{3.18}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

For $h(z) = f(z)$ and $\alpha = m-1 = 0$ in Corollary 3.1 we obtain following criterion which will be useful for univalency of the general integral operators.

Corollary 3.2. *Let c, β be complex numbers such that $|c| < 1$, $|\beta - 1| < 1$ and let the functions $f, g \in \mathcal{A}$. If the inequality*

$$\left| c\,|z|^2 + (1-|z|^2)\left[(\beta-1)\frac{zg'(z)}{g(z)} + \frac{zf''(z)}{f'(z)}\right]\right| \leqslant 1$$

is true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Remark 3.1. For special values of parameters β, c and m our results reduce to several well-known results as follows:

1. Putting $\beta = 1$ and $\alpha = 0$ in Corollary 3.2, then we obtain the the result of Ahlfors [2]. For $c = 0$, Ahlfors's criterion reduces to a criterion found earlier by Becker [3].
2. Putting $h(z) = f(z)$, $\beta = m = 1$ and $c = 0$ in Corollary 3.1, we obtain the result of Pascu [12].

3. Putting $h(z)=f(z)$, $\beta=1$ and $c=0$ in Corollary 3.1, we obtain the result of Răducanu *et al.* [15]

For $h(z)=f(z)$ and $n=1$ in Theorem 3.1, we have the following corollary.

Corollary 3.3. *Let $f,g\in\mathcal{A}$. Also let $m\in\mathbb{R}_+$ and complex numbers α, β, c be as in Theorem 3.1. If the inequalities*

$$\left|\left(\frac{2(1+c)}{m+1}-1\right)-\frac{zf''(z)-\alpha}{f'(z)}\right|<\left|1-\frac{zf''(z)-\alpha}{f'(z)}\right| \tag{3.19}$$

and

$$\left||z|^{m+1}\left(\frac{(1+c)f'(z)}{zf''(z)+f'(z)-\alpha}-1\right)\right.$$
$$\left.+(1-|z|^{m+1})\left[(\beta-1)\frac{zg'(z)}{g(z)}+\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)-\alpha}\right]-\frac{m-1}{2}\right|\leqslant\frac{m+1}{2}$$

are satisfied for all $z\in\mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

The condition (3.6) of Theorem 3.1 can be replaced with a simpler one.

Theorem 3.2. *Let $f,g\in\mathcal{A}$. Also let $m\in\mathbb{R}_+$ and complex numbers α, β, c be as in Theorem 3.1. If the inequalities*

$$\left|\left(\frac{(1+c)f'(z)}{[\mathcal{R}^nh(z)]'-\alpha}-1\right)-\frac{m-1}{2}\right|<\frac{m+1}{2} \tag{3.20}$$

and

$$\left|(\beta-1)\frac{zg'(z)}{g(z)}+\frac{z[\mathcal{R}^nh(z)]''}{[\mathcal{R}^nh(z)]'-\alpha}-\frac{m-1}{2}\right|\leqslant\frac{m+1}{2} \tag{3.21}$$

are true for all $z\in\mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended

Proof. Making use of (3.5) and (3.21) we obtain

$$\begin{aligned}
&\left||z|^{m+1}\left(\frac{(1+c)f'(z)}{[R^nh(z)]'-\alpha}-1\right)+(1-|z|^{m+1})\right.\\
&\left.\left[(\beta-1)\tfrac{zg'(z)}{g(z)}+\tfrac{z[R^nh(z)]''}{[R^nh(z)]'-\alpha}\right]-\tfrac{m-1}{2}\right|\\
=\;&\left||z|^{m+1}\left(\frac{(1+c)f'(z)}{[R^nh(z)]'-\alpha}-1-\frac{m-1}{2}\right)\right.\\
&\left.+(1-|z|^{m+1})\left[(\beta-1)\tfrac{zg'(z)}{g(z)}+\tfrac{z[\mathcal{R}^nh(z)]''}{[\mathcal{R}^nh(z)]'-\alpha}-\tfrac{m-1}{2}\right]\right|\\
\leq\;&|z|^{m+1}\tfrac{m+1}{2}+(1-|z|^{m+1})\tfrac{m+1}{2}=\tfrac{m+1}{2}.
\end{aligned}$$

The conditions of Theorem 3.1 being satisfied it follows that the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$. □

For $n=c=\alpha=0$, $\beta=2$, $m=3$, $h(z)=f(z)$ and $g(z)=z$, Theorem 3.2 yields following result.

Remark 3.2. Let $f\in\mathcal{A}$. If the inequality

$$\left|\frac{zf''(z)}{f'(z)}\right|\leqslant 2,$$

is true for all $z\in\mathcal{U}$, then the function

$$\mathcal{F}_2(z)=\left(2\int\limits_0^z uf'(u)du\right)^{1/2}$$

is analytic and univalent in $\mathcal{U}$.

We consider $\alpha,\ \beta$ and c be real numbers, such that $\alpha<0$. For $h(z)=f(z)$ and $n=0$, by elementary calculations we obtain that inequality (3.5) is equivalent to

$$\Re f'(z)>\frac{m-c}{\alpha(m+1)}\left|f'(z)\right|^2,\quad m\geq c;\ z\in\mathcal{U}. \tag{3.22}$$

Corollary 3.4. *Let $f\in\mathcal{A}$. Also let $m\in\mathbb{R}_+$ and $\alpha,\ \beta,\ c$ be real numbers such that*

$$\alpha<0,\quad c\neq-1,\ c\leq m,\ -1<c\leq m-\alpha(m+1),\qquad \left|\beta-\frac{m+1}{2}\right|<\frac{m+1}{2}.$$

If the inequalities

$$\Re f'(z)>\frac{m-c}{\alpha(m+1)}\left|f'(z)\right|^2 \tag{3.23}$$

and

$$\left|(\beta-1)\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)-\alpha}-\frac{m-1}{2}\right|\leqslant\frac{m+1}{2} \tag{3.24}$$

are satisfied for all $z\in\mathcal{U}$, then the function $f(z)$ is univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. If we take the inequality (3.22) and $f\in\mathcal{A}$ instead of the inequality (3.20) and $g,h\in\mathcal{A}$, respectively, and $n=0$ in Theorem 3.2, Corollary 3.4 can be show easily. □

Remark 3.3. Consider $\beta=1$ and $\alpha<0$ in Corollary 3.4. If in the inequality (3.23) we let $\alpha\to-\infty$, we obtain that

$$\Re f'(z)>0,\quad z\in\mathcal{U}.$$

Since (3.24) holds true for $\beta=1$ and $\alpha\to-\infty$ it follows from Corollary 3.4 that the function f is univalent in $\mathcal{U}$. Therefore, we can conclude that the univalence criterion due to *Alexander-Noshiro-Warshawski* [1], [10], [20] is a limit case of Corollary 3.4.

For suitable values of $\alpha,\ \beta,\ c,\ n,\ m,\ h(z)$ and $g(z)$ in Theorem 3.2, we can obtain different univelence criteria.

Reasoning along the same line as in the proof of the Theorem 2.1 for the Loewner chain

$$\mathcal{L}(z,t) \tag{3.25}$$

$$=\left\{\beta\int\limits_0^{e^{-at}z}g^{\beta-1}(s)f'(s)ds+\frac{\beta(e^{bt}-e^{-at})}{1+c}zg^{\beta-1}(e^{-at}z)\left(\frac{\mathcal{R}^nf(e^{-at}z)-\alpha}{\mathcal{R}^vh(e^{-at}z)-\alpha}\right)\right\}^{\frac{1}{\beta}}$$

we obtain the following theorem. We omit the details.

Theorem 3.3. *Let $f, g, h \in \mathcal{A}$. Also let $m \in \mathbb{R}_+$ and α, β, c be complex numbers such that*

$$c \neq -1, \quad \left|c - \frac{m-1}{2}\right| \leqslant \frac{m+1}{2}, \quad \left|\beta - \frac{m+1}{2}\right| < \frac{m+1}{2}.$$

If the inequalities

$$\left|\left((1+c)f'(z)\frac{\mathcal{R}^v h(z) - \alpha}{\mathcal{R}^n f(z) - \alpha} - 1\right) - \frac{m-1}{2}\right| < \frac{m+1}{2} \tag{3.26}$$

and

$$\begin{aligned} &\Bigg| |z|^{m+1}\left((1+c)f'(z)\frac{\mathcal{R}^v h(z) - \alpha}{\mathcal{R}^n f(z) - \alpha} - 1\right) \\ &+ (1-|z|^{m+1})\left[(\beta-1)\frac{zg'(z)}{g(z)} + \frac{z[\mathcal{R}^n f(z)]'}{\mathcal{R}^n f(z) - \alpha} - \frac{z[\mathcal{R}^v h(z)]'}{\mathcal{R}^v h(z) - \alpha}\right] - \frac{m-1}{2}\Bigg| \leqslant \frac{m+1}{2} \end{aligned} \tag{3.27}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

For $\alpha = 0$ in Theorem 3.3, we obtain new result as follows:

Corollary 3.5. *Let $f, g, h \in \mathcal{A}$. Also let $m \in \mathbb{R}_+$ and complex numbers β, c be as in Theorem 3.3. If the inequalities*

$$\left|\left((1+c)f'(z)\frac{\mathcal{R}^v h(z)}{\mathcal{R}^n f(z)} - 1\right) - \frac{m-1}{2}\right| < \frac{m+1}{2}$$

and

$$\begin{aligned} &\Bigg| |z|^{m+1}\left((1+c)f'(z)\frac{\mathcal{R}^v h(z)}{\mathcal{R}^n f(z)} - 1\right) \\ &+ (1-|z|^{m+1})\left[(\beta-1)\frac{zg'(z)}{g(z)} + (n+1)\frac{\mathcal{R}^{n+1} f(z)}{\mathcal{R}^n f(z)} - (v+1)\frac{\mathcal{R}^{v+1} h(z)}{\mathcal{R}^v h(z)} - n + v\right] \\ &- \frac{m-1}{2}\Bigg| \leqslant \frac{m+1}{2} \end{aligned}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

For $v = 2,\ n = 1$ and $v = 0,\ n = 2$ in Corollary 3.5 , we obtain Corollary 3.6 and Corollary 3.7, respectively.

Corollary 3.6. *Let $f, g, h \in \mathcal{A}$. Also let $m \in \mathbb{R}_+$ and complex numbers β, c be as in Theorem 3.3. If the inequalities*

$$|(1+c)\left(2h'(z) + zh''(z)\right) - m - 1| < m+1 \tag{3.28}$$

and

$$\left| |z|^{m+1}\left((1+c)\left(h'(z) + \frac{z}{2}h''(z)\right) - 1\right)\right.$$

$$+ (1-|z|^{m+1})\left[(\beta-1)\frac{zg'(z)}{g(z)}+\frac{zf''(z)}{f'(z)}-\frac{3z^2h''(z)+z^3h'''(z)}{2zh'(z)+z^2h''(z)}\right]-\frac{m-1}{2}\Bigg| \leqslant \frac{m+1}{2}, \tag{3.29}$$

are true for all $z \in \mathcal{U}$, *then the function* $\mathcal{F}_\beta(z)$ *defined by (3.7) is analytic and univalent in* $\mathcal{U}$, *where the principal branch is intended.*

Corollary 3.7. *Let* $f, g, h \in \mathcal{A}$. *Also let* $m \in \mathbb{R}_+$ *and complex numbers* β, c *be as in Theorem 3.3. If the inequalities*

$$\left|\left(2(1+c)f'(z)\frac{h(z)}{2zf'(z)+z^2f''(z)}-1\right)-\frac{m-1}{2}\right| < \frac{m+1}{2} \tag{3.30}$$

and

$$\Bigg||z|^{m+1}\left(2(1+c)f'(z)\frac{h(z)}{2zf'(z)+z^2f''(z)}-1\right) \tag{3.31}$$

$$+ (1-|z|^{m+1})\left[1+(\beta-1)\frac{zg'(z)}{g(z)}+\frac{3z^2f''(z)+z^3f'''(z)}{2zf'(z)+z^2f''(z)}-\frac{zh'(z)}{h(z)}\right]-\frac{m-1}{2}\Bigg|$$

$$\leqslant \frac{m+1}{2}$$

are true for all $z \in \mathcal{U}$, *then the function* $\mathcal{F}_\beta(z)$ *defined by (3.7) is analytic and univalent in* $\mathcal{U}$, *where the principal branch is intended.*

Remark 3.4. Putting $\beta = m = n = 1$, $\alpha = c = v = 0$ and $h(z) = f(z)$ in Theorem 3.3, we get the result of Kanas and Lecko [9].

Corollary 3.8. *Let* $f, g \in \mathcal{A}$. *Also let* $m \in \mathbb{R}_+$ *and complex numbers* β, c *be as in Theorem 3.3. If the inequalities*

$$\left|(1+c)f'(z)-\frac{m+1}{2}\right| < \frac{m+1}{2} \tag{3.32}$$

and

$$\left||z|^{m+1}\left((1+c)f'(z)-1\right)+(1-|z|^{m+1})(\beta-1)\frac{zg'(z)}{g(z)}-\frac{m-1}{2}\right| \leqslant \frac{m+1}{2} \tag{3.33}$$

are true for all $z \in \mathcal{U}$, *then the function* $\mathcal{F}_\beta(z)$ *defined by (3.7) is analytic and univalent in* $\mathcal{U}$, *where the principal branch is intended.*

Proof. It results from Theorem 3.3 with $\alpha \to \infty$. □

Corollary 3.9. *Let* $f, g \in \mathcal{A}$. *Also let* $m \in \mathbb{R}_+$ *and complex numbers* β, c *be as in Theorem 3.3. If the inequalities*

$$\left|(1+c)f'(z)-\frac{m+1}{2}\right| < \frac{m+1}{2}$$

and

$$\left|(\beta-1)\frac{zg'(z)}{g(z)}-\frac{m-1}{2}\right| \leqslant \frac{m+1}{2},$$

are true for all $z \in \mathcal{U}$, *then the function* $\mathcal{F}_\beta(z)$ *defined by (3.7) is analytic and univalent in* $\mathcal{U}$, *where the principal branch is intended.*

Proof. Corollary 3.9 can be demonstrated the same line as in the proof of the Theorem 3.2. We omit the details. □

4. Univalence criteria and the Sălăgean derivative

Sălăgean (see [18]) introduced an operator S^n ($n \in \mathbb{N}_0$) defined, for a function $f \in \mathcal{A}$, by

$$\begin{aligned} S^0 f(z) &= f(z), \\ S^1 f(z) &= Sf(z) = zf'(z), \\ S^2 f(z) &= S(Sf(z)) = zf'(z) + z^2 f''(z), \\ S^n f(z) &= S(S^{n-1} f(z)). \end{aligned}$$

Replacing the Ruscheweyh derivative by the Sălăgean derivative in the construction of the Loewner chains (3.8) and (3.25) , applying Theorem 2.1 and using the well known condition $z\left(S^n f(z)\right)' = S^{n+1} f(z)$ we obtain following Theorems 4.1 and 4.2.

Theorem 4.1. *Let $f, g, h \in \mathcal{A}$. Also let $m \in \mathbb{R}_+$ and α, β, c be complex numbers such that*

$$\alpha \neq 1, \quad c \neq -1, \quad \left|\frac{1+c}{1-\alpha} - \frac{m+1}{2}\right| \leqslant \frac{m+1}{2}, \qquad \left|\beta - \frac{m+1}{2}\right| < \frac{m+1}{2}.$$

If the inequalities

$$\left|\left(\frac{(1+c)f'(z)}{[S^n h(z)]' - \alpha} - 1\right) - \frac{m-1}{2}\right| < \frac{m+1}{2}$$

and

$$\left| |z|^{m+1}\left(\frac{(1+c)f'(z)}{[S^n h(z)]' - \alpha} - 1\right) + (1-|z|^{m+1})\left[(\beta-1)\frac{zg'(z)}{g(z)} + \frac{z[S^n h(z)]''}{[S^n h(z)]' - \alpha}\right] - \frac{m-1}{2}\right| \leqslant \frac{m+1}{2}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Theorem 4.2. *Let $f, g, h \in \mathcal{A}$. Also let $m \in \mathbb{R}_+$ and α, β, c be complex numbers such that*

$$c \neq -1, \quad \left|c - \frac{m-1}{2}\right| \leqslant \frac{m+1}{2}, \qquad \left|\beta - \frac{m+1}{2}\right| < \frac{m+1}{2}.$$

If the inequalities

$$\left|\left((1+c)f'(z)\frac{S^v h(z) - \alpha}{S^n f(z) - \alpha} - 1\right) - \frac{m-1}{2}\right| < \frac{m+1}{2}$$

and

$$\left| |z|^{m+1}\left((1+c)f'(z)\frac{S^v h(z) - \alpha}{S^n f(z) - \alpha} - 1\right) + (1-|z|^{m+1})\left[(\beta-1)\frac{zg'(z)}{g(z)} + \frac{S^{n+1} f(z)}{S^n f(z) - \alpha} - \frac{S^{v+1} h(z)}{S^v h(z) - \alpha}\right] - \frac{m-1}{2}\right| \leqslant \frac{m+1}{2}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Different corollaries from Theorem 4.1 and 4.2 can be obtained under suitable choices of m, β, v, α, n and $h(z), g(z)$. For example, if $v = 2$, $n = 1$ and $\alpha = 0$ then Theorem 4.2 reduces to Corollary 4.1.

Corollary 4.1. *Let $f, g \in \mathcal{A}$. Also let $m \in \mathbb{R}_+$ and complex numbers β, c be as in Theorem 4.2. If the inequalities*

$$\left|(1+c)(h'(z)+zh''(z)) - \frac{m+1}{2}\right| < \frac{m+1}{2}$$

and

$$\Bigg| |z|^{m+1}\left((1+c)(h'(z)+zh''(z))-1\right) + \left(1-|z|^{m+1}\right)\left[(\beta-1)\frac{zg'(z)}{g(z)} + \frac{zf''(z)}{f'(z)} - \frac{2z^2h''(z)+z^3h'''(z)}{zh'(z)+z^2h''(z)}\right] - \frac{m-1}{2}\Bigg| \leqslant \frac{m+1}{2}$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.7) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Remark 4.1. In its special case when m, β, v, α, n, $h(z)$ and $g(z)$ in Theorem 3.1 - 4.2 we obtain the some results of Kanas and Srivastava [8].

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