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NONAXISYMMETRIC FREE CONVECTION FLOW OVER A ROTATING DISK IN A VISCOELASTIC FLUID WITH MAGNETIC FIELD*

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Abstract The non axisymmetric motion produced by a buoyancy-induced secondary flow of a viscoelastic fluid over an infinite rotating disk in a vertical plane with a magnetic field applied normal to the disk has been studied. The governing Navier Stokes equations and the energy equation admit a self similar solution. The system of ordinary differential equations has been solved numerically using Runge-Kutta Gill subroutine. The turning moment for the viscoelastic fluid is found to be less than that of the Newtonian fluid but the turning moment is increased due to the magnetic parameter. The resultant force due to the buoyancy-induced secondary flow increases with the magnetic parameter but reduces as the viscoelastic parameter increases. The quantity of fluid, which is pumped outwards due to the centrifuging action of the disk, for the viscoelastic fluid is more than that of the Newtonian fluid. The buoyancy-induced secondary flow boundary layer is much thicker than the primary boundary layer thickness. The thermal boundary layer due to the primary flow increases with the magnetic parameter decreases as the viscoelastic parameter increases. The heat transfer increases with the viscoelastic parameter but decreases as the magnetic parameter increases. The effect of the viscoelastic parameter is more pronounced on the secondary flow than on the primary flow.

Keywords Viscoelastic fluid, three-dimensional flow, rotating disk, non Axisymmetric, free convection, magnetic field.

MSC(2000) 93C73, 76U05 , 76W05, 76A05,76A10,76M20.

1. Introduction

The phenomenon of free convection arises in a fluid when the temperature changes cause density variation which lead to buoyancy forces. This process of heat transfer is encountered in the natural world such as in power transformers, nuclear reactors, etc. Excellent reviews of free convection flows have been given by Ede [8], Gebhart ([10, 11]). The axisymmetric forced convection flow over a rotating disk with or without magnetic field has been studied by von *kármán* [13] ,Cochran [7], Sparrow and Cess [23], Benton [4] and kumar [12]. In rotating flows the axisymmetric character of the flow is destroyed when translational velocities or buoyancy forces are

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applied to a symmetric flow. Such non-axisymmetric flows over rotating disks have been studied by Rott and Lewellen [21] Chawla and Verma [6] and Thacker [25]. Also, the non symmetric flow over a rotating plates and disks without magnetic field and buoyancy forces has been studied by Parter [18] and Rajagopal and Lai ([15, 16]). Turkyilmazoglu in a series of four research articles, considered various aspects of the boundary layer flow due to a infinitely rotating disk in an incompressible viscous fluid. He has obtained exact solutions, analytical solutions as well as the numerical solutions for the rotating disk problem considering the effects of uniform suction and blowing and uniform magnetic [26, 27, 29]. In [28] the three dimensional boundary layer flow of an electrically conducting fluid on a radially stretchable rotating disk is studied.

In recent years, the study of non-Newtonian fluids has gained importance due to their increasing applications in industry. The steady axisymmetric forced convection flow without magnetic field for a viscoelastic fluid has been considered by Elliott [9]. Various aspects of non-Newtonian fluid have been discussed in Pipkin [19], Beard [3] and Tanner [24]. Recently Rajagopal [20] has presented an excellent review of the forced convection flow of a viscoelastic fluid between rotating disks. Recently, Ariel [2] analyzed the two dimensional stagnation point flow of an elastico-viscous fluid with partial slip. Analysis of viscoelastic fluid over a stretching sheet subject to a transverse magnetic field with heat and mass transfer has been studied by Aiboud [1]. Unsteady rotating flow over an impulsively rotating infinite disk with axial magnetic field and suction has been studied by Kumari and Nath [14].

The aim of the present study is to consider the non axisymmetric flow of a viscoelastic fluid over a disk rotating in a vertical plane in the presence of a buoyancy force and a magnetic field. The non axisymmetric motion arises due to the buoyancy induced cross or secondary flow. A set of transformations has been applied which uncouple the momentum and energy equation that result in an independent set of equation that govern the (a) primary axisymmetric flow with an axial magnetic field. (b) an energy equation dependent on the primary flow and (c) secondary cross-flow dependent on both the primary flow and the energy. The system of ordinary differential equations governed by a two-point boundary value problem has been solved numerically using Runge-Kutta-Gill integration scheme. The particular cases of the present results have been compared with those of Cochran [7], Sparrow and Cess [23], Chawla and Verma [6], Thacker [25] and Elliott [9].

2. Governing Equation

We consider a cartesian x,y,z co-ordinate system in a non rotating reference frame where the x-axis is aligned with the negative direction of gravity g, and the z-axis is horizontal. An infinite vertical disk is placed at z=0 in a viscous incompressible viscoelastic fluid. The disk rotates around z-axis with a constant angular velocity ω. A magnetic field of strength B is applied normal to the disk surface. The disk is at a constant temperature T_w and the fluid temperature far from the disk is $T_\infty (T_w > T_\infty)$. The effects of the induced magnetic field, viscous dissipation and joule heating have been neglected (1962). The Navier Stokes equations and the energy equation governing the steady incompressible free convection flow of a viscoelastic fluid (Walter liquid B model) under the Boussinesq approximation can be expressed as

$$u_x + v_y + w_z = 0, \tag{2.1}$$

$$\begin{aligned} uu_x + vu_y + wu_z = {} & -\rho^{-1}p_x + \nu\left(u_{xx} + u_{yy} + u_{zz}\right) + g\beta(T - T_\infty) \\ & - \frac{\sigma B^2 u}{\rho} - \big(\frac{K_0}{\rho}\big)\Big[2\big(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\big)\left(\nabla^2 u\right) \\ & - \nabla^2(uu_x + vu_y + wu_z)\Big], \end{aligned} \tag{2.2}$$

$$\begin{aligned} uv_x + vv_y + wv_z = {} & -\rho^{-1}p_y + \nu\left(v_{xx} + v_{yy} + v_{zz}\right) - \frac{\sigma B^2 v}{\rho} \\ & - \big(\frac{K_0}{\rho}\big)\Big[2\big(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\big)(\nabla^2 v) - \nabla^2\left(uv_x + vv_y + wv_z\right)\Big], \end{aligned} \tag{2.3}$$

$$\begin{aligned} uw_x + vw_y + ww_z = {} & -\rho^{-1}p_z + \nu\left(w_{xx} + w_{yy} + w_{zz}\right) \\ & - \big(\frac{K_0}{\rho}\big)\Big[2\big(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\big)(\nabla^2 w) \\ & - \nabla^2\left(uw_x + vw_y + ww_z\right)\Big], \end{aligned} \tag{2.4}$$

$$uT_x + vT_y + wT_z = \alpha\left(T_{xx} + T_{yy} + T_{zz}\right), \tag{2.5}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{2.6}$$

The boundary conditions are given by

$$\begin{aligned} & u = -y\omega, \quad v = x\omega, \quad w = 0, \quad T = T_w \quad \text{at} \quad z = 0, \\ & u \to 0, \quad v \to 0, \quad T \to T_\infty \quad \text{as} \quad z \to \infty. \end{aligned} \tag{2.7}$$

Here u, v and w are the velocity components in the $x-, y-$ and $z-$direction respectively; p is the pressure; ρ is the density; ν is the kinematic viscosity; g is the acceleration due to gravity; α is the thermal diffusivity of the fluid; β is the coefficient of thermal expansion; K_0 is the viscoelastic parameter; B is the applied magnetic field; T is the temperature; σ is the electrical conductivity; and the subscripts x, y and z denote derivations with respect to x, y and z, respectively.

As mentioned earlier due to the buoyancy force the flow is no longer axisymmetric but becomes non-symmetric. However, it is possible to find transformations which uncouple the momentum and energy equations and reduce the equations (2.1)-(2.5) to a system of ordinary differential equations. The transformations are given by Sparrow and Cess [23] and Thacker [25].

$$\begin{aligned} & u = \omega\left[-2^{-1}xF'(\eta) - yG(\eta)\right] + \left[g\beta\left(T_w - T_\infty\right)/\omega\right]H(\eta), \\ & u = \omega\left[xG(\eta) - 2^{-1}yF'(\eta)\right] + \left[g\beta\left(T_w - T_\infty\right)/\omega\right]N(\eta), \\ & w = (\nu\omega)^{1/2}F(\eta), \quad p = -\rho\omega\nu P(\eta), \quad \eta = \left(\frac{\omega}{\nu}\right)^{1/2}z, \\ & T - T_\infty = \left(T_w - T_\infty\right)\Theta(\eta), \quad K^* = \frac{K_0\omega}{\mu}, \quad M = \frac{\sigma B^2}{\rho\omega}. \end{aligned} \tag{2.8}$$

Applying the above transformations to equations (2.1)-(2.5), we find that (2.1) is satisfied identically and (2.2)-(2.5) reduce to

$$F''' - FF'' + \frac{(F')^2}{2} - 2G^2 - MF' - K^*\Big[\big(\frac{(F')^2}{2} - FF'' - 2G^2\big)'' - 2\big(\frac{F'F''}{2} - FF^{IV} - 2GG''\big)\Big] = 0, \tag{2.9}$$

$$G'' - FG' + F'G - MG - K^*\Big[(F'G - FG')'' - 2\big(\frac{F'G''}{2} + \frac{GF'''}{2} - FG'''\big)\Big] = 0, \tag{2.10}$$

$$Pr^{-1}\Theta'' - F\Theta' = 0, \tag{2.11}$$

$$H'' - FH' + \frac{F'H}{2} + GN - MH + \Theta - K^*\left[\left(\frac{F'H}{2} + GN - FH'\right)'' -2\left(\frac{F'''H}{2} + G''N - FH'''\right)\right] = 0, \tag{2.12}$$

$$N'' - FN' + \frac{F'N}{2} + GH - MN - K^*\left[\left(\frac{F'N}{2} + GH - FN'\right)'' -2\left(\frac{F'''N}{2} + G''H - FN'''\right)\right] = 0, \tag{2.13}$$

$$P' + FF' - F'' = K^*\left[(FF')'' - 2FF'''\right]. \tag{2.14}$$

The boundary conditions are given by

$$\begin{aligned} &F = F' = 0, \quad G = \Theta = 1, \quad H = N = P = 0, && \text{at} \quad \eta = 0, \\ &F' = G = \Theta = H = N = 0, && \text{as} \quad \eta \to \infty. \end{aligned} \tag{2.15}$$

Here F', G and F are the dimensionless velocity components along $x-, y-$, and $z-$ directions, respectively; P is the dimensionless pressure; Θ is the dimensionless temperature; H and N are the buoyancy-induced cross flow velocity components along $x-$ and $y-$ directions, respectively; M is the dimensionless magnetic parameter; Pr is the Prandtl number; K^* is the dimensionless viscoelastic parameter; η is the similarity variable; and prime denotes derivative with respect to η.

Equations (2.9)-(2.11) represent the steady forced convection flow over a rotating disk which is symmetric about the $z-$axis. Equations (2.12) and (2.13) represent nonsymmetric nature of the flow. Equation (2.14) represents the dimensionless pressure and can easily be determined after F along with its derivatives is known from (2.9) and (2.10). Equations (2.9) and (2.10) are non linear and are uncoupled from (2.11) -(2.13) which are linear. Equations (2.9) -(2.11) for $K^* = 0$ (Newtonian fluid) reduce to those of Sparrow and Cess [23] who studied the flow and heat transfer on a rotating disk in the presence of an applied magnetic field. Also equations (2.9) and (2.10) for $K^* = M = 0$ reduce to those of van *Kármán* [13], Cohran [7] and Benton [4] who studied the symmetric primary flow over an infinite rotating disk. Equations (2.9)-(2.13) for $K^* = M = 0$ reduce to those of Chawla and Verma [6] who studied the free convection flow over a rotating disk without a magnetic field. Also (2.9) - (2.13) for $K^* = 0$ reduce to those of Thacker [25] who investigated the free convection flow over a rotating disk taking into account the effect of a magnetic field. Furthermore (2.9) and (2.10) for M=0 reduce to those of Elliott [9] who considered the axisymmetric flow of a viscoelastic fluid over a rotating disk with out a buoyancy force.

Among the quantities of practical interest is the torque required to maintain a steady rotation of the disk. Such a torque is needed to overcome the tangential shear stress imposed by the fluid on the disk surface. The tangential shear stress τ_1 is expressed as Sparrow and Cess [23] and Elliott [9]

$$\tau_1 = \rho \left(\nu\omega^3\right)^{1/2} G'(0). \tag{2.16}$$

In dimensionless form the skin friction coefficient in the tangential direction C_f can be written as

$$C_f = 2\tau_1/\left(\rho r^2\omega^2\right) = -2\left(Re_r\right)^{-1/2} G'(0), \quad Re_r = \omega r^2 \nu. \tag{2.17}$$

Similarly, the radial shear stress τ_2 is expressed in the form Sparrow and Cess [23] and Elliott [9]

$$\tau_2 = -\rho\left(\nu\omega^3\right)^{1/2} F''(0). \tag{2.18}$$

Also in dimensionless form the skin friction coefficient in the radial direction $\overline{C_f}$ is expressed as

$$\overline{C_f} = 2\tau_2/(\rho r^2\omega^2) = -2\left(Re_r\right)^{-1/2} F''(0). \tag{2.19}$$

Although the results are strictly applicable to an infinite disk only, we may use the some results for a finite disk, provided that the radius R is large compared with thickness S of the layer carried with the disk. On a disk of radius R, the torque T_p associated with the primary flow is

$$T_p = -2^{-1}\rho\pi R^4(\nu\omega^3)^{1/2}G'(0). \tag{2.20}$$

In dimensionless form, the torque or moment coefficient C_T is given by

$$\begin{aligned} C_T &= 4T_p\left(\rho\omega^2 R^5\right) = -2\pi\left(Re_R\right)^{-1/2} G'(0), \\ Re_R &= \omega R^2/\nu. \end{aligned} \tag{2.21}$$

Due to the buoyancy induced secondary flow on the disk with R (neglecting the edge effects), the resultant force RF is given by Sparrow and Cess [23] and Thacker [25]

$$RF = \rho g\beta\pi R^2\left(T_w - T_\infty\right)\left(\nu/\Omega\right)^{1/2}\left[\left(H'(0)\right)^2 + \left(N'(0)\right)^2\right]^{1/2}, \tag{2.22}$$

where $H'(0)$ and $N'(0)$ are the buoyancy induced shear stress in $x-$ and $y-$ directions, respectively. In dimensionless form, the resultant force $\overline{RF}$ is expressed as

$$\overline{RF} = \pi Gr_R\left(Re_R\right)^{-5/2}\left[\left(H'(0)\right)^2 + \left(N'(0)\right)^2\right]^{1/2}, \tag{2.23}$$

where

$$\overline{RF} = RF/\left(\rho\omega^2 R^4\right), \quad Gr_R = g\beta\left(T_w - T_\infty\right)R^3/\nu^2. \tag{2.24}$$

The quantity of fluid Q which is pumped outwards as a result of the centrifuging motion on one side of the disk of radius is

$$Q = 2\pi\omega R^3\left(Re_R\right)^{-1/2} F(\infty). \tag{2.25}$$

In dimensionless form it can be expressed as

$$\overline{Q} = Q/\omega R^3 = \pi \left(Re_R\right)^{-1/2} F(\infty). \tag{2.26}$$

The heat transfer coefficient in terms of Nusslet number is expressed as

$$Nu = -R\left(\frac{\partial T}{\partial z}\right)_{z=0} / (T_w - T_\infty) = -\left(Re_R\right)^{1/2} G'(0). \tag{2.27}$$

3. Method of Solution

It may be remarked that the presence of elasticity ($K^* > 0$) gives differential equations (equations (2.9), (2.10), (2.12)-(2.14)) which are one order higher than in the viscous case ($K^* = 0$). This results in an additional boundary condition being required in order to obtain the solution. Hence following Beard and Walters [3] and Elliott [9], we use a regular perturbation technique and assume the solution of the form

$$\begin{aligned}
F &= F_0 + K^* F_1 + \left(K^*\right)^2 F_2 + \left(K^*\right)^3 F_3 + \cdots, \\
G &= G_0 + K^* G_1 + \left(K^*\right)^2 G_2 + \left(K^*\right)^3 G_3 + \cdots, \\
\Theta &= \Theta_0 + K^* \Theta_1 + \left(K^*\right)^2 \Theta_2 + \left(K^*\right)^3 \Theta_3 + \cdots, \\
H &= H_0 + K^* H_1 + \left(K^*\right)^2 H_2 + \left(K^*\right)^3 H_3 + \cdots, \\
N &= N_0 + K^* N_1 + \left(K^*\right)^2 N_2 + \left(K^*\right)^3 N_3 + \cdots, \\
P &= P_0 + K^* P_1 + \left(K^*\right)^2 P_2 + \left(K^*\right)^3 P_3 + \cdots,
\end{aligned} \tag{3.1}$$

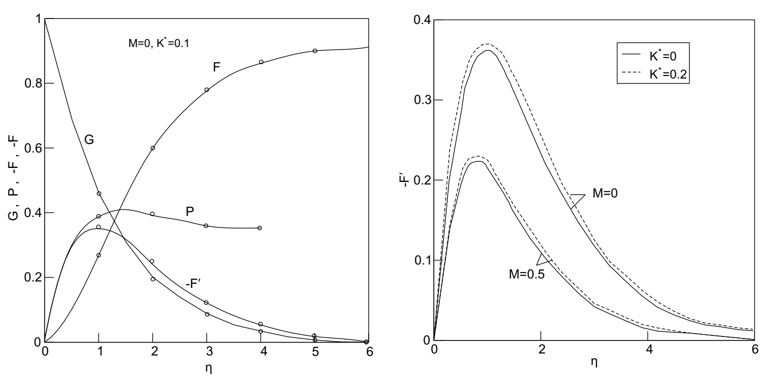
which is valid for ($K^* << 1$). We substitute (3.1) in (2.9)-(2.15) and equate the coefficients of various powers of K^* which results in the system of equations given in the Appendix. It may be noted that the first two equations (see Appendix) are non linear and the remaining equations are linear. The non linear equations under boundary conditions have been solved by shooting method with Newton's correction formula for the guessed values of the unknown boundary conditions. The numerical integration is carried out using Runge-Kutta-Gill subroutine. The remaining linear equation have been solved as an initial value problem using the superposition principle as explained in Na [17]. The integration has been performed by Runge-Kutta-Gill subroutine as in the case of non linear equations.

4. Results and Discussion

In order to test the accuracy of our method we have compared the radial, tangential and axial velocity profiles (F', G, F) for the axially symmetric flow when $M = K^* = 0$ with those of Cochran [7]. For Newtonian fluids ($K^* = 0$), the radial and tangential shear stresses for primary flow $(-F''(0), -G'(0))$ and heat transfer $(-\Theta'(0))$ have been compared with those of Sparrow and Cess [23]. The heat transfer $(-\Theta'(0))$ and the shear stresses due to cross or secondary flow $(H'(0), -N'(0))$ for $M = K^* = 0$ have been compared with those of Chawla and Verma [6] and for $K^* = 0,\ M > 0$ with those of Thacker [25]. Also for $M = 0,\ K^* > 0$ the velocity profiles (F', G, F) have been compared with those of Elliott [9]. In all the cases the results are found to be in excellent agreement. The maximum difference is found to be less than one per cent. The comparison shown in Tables 1-3 and Fig 1(a). For the sake of brevity, the comparison with those of Cochran [7] is not presented here.

M	Present Results			Sparrow and Cess [23]		
	$-F''(0)$	$-G'(0)$	$-\Theta'(0)$	$-F''(0)$	$-G'(0)$	$-\Theta'(0)$
0	1.0211	0.6161	0.3964	1.021	0.616	0.3960
0.5	0.7701	0.8492	0.2824	0.770	0.849	0.2820
1.0	0.6201	1.0691	0.1943	0.619	1.069	0.1940
2.0	0.4613	1.4422	0.0985	0.461	1.442	0.0982
3.0	0.3812	1.7483	0.0592	0.381	1.748	0.0588
4.0	0.3309	2.0104	0.0399	0.331	2.010	0.0395

Table 1. Comparison of radial and tangential shear stresses for primary flow $(-F''(0), -G'(0))$ and heat transfer $(-\Theta'(0))$ when $K^* = 0,\ Pr = 1$.



(a) (b)

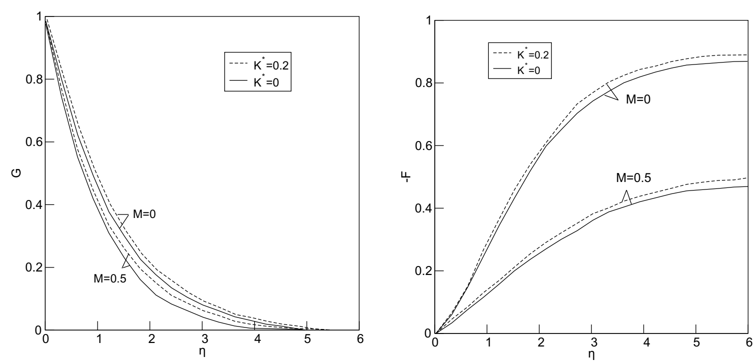
Figure 1. (a) Comparison of radial, tangential and axial velocity profiles $(-F', G, -F)$ and pressure profiles for (P) for $M = 0,\ K^* = 0.1$. (b) Radial velocity profile $(-F')$ for M=0, 0.5, K^*=0, 0.2.

The effects of the viscoelastic parameter K^* and the magnetic parameter M on the velocity profiles (F', G, F) of the primary Von K*á*rm*á*n flow (i.e., axially symmetric flow)are shown in Figures 1(b), 2(a) and 2(b). It is observed that there is a general reduction in the fluid velocity profiles (F', G, F) with increasing magnetic parameter M. Also the thickness of the velocity boundary layer decreases with the increase in the value of the magnetic parameter. The effect of the viscoelastic parameter K^* is found to be just opposite. Similar trend for $K^* = 0,\ M > 0$ has been observed by Sparrow and Cess [23] and for $M = 0,\ K* = 0$ by Elliott [9]. The above behavior of velocity field can be explained by examining the details of the flow field. The rotating disk acts like a fan, drawing fluid axially inward from the surroundings towards the disk surface. Since the surface of the disk is impermeable, the incoming fluid is turned and discharges in the radial direction. Thus there is a close relationship between the axial inflow and the radial outflow. Since the magnetic field is axial there are no magnetic forces operating to retard the axial flow. But the radial component of the magnetic force opposes the radial velocity. The reduction in the radial velocity tends to decrease the incoming axial

Pr	Present Results			Thacker [25]		
	$-\Theta'(0)$	$H'(0)$	$-N'(0)$	$-\Theta'(0)$	$H'(0)$	$-N'(0)$
0.7	0.3231	1.0673	0.3618	0.3231	1.0673	0.3617
1.0	0.3964	0.9176	0.2749	0.3962	0.9178	0.2748
3.0	0.6827	0.6131	0.1275	0.6826	0.6131	0.1278
5.0	0.8536	0.5163	0.0919	0.8533	0.5161	0.0916

Table 2. Comparison of heat transfer $(-\Theta'(0))$ and buoyancy induced cross-flow shear stresses $(H'(0), -N'(0))$ when $M = K^* = 0$.

velocity. Since there is less fluid flow, the turning from axial to radial velocity takes place closer to the disk surface. Thus the inflow velocity remains constant to within smaller distances of the disk surface as the magnetic field increases. The magnetic field induces a magnetic force in the tangential direction which tends to oppose the tangential fluid velocity. Thus the tangential velocity is reduced and the boundary layer becomes thinner.



(a) (b)

Figure 2. (a) Tangential velocity profile G for M=0, 0.5, K^*=0, 0.2 (b) Axial velocity profile (-F) for M=0, 0.5, K^*=0, 0.2.

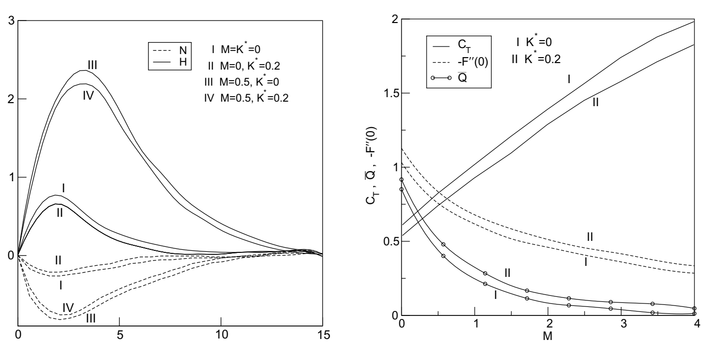
The effects of the viscoelastic parameter K^* and the magnetic parameter M on the buoyancy induced secondary (cross) flow profiles (H, N) and the temperature profile (Θ) are presented in Figure 3(a). It is seen that the secondary flow profile in x-direction $H > 0$ and the secondary flow profile in the y-direction $N < 0$ in the entire region of the flow field. Also $max\ |H| >> |N|$. For example, when $M = 0.5,\ K^* = 0.2,\ Pr = 0.7\ max\ |H|$ is about 0.3 times more than $max\ |N|$. Similar trend has been observed by Chawla and Verma [6] and Thacker [25] for the Newtonian fluid $(K^* = 0)$. For a fixed K^*, the thickness of the cross flow induced thermal boundary layer increases as the magnetic parameter M increases. Similarly, for a fixed M, the thickness of the secondary flow induced thermal boundary layer increases as the viscoelastic parameter K^* increases.

Also the buoyancy induced cross (secondary) flow boundary layer is more thicker than the primary boundary layer thickness. For example, for $K^* = 0.2,\ M = 0.5,\ Pr = 0.7$, the secondary flow boundary layers are about 2.5 times thicker than those of the primary flow. The effects of the viscoelastic parameter K^* and the magnetic parameter M on the turning moment C_T, the quantity of the fluid

M	Pr	Present Results			Thacker [25]		
		$-\Theta'(0)$	$H'(0)$	$-N'(0)$	$-\Theta'(0)$	$H'(0)$	$-N'(0)$
0.5	0.7	0.2207	01.7863	0.8419	0.2209	01.7864	0.8418
0.5	1.0	0.2829	01.4291	0.5984	0.2827	01.4295	0.5981
0.5	3.0	0.5392	00.8184	0.2295	0.5389	00.8180	0.2299
0.5	5.0	0.6945	00.6596	0.1541	0.6948	00.6599	0.1544
1.0	0.7	0.1447	03.4641	1.7878	0.1449	03.4642	1.7875
1.0	1.0	0.1927	02.5755	1.2321	0.1930	02.5759	1.2318
1.0	3.0	0.4155	01.1854	0.4061	0.4153	01.1859	0.4065
1.0	5.0	0.5586	00.8891	0.2536	0.5589	00.8889	0.2533
3.0	0.7	0.0414	20.1719	6.0942	0.0417	20.1435	6.0947
3.0	1.0	0.0589	14.1172	4.2264	0.0587	14.1388	4.2292
3.0	3.0	0.1607	04.8251	1.3424	0.1609	04.8257	1.3428
3.0	5.0	0.2475	02.9827	0.7771	0.2478	02.9831	0.7775

Table 3. Comparison of heat transfer ($-\Theta'(0)$) and buoyancy induced cross-flow shear stresses ($H'(0), -N'(0)$) when $K^* = 0, M > 0$.

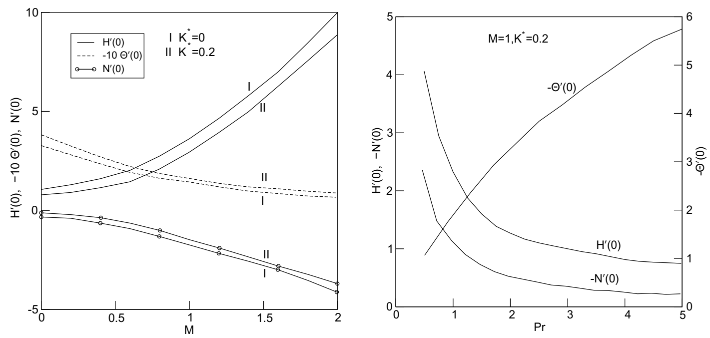
that is pumped outwards from the disk ($\overline{Q}$) and the radial shear stress ($-F''(0)$) are presented in Figure 3(b). It is seen that for a fixed M, C_T decreases as K^* increases which implies that C_T for the viscoelastic fluid ($K^* > 0$) is less than that of the Newtonian fluid ($K^* = 0$). However, $\overline{Q}$ and $-F''(0)$ increase with K^*. The effect of the magnetic parameter M is found to be just opposite to that of K^*. Similar trend has been observed by Sparrow and Cess [23] for $K^* = 0$ and by Elliott [9] for $M = 0$.



(a) (b)

Figure 3. (a) Buoyancy induced secondary flow profiles (H,N) for Pr=0.7, M=0, 0.5, K^*=0, 0.2. (b) Turning moment (C_T), amount of fluid that is pumped outwards from the disk ($\overline{Q}$), and tangential shear stress ($-F''(0)$) for K^* =0, 0.2.

The effects of the viscoelastic parameter K^* and the magnetic parameter M on the heat transfer ($-\Theta'(0)$) and the buoyancy induced cross (secondary) flow shear stresses ($H'(0)), -N'(0)$) are shown in Figure 4(a). The heat transfer ($-\Theta'(0)$), increases with K^* due to the reduction in the thermal boundary layer thickness. On the other hand, $-\Theta'(0)$ decreases as M increases due to the increase in the thermal boundary layer thickness. Similar trend has been observed by Thacker [25] for Newtonian fluids ($K^* = 0$). The shear stress components $H'(0)$ and $-N'(0)$



(a) (b)

Figure 4. (a) Buoyancy induced shear stress components $(H'(0), -N'(0))$ and heat transfer parameter $(-\Theta'(0))$ for Pr=0.7, K^*=0, 0.2 (b) Buoyancy induced shear stress components $(H'(0), -N'(0))$ and heat transfer parameter $(-\Theta'(0))$ for M=1, K^*=0.2.

associated with the buoyancy-induced secondary flow increase with M but reduce as K^* increases. Also the resultant force $\overline{FR}$ increases with M but reduces with K^*.

The effect of the Prandtl number on the heat transfer $(-\Theta'(0))$ and the shear stress components due to the buoyancy induced cross flow (or secondary flow) $H'(0)$ and $-N'(0)$ is shown in Figure 4(b). It is found that the heat transfer $(-\Theta'(0))$ increases with Pr. Since a higher Prandtl number fluid has a relatively lower thermal conductivity which reduces conduction and thereby increases the variations. This results in thinner thermal boundary layer and increase in the convective heat transfer at the wall. However, the shear stress components due to the secondary flow $(H'(0), -N'(0))$ decrease as Pr increases. Similar trend has been observed by Chawla and Verma [6].

5. Conclusion

Many flows generated by rotation or free convection in axisymmetric enclosures with axisymmetric boundary conditions break into non-axisymmetric patterns above a certain threshold of the governing parameters. Hence the non axisymmetric motion produced by a buoyancy induced secondary flow of a viscoelastic fluid over a rotating disk has been studied. A magnetic filed of strength B is applied normal to the disk surface. The buoyancy induced secondary flow boundary layer due to the rotating disk is much thicker than the primary boundary layer thickness. The quantity of fluid which is pumped outwards due to the centrifuging action of the rotating disk is more for viscoelastic fluid compared to Newtonian fluid. The effect of magnetic parameter and the viscoelastic parameter on the velocity and temperature profiles as well as the skin friction and heat transfer rates are presented.

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A. Appendix

$(K^*)^0$:

$$F_0''' - F_0F_0'' + (F_0')^2/2 - 2G_0^2 - MF_0' = 0, \tag{A.1}$$
$$G_0'' - F_0G_0' + F_0'G_0 - MG_0 = 0, \tag{A.2}$$
$$Pr^{-1}\Theta_0'' - F_0\Theta_0' = 0, \tag{A.3}$$
$$H_0'' - F_0H_0' + F_0'H_0/2 + G_0N_0 + \Theta_0 = 0, \tag{A.4}$$
$$N_0'' - F_0N_0' + F_0'N_0/2 - G_0H_0 = 0, \tag{A.5}$$
$$P_0' + F_0F_0' - F_0'' = 0. \tag{A.6}$$

$(K^*)^1$:

$$\begin{aligned}&F_1''' - F_0F_1'' - F_0''F_1 + F_0'F_1' - 4G_0G_1 - MF_1'\\ &= \left[F_0F_0^{IV} - 2F_0'F_0''' - 4\left(G_0'\right)^2\right],\end{aligned} \tag{A.7}$$
$$\begin{aligned}&G_1'' - F_0G_1' - G_0'F_1 + F_0'G_1 + G_0F_1' - MG_1\\ &= \left[F_0G_0''' - 2F_0'G_0'' + F_0''G_0'\right],\end{aligned} \tag{A.8}$$
$$Pr^{-1}\Theta_1'' - F_0\Theta_1' - F_1\Theta_0' = 0, \tag{A.9}$$
$$\begin{aligned}&H_1'' - F_0H_1' - H_0'F_1 + \left(F_0'H_1 + H_0F_1'\right)/2 + G_0N_1 + N_0G_1 + \Theta_1\\ &= \left[F_0H_0''' - (3/2)\,F_0'H_0'' - F_0'''H_0/2 + G_0N_0'' + 2G_0'N_0' - G_0''N_0\right],\end{aligned} \tag{A.10}$$
$$\begin{aligned}&N_1'' - F_0N_1' - N_0'F_1 + \left(F_0'N_1 + N_0F_1'\right)/2 - G_0H_1 - N_0G_1\\ &= \left[F_0N_0''' - (3/2)\,F_0'N_0'' - F_0'''N_0/2 - G_0H_0'' - 2G_0'H_0' + G_0''H_0\right],\end{aligned} \tag{A.11}$$
$$P_1' = F_1'' - F_0F_1' - F_0'F_1 + F_0F_0''' + 3F_0'F_0''. \tag{A.12}$$

$(K^*)^2$:

$$\begin{aligned}&F_2''' - \left(F_0F_2'' + F_0''F_2 + F_1F''\right) - \left(2F_0'F_2' + (F_1')^2\right)/2 - 2\left(2G_0G_2 + G_1^2\right) - MF_2\\ &= \left[F_0F_1^{IV} + F_0^{IV}F_1 - 2\left(F_0'F_1''' + F_0'''F_1'\right) - 8G_0'G_1'\right],\end{aligned} \tag{A.13}$$
$$\begin{aligned}&G_2'' - \left(F_0G_2' + G_0'F_2\right) + \left(F_0'G_2 + G_0F_2'\right) - MG_2\\ &= \left[F_0G_1''' + G_0'''F_1 + F_0''G_1' + G_0'F_1'' - 2\left(F_0'G_1'' + G_0''F_1'\right)\right],\end{aligned} \tag{A.14}$$
$$Pr^{-1}\Theta_2'' - \left(F_0\Theta_2' + \Theta_0'F_2\right) = -2F_1\Theta_1', \tag{A.15}$$
$$\begin{aligned}&H_2'' - \left(F_0H_2' + H_0'F_2 + F_1H_1'\right) + \left(F_0'H_2 + H_0F_2' + F_1'H_1\right)/2\\ &+ \left(G_0N_2 + N_0G_2 + G_1N_1\right) + \Theta_2 = \left[F_0H_1''' + H_0'''F_1 - (3/2)\left(F_0'H_1'' + H_0''F_1'\right)\right.\\ &\left.- \left(F_0'''H_1 + H_0F_1'''\right)/2 + G_0N_1'' + N_0''G_1 + 2\left(G_0'N_1' + N_0'G_1'\right) - G_0''N_1 - N_0G_1''\right]\end{aligned} \tag{A.16}$$
$$\begin{aligned}&N_2'' - \left(F_0N_2' + N_0'F_2 + F_1N_1'\right) + \left(F_0'N_2 + N_0F_2' + F_1'N_1\right)/2\\ &- \left(G_0H_2 + H_0G_2 + G_1H_1\right) = \left[F_0N_1''' + N_0'''F_1 - (3/2)\left(F_0'N_1'' + N_0''F_1'\right)\right.\\ &\left.- \left(F_0'''N_1 + N_0F_1'''\right)/2 - \left(G_0H_1'' + H_0''G_1\right) - 2\left(G_0'H_1' + H_0'G_1'\right) + G_0''H_1 + H_0G_1''\right]\end{aligned} \tag{A.17}$$
$$P_2' + F_0F_2' + F_0'F_2 + F_1F_1' - F_2'' = \left[F_0F_1''' + F_0'''F_1 + 3\left(F_0'F_1'' + F_0''F_1'\right)\right] \tag{A.18}$$

$(K^*)^3$:

$$\begin{aligned}
&F_3''' - (F_0F_3'' + F_0''F_3 + F_1F_2'' + F_2F_1'') + (F_0'F_3' + F_1F_2') \\
&- 4\,(G_0G_3 + G_1G_2) - MF_3' - [F_0F_2^{IV} + F_0^{IV}F_2 + F_1F_1^{IV} \\
&- 2\,(F_0'F_2''' + F_0'''F_2' + F_1'F_1''') - 4\left((G_1')^2 + G_0'G_2'\right)] = 0,
\end{aligned} \tag{A.19}$$

$$\begin{aligned}
&G_3'' - (F_0G_3' + G_0'F_3 + F_1G_2' + F_2G_1') + (F_0'G_3 + G_0F_3' + F_1'G_2 + F_2'G_1) \\
&- MG_3 - [F_0G_2''' + G_0'''F_2 + F_1G_1''' - 2\,(F_0'G_2'' + G_0''F_2' + F_1'G_1'') \\
&+ (F_0''G_2' + G_0'F_2'' + F_1''G_1')] = 0,
\end{aligned} \tag{A.20}$$

$$Pr^{-1}\Theta_3'' - (F_0\Theta_3' + F_3\Theta_0' + F_1\Theta_2' + F_2\Theta_1') = 0, \tag{A.21}$$

$$\begin{aligned}
&H_3'' - (F_0H_3' + H_0'F_3 + F_1H_2' + F_2H_1') + 2^{-1}\,(F_0'H_3 + H_0F_3' + F_1'H_2 + F_2'H_1) \\
&+ (G_0N_3 + N_0G_3 + G_1N_2 + G_2N_1) + \Theta_3 - [F_0H_2''' + H_0'''F_2 + F_1H_1''' \\
&-(3/2)\,(F_0'H_2'' + H_0''F_2' + F_1'H_1'') - (1/2)\,(F_0'''H_2 + H_0F_2''' + F_1'''H_1) \\
&+ (G_0N_2'' + N_0''G_2 + G_1N_1'') + 2\,(G_0'N_2' + N_0'G_2' + G_1'H_1') - \\
&(G_0''N_2 + N_0G_2'' + G_1''N_1)] = 0,
\end{aligned} \tag{A.22}$$

$$\begin{aligned}
&N_3'' - (F_0N_3' + N_0'F_3 + F_1N_2' + F_2N_1') + 2^{-1}\,(F_0'N_3 + N_0F_3' + F_1'N_2 + F_2'N_1) \\
&- (G_0H_3 + H_0G_3 + G_1H_2 + G_2H_1) - [F_0N_2''' + N_0'''F_2 + F_1N_1''' \\
&- (3/2)\,(F_0'N_2'' + N_0''F_2' + F_1'N_1'') - (1/2)\,(F_0'''N_2 + N_0F_2''' + F_1'''N_1) \\
&- (G_0H_2'' + H_0''G_2 + G_1H_1'') - 2\,(G_0'H_2' + H_0'G_2' + G_1'H_1') \\
&+ (G_0''H_2 + H_0G_2'' + G_1''H_1)] = 0,
\end{aligned} \tag{A.23}$$

$$\begin{aligned}
&P_3' + (F_0F_3' + F_0'F_3 + F_1F_2' + F_2F_1') - F_3'' - [(F_0F_2'' + F_0'''F_2 + F_1F_1''') \\
&+3\,(F_0'F_2'' + F_0''F_2' + F_1'F_1'')]\,.
\end{aligned} \tag{A.24}$$

The boundary conditions are given by

$$F(0) = F'(0) = G_0 - 1 = \Theta_0 - 1 = P_0 = H_0 = N_0 = 0, \quad \text{at} \quad \eta = 0, \tag{A.25}$$

$$F_0' = G_0 = \Theta_0 = H_0 = N_0 = 0, \quad \text{as} \quad \eta \to \infty. \tag{A.26}$$

For n=1,2,3.

$$F_n = F_n' = G_n = \Theta_n = P_n = H_n = N_n = 0, \quad \text{at} \quad \eta = 0, \tag{A.27}$$

$$F_n' = G_n = \Theta_n = H_n = N_n = 0, \quad \text{as} \quad \eta \to \infty. \tag{A.28}$$

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