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THE EXISTENCE OF PULLBACK EXPONENTIAL ATTRACTORS FOR NONAUTONOMOUS DYNAMICAL SYSTEM AND APPLICATIONS TO NONAUTONOMOUS REACTION DIFFUSION EQUATIONS*

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Abstract First we establish some sufficient conditions for the existence of pullback exponential attractors by using $\omega-$limit compactness in the framework of process. Then we provide a new method to prove the existence of pullback exponential attractors. As a simple application, we prove the existence of pullback exponential attractors for nonautonomous reaction diffusion equations in H_0^1.

Keywords Dynamical system, pullback exponential attractors, pullback ω-limit compactness, reaction diffusion equation.

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1. Introduction

One of the most important problems in infinite dimension dynamical or semigroup generated by autonomous partial differential equations is to prove the existence of global attractor(see [1,13,17,22,24]), which is a compact invariant set attracting all bounded subsets of the phase space. To be more precise, let $S(t) : X \to X, t \geq 0$, be an operator in a metric space (X, d), we call the family $\{S(t)|t \geq 0\}$ be a semigroup in X if it satisfied the properties

$$S(t)S(\tau) = S(t + \tau), t, \tau \geq 0, \quad S(0) = Id,$$

where Id denotes the identity operator in X.

The subset $\mathcal{A} \subset X$ is the global attractor for the semigroup $\{S(t)|t \geq 0\}$ if $\mathcal{A} \neq \emptyset$ is compact, strictly invariant, that is $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and every bounded subset $D \subset X$

$$\lim_{t\to\infty} dist(S(t)D, \mathcal{A}) = 0.$$

Here, $dist(\cdot, \cdot)$ is the Hausdorff semidistance in X; that is $dist(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b)$. However, global attractor attract any bounded set of phase space, but the attraction to it may be arbitrarily slow. In order to describe the attracts speed, the

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concept of exponential attractors(see [7, 9–11, 18, 25, 26]) is put forward, which is a compact, positively invariant set of finite fractal dimension(see Definition 2.6) and exponentially attracts each bounded subset. The subset $\mathcal{M} \subset X$ is the exponential attractor for the semigroup $\{S(t)|t \geq 0\}$ if $\mathcal{M} \neq \emptyset$ is compact and has finite fractal dimension, semi-invariant, that is $S(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$, and there exists a constant $k > 0$ such that, for every bounded $B \subset X$, there exists a constant $c = c(B)$ and

$$dist(S(t)B, \mathcal{M}) \leq ce^{-kt}.$$

We note that the existence of an exponential attractor $\mathcal{M}$ for semigroup implies the existence of global attractor $\mathcal{A}$ and $\mathcal{A} \subset \mathcal{M}$. In contrast to the global attractor, an exponential attractor is not uniquely defined.

Recently, many authors have paid much attention to more general nonautonomous differential equations and the processes generated by them. Different approaches were made to find the counterpart of the global attractors in this case. Pullback attractors(see Definition 2.1, Definition 2.3, or [2–4, 14, 15, 19, 21, 23]), as a suitable notion, describes nonautonomous dynamical systems, which is a minimal family of compact invariant sets under the process and pullback attracting each bounded subset of the phase space.

As the global attractors, the attraction of the pullback attractors to it may be arbitrarily slow. Like autonomous case, to overcome this drawback creates the notion of the pullback exponential attractors(see Definition 2.7 or [5, 14]) which is a family of nonempty compact and positively semi-invariant sets under the process that fractal dimension uniformly bounded for all times and pullback attracts each bounded subset of the phase space at an uniform exponential rate.

In [5, 14], the authors proved the existence of pullback exponential attractors under some suitable conditions and applied it to some semilinear parabolic problem. These methods need the process $U(t, \tau)$ satisfies some strictly conditions. For example, in [5, 14], which need the process satisfies

$$\sup_{\tau \in T} \|U(\tau, \tau - T_B)u_1 - U(\tau, \tau - T_B)u_2\|_V \leq k\|u_1 - u_2\|_W, u_1, u_2 \in B,$$

here V is compactly embedded in W, B is the uniformly bounded absorbing set for the process and T_B is the time corresponding to the absorbing set B. In fact, we could not get the inequality for reaction diffusion equation.

As far as I know, only a few articles [5, 14, 20] study the existence of pullback exponential attractors, to many equations it is difficulty to get the existence of pullback exponential attractors by using these methods. In this paper, we study the asymptotic behavior of nonautonomous dynamical system in the framework of process. We prove that the process exists pullback exponential attractor under suitable conditions by using pullback ω-limit compactness method. Then we present a new method to prove the existence of pullback exponential attractors. As a simple application, we prove the existence of pullback exponential attractors for nonautonomous reaction diffusion equations.

The paper is organized as follows: In section 2, we recall some basic concept about pullback attractors and pullback exponential attractors for a process. In section 3, we prove the existence of pullback exponential attractors in Banach space and provided a new method to verify the existence of it. In section 4, we apply our result to a non-autonomous reaction diffusion system and get the existence of pullback exponential attractors.

2. Preliminaries

Let X be a complete metric space, $B(X)$ be the set of all bounded subsets of X, and a two-parameter family of mappings $\{U(t,\tau)|t \geq \tau\} = \{U(t,\tau) : t \geq \tau, t, \tau \in \mathbb{R}\}$ act on X : $U(t,\tau) : X \to X, t \geq \tau, \tau \in \mathbb{R}$.

Definition 2.1. A two-parameter family of mappings $\{U(t,\tau)\}$ is said to be a process in X, if

(1) $U(t,s)U(s,\tau) = U(t,\tau), \forall t \geq s \geq \tau$,

(2) $U(\tau,\tau) = Id$, is the identity operator, $\tau \in \mathbb{R}$.

The pair $(U(t,\tau), X)$ is generally referred to as a nonautonomous dynamical system, and $(U(n,m), X)(n,m \in \mathbb{N})$ is called a non-autonomous discrete dynamical system generated by $(U(t,\tau), X)$. If $x \to U(t,\tau)x$ is continuous in X, we say that the process is continuous process; if $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$ as $x_n \to x$, we say that the process is the norm to weak continuous process. Obviously, continuous process is also a norm to weak continuous process.

Definition 2.2. A family of bounded sets $\{B(t)|t \in \mathbb{R}\} \subset B(X)$ are called pullback absorbing sets for the process $(U(t,\tau), X)$ if for any $t \in \mathbb{R}$, and any bounded set $B \subset X$, there exists a $\tau_0(t,B) \leq t$ such that $U(t,\tau)B \subset B(t)$ for all $\tau \leq \tau_0$.

Definition 2.3. The family $\mathcal{A} = \{\mathcal{A}(t)|t \in \mathbb{R}\} \subset B(X)$ is said to be a pullback attractor for $U(t,\tau)$ if

(1) $\mathcal{A}(t)$ is compact for all $t \in \mathbb{R}$.

(2) $\mathcal{A}$ is invariant, i.e.,

$$U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t) \quad \text{for all} \quad t \geq \tau.$$

(3) $\mathcal{A}$ is pullback attracting, i.e.,

$$\lim_{\tau \to -\infty} dist((U(t,\tau)B, \mathcal{A}(t)) = 0, \quad \text{for all} \;\; B \in B(X), \text{ and all } t \in \mathbb{R}.$$

(4) if $\{C(t)\}_{t\in\mathbb{R}}$ is another family of closed attracting sets, then $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Now, we briefly review the basic concept about the Kuratowski measure of non-compactness and restate its basic property, which will be used to characterize the existence of pullback exponential attractors for non-autonomous dynamical system.

Let X be a Banach space and B be a bounded subset of X. The Kuratowski measure of non-compactness $\alpha(B)$ of B is defined by

$$\alpha(B) = \inf\{\delta > 0 \;|B \text{ admits a finite cover by sets of diameter} \leq \delta\}.$$

The following summarizes some of the basic properties of this measure of non-compactness.

Lemma 2.1 ([8])**.** *Let $B,\ B_1,\ B_2 \subset X$. Then*

(1) $\alpha(B) = 0$ *if, and only if,* $\overline{B}$ *is compact;*

(2) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$*;*

(3) $\alpha(B_1) \leq \alpha(B_2)$ *for* $B_1 \subset B_2$*;*

(4) $\alpha(B_1 \bigcup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$*;*

(5) If $F_1 \supset F_2 \ldots$ *are non-empty closed sets in* X *such that* $\alpha(F_n) \to 0$ *as* $n \to \infty$*, then* $F = \bigcap_{n=1}^{\infty} F_n$ *is nonempty and compact.*

In addition, let X*be an infinite dimensional Banach space with a decomposition* $X = X_1 \oplus X_2$ *and let* $P : X \to X_1$*,* $Q : X \to X_2$ *be projectors with* $dim X_1 < \infty$*. Then*

(6) $\alpha(B(\varepsilon))) = 2\varepsilon$*, where* $B(\varepsilon)$ *is a ball of radius* ε*.*

(7) $\alpha(B) < \varepsilon$ *for any bounded subset* B *of* X *for which the diameter of* QB *is less than* ε*.*

Definition 2.4. A process $\{U(t,\tau)\}$ is called pullback $\omega-$limit compact if for any $\varepsilon > 0$ and $B \in B(X)$, there exists a $\tau_0(t,B) \leq t$ such that $\alpha(\bigcup\limits_{\tau\leq\tau_0} U(t,\tau)B) \leq \varepsilon$.

Definition 2.5. A process $\{U(t,\tau)\}$ is called pullback $\omega - \mathcal{D}-$limit compact for $\{B(t)|t \in \mathbb{R}\}$if for any $\varepsilon > 0$, there exists a $\tau_0(t,\widehat{B}) \leq t$ such that $\alpha(\bigcup\limits_{\tau\leq\tau_0} U(t,\tau)B(\tau))$ $\leq \varepsilon$.

Lemma 2.2 ([4, 15, 23])**.** *Assume that the process* $\{U(t,\tau)|t \geq \tau\}$ *is pullback* $\omega-$*limit compact, then for any sequence* $\{\tau_n\} \subset (-\infty, t]$*,* $\tau_n \to -\infty$ *as* $n \to +\infty$*, and any sequence* $\{x_n\} \subset B$*, there exists a convergence subsequence of* $\{U(t,\tau_n)x_n\}$ *whose limit lies in* $\omega(B,t)$*, here* $\omega(B,t)$ *defined by*

$$\omega(B,t) = \bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s} U(t,\tau)B}.$$

Lemma 2.3 ([4,15,23])**.** *Assume that the process* $\{U(t,\tau)|t \geq \tau\}$ *is pullback* $\omega - \mathcal{D}-$*limit compact for* $\{B(t)|t \in \mathbb{R}\}$*, then for any sequence* $\{\tau_n\} \subset (-\infty, t]$*,* $\tau_n \to -\infty$ *as* $n \to +\infty$*, and any sequence* $\{x_n \in B(\tau_n)\}$*, there exists a convergence subsequence of* $\{U(t,\tau_n)x_n\}$ *whose limit lies in* $\omega(\widehat{B},t)$*, here* $\omega(\widehat{B},t)$ *defined by*

$$\omega(\widehat{B},t) = \bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s} U(t,\tau)B(\tau)}.$$

Theorem 2.1 ([4,15,23])**.** *Let* $\{U(t,\tau)|t \geq \tau\}$ *be a continuous or norm-to-weak continuous process and* $\{U(t,\tau)|t \geq \tau\}$ *is pullback* $\omega-$*limit compact,* $\{B(t)|t \in \mathbb{R}\} \subset B(X)$ *be a family of pullback bounded absorbing sets for the process.Then the process* $\{U(t,\tau)|t \geq \tau\}$ *exists pullback attractor* $\mathcal{A} = \{\mathcal{A}(t)|t \in \mathbb{R}\}$*, and*

$$\mathcal{A}(t) = \bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s} U(t,\tau)B(t)} = \bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s}\{U(t,\tau)B|B \in B(X)\}}.$$

Definition 2.6. For any $\varepsilon > 0$, let $n(\mathcal{M},\varepsilon)$ denote the minimum number of ball of X of radius ε which is necessary to cover $\mathcal{M}$. The fractal dimension of $\mathcal{M}$, which is also called the capacity of $\mathcal{M}$, is the number

$$\dim_f \mathcal{M} = \overline{\lim_{\varepsilon\to 0^+}}\frac{\ln n(\mathcal{M},\varepsilon)}{\ln\dfrac{1}{\varepsilon}}.$$

Definition 2.7. Let $\{U(t,\tau)|t \geq \tau\}$ or $\{U(n,m)|n \geq m\}$ be a process in a metric space X. We call the family $\mathcal{M} = \{\mathcal{M}(t)|t \in \mathbb{R}\}$ or $\mathcal{M} = \{\mathcal{M}(n)|n \in \mathbb{Z}\}$ be a pullback exponential attractor for $U(t,\tau)$ or $U(n,m)$ if

(1) The sets $\mathcal{M}(t) \in B(X)$ or $\mathcal{M}(n) \in B(X)$ are compact in X,$\forall t \in \mathbb{R}$ or $\forall n \in \mathbb{Z}$.

(2) It is positively semi-invariant, that is

$$U(t,\tau)\mathcal{M}(\tau) \subset \mathcal{M}(t), \forall t \geq \tau \text{ or } U(n,m)\mathcal{M}(m) \subset \mathcal{M}(n), \forall n \geq m.$$

(3) The fractal dimension of $\mathcal{M}(t)$ or $\mathcal{M}(n)$ are uniformly bounded in X, that is, there exists $F > 0$ such that

$$\dim_f \mathcal{M}(t) \leq F, \forall t \in \mathbb{R} \text{ or } \dim_f \mathcal{M}(n) \leq F, \forall n \in \mathbb{N}.$$

(4) The sets $\{\mathcal{M}(t)|t \in \mathbb{R}\}$ or $\{\mathcal{M}(n)|n \in \mathbb{Z}\}$ pullback exponential attracts bounded subset of X; that is, there exists a constant $l > 0$, for every bounded subset $B \in B(X)$ and $t \in \mathbb{R}$ or $n \in \mathbb{Z}$, there exists $k > 0$ such that $dist(U(t,\tau)B, \mathcal{M}(t)) \leq ke^{-l(t-\tau)}$ or $dist(U(n,m)B, \mathcal{M}(n)) \leq ke^{-l(n-m)}$.

Where $dist$ denotes the non-symmetric Hausdorff semidistance between sets, that is $dist(A,B) = \sup_{a\in A} \inf_{b \in B} d(a,b)$.

Lemma 2.4 ([6]). *Let B_R be a ball of the radius R in R^d equipped with Euclidean norm $|\cdot|$. Then for any $\varepsilon > 0$ there exist a finite set $\{x_k : k = 1, 2, \ldots, n_\varepsilon\} \subset B_R$ such that*

$$B_R \subset \bigcup_{k=1}^{n_\varepsilon} \{x \in R^d : |x - x_k| < \varepsilon\} \text{ and } n_\varepsilon \leq (1 + \frac{2R}{\varepsilon})^d.$$

3. Pullback exponential attractors for nonautonomous dynamical system

3.1. Pullback exponential attractors for discrete processes in Banach space

We assume X is a Banach space, $\{U(t,\tau)|t \geq \tau\}$ is a process in X, $\{U(n,m)|n \geq m\}$ is a discrete process in X.

We first construct a pullback exponential attractor for discrete processes $\{U(n,m)| n, m \in \mathbb{Z}, n \geq m\}$, and we assume that the process $\{U(n,m)|n \geq m\}$ satisfies the following properties:

(H1) For the process $\{U(n,m)|n \geq m\}$ there exists a family of bounded absorbing sets $\{B(n)|n \in \mathbb{Z}\}$ and it is positively semi-invariant for the process $\{U(n,m)|n \geq m\}$, that is

$$U(n,m)B(m) \subset B(n), \text{ for all } n \geq m.$$

(H2) There exist $0 < \theta < 1$,$K, M > 0$ and $x_1, x_2, \cdots, x_N \in U(n,m)B(m)$ such that $\{B(x_i, \theta^{n-m})|i = 1, 2, \cdots, N\}$ is a covering of the $U(n,m)B(m)$ and $N \leq KM^{n-m}$ for any $n \geq m$.

(H3) $\forall B \in B(X)$, $\exists T > 0, \forall n \geq m$, $U(n, m-T)B \subset B(n)$.

Lemma 3.1. *Let $\{U(n,m)|n \geq m\}$ be a process in X and the assumption (H1) and (H2) be satisfied. Then the process is pullback $\omega-\mathcal{D}-$limit compact for $\{B(n)|n \in \mathbb{Z}\}$.*

Proof. Though the assumption (H2), $U(n,m)B(m) \subset \bigcup\limits_{i=1}^{N} B(x_i;\theta^{n-m})$, and the definition of the measure of non-compactness, we obtain that $\alpha(U(n,m)B(m)) \leq 2\theta^{n-m}$. Since $0<\theta<1$, we get that $\lim\limits_{m\to-\infty} \alpha(U(n,m)B(m)) = 0$, which imply that the process $\{U(n,m)|n \geq m\}$ is pullback $\omega-\mathcal{D}-$limit compact for $\{B(n)|n \in \mathbb{Z}\}$. $\square$

Lemma 3.2. *Let$\{U(n,m)|n \geq m\}$ be a process in X and the assumption (H1) and (H2) be satisfied. Then the process exists a family of positively semi-invariant compact sets $\{\mathcal{D}(n)|n \in \mathbb{Z}\}$ such that*

$$\mathcal{D}(n) = \bigcap_{m\leq n} \overline{\bigcup_{p\leq m} U(n,p)B(p)}.$$

Proof. By Lemma 3.1, we know that the process $\{U(n,m)|n \geq m\}$ is pullback $\omega-\mathcal{D}-$limit compact for $\{B(n)|n \in \mathbb{Z}\}$, Lemma 2.3 implies that $\mathcal{D}(n)$ is compact. Next, we will prove that the process is positively semi-invariant.

By Lemma 2.3, $\forall\psi \in \mathcal{D}(m)$, there exist two sequences $\tau_k \in (-\infty,T](\tau_k \to -\infty$ as $k\to+\infty)$ and $\phi_k \in B(k)$ such that $\psi = \lim\limits_{k\to+\infty} U(m,\tau_k)\phi_k$. $\forall n \geq m$, $U(n,m)U(m,\tau_k)\phi_k = U(n,\tau_k)\phi_k \to (\rightharpoonup)U(n,m)\psi$. Since $\{U(n,m)|n \geq m\}$ is pullback $\omega-\mathcal{D}-$limit compact for $\{B(n)|n \in \mathbb{Z}\}$, by Lemma 2.3 $U(n,\tau_k)\phi_k$ has a convergent subsequence $U(n,\tau_{k_i})\phi_{k_i}$, let $U(n,\tau_{k_i})\phi_{k_i} \to \varphi$, by Lemma 2.3 we know that $\varphi \in \mathcal{D}(n)$. Obviously $U(n,m)\psi = \varphi$, which implies that $U(n,m)\mathcal{D}(m) \subset \mathcal{D}(n)$. $\square$

Lemma 3.3. *Let$\{U(n,m)|n \geq m\}$ be a process in X and the assumption (H1)-(H3) be satisfied. Then the process is pullback $\omega-$limit compact.*

Proof. $\forall B \in B(X)$, by the assumption (H3), there exists $T \in \mathbb{Z}^+$, such that, $U(n,m)B = U(n,m+T)U(m+T,m)B \subset U(n,m+T)B(m+T)$. By (3) of Lemma 2.1, we have

$$\alpha(U(n,m)B) \leq \alpha(U(n,m+T)B(m+T)) \leq 2\theta^{(n-(m+T))} \to 0, \text{ as } m \to -\infty.$$

We get $\{U(n,m)|n \geq m\}$ is pullback $\omega-$limit compact. $\square$

By Lemma 3.3 and Theorem 2.1 we know that the process exists pullback attractor.

Theorem 3.1. *Let$\{U(n,m)|n \geq m\}$ be a process in X and the assumption (H1)-(H3) be satisfied. Then the process exists pullback attractor $\{\mathcal{A}(n)|n \in \mathbb{Z}\}$.*

Theorem 3.2. *Let $\{U(n,m)|n \geq m\}$ be a discrete process in X and the assumptions (H1)-(H3) be satisfied. Then there exists a pullback exponential attractor $\{\mathcal{M}(n)|n \in \mathbb{Z}\}$ for the process $\{U(n,m)|n,m \in \mathbb{Z}, n \geq m\}$, and the fractal dimension of $\{\mathcal{M}(n)|n \in \mathbb{Z}\}$ can be estimated by*

$$\dim_f^X(\mathcal{M}(n)) \leq \frac{\ln M}{\ln \frac{1}{\theta}}, \forall n \in \mathbb{Z}.$$

Proof. By the assumption (H2), for any $k \in \mathbb{Z}$,

$r = 1$, there exists $W_k^0 = \{x_{k,k}^1, x_{k,k}^2, \cdots, x_{k,k}^{N_k^0}\} \subset B(k)$ such that $U(k,k)B(k) \subset \bigcup_{i=1}^{N_k^0} B(x_{k,k}^i, 1)$ and $N_k^0 \le K$;

for $r = \theta$, there exists $W_k^1 = \{x_{k,k-1}^1, x_{k,k-1}^2, \cdots, x_{k,k-1}^{N_k^1}\} \subset B(k)$ such that $U(k,k-1)B(k-1) \subset \bigcup_{i=1}^{N_k^1} B(x_{k,k-1}^i, \theta)$ and $N_k^1 \le KM$;

for $r = \theta^2$, there exists $W_k^2 = \{x_{k,k-2}^1, x_{k,k-2}^2, \cdots, x_{k,k-2}^{N_k^2}\} \subset B(k)$ such that $U(k,k-2)B(k-2) \subset \bigcup_{i=1}^{N_k^2} B(x_{k,k-2}^i, \theta^2)$ and $N_k^2 \le KM^2$;

$\vdots$

for $r = \theta^m$, there exists $W_k^m = \{x_{k,k-m}^1, x_{k,k-m}^2, \cdots, x_{k,k-m}^{N_k^m}\} \subset B(k)$ such that $U(k,k-m)B(k-m) \subset \bigcup_{i=1}^{N_k^m} B(x_{k,k-m}^i, \theta^m)$ and $N_k^m \le KM^m$; $\cdots$.

Let $M(k) = \bigcup_{n=0}^{+\infty} \bigcup_{i=0}^{n} U(k,k-i)W_{k-i}^{n-i}$, we get

$$\begin{aligned}
M(k+1) &= \bigcup_{n=0}^{+\infty} \bigcup_{i=0}^{n} U(k+1,k+1-i)W_{k+1-i}^{n-i} \\
&= \bigcup_{n=0}^{+\infty} \bigcup_{i=0}^{n} U(k+1,k)U(k,k+1-i)W_{k+1-i}^{n-i} \\
&\supset \bigcup_{n=1}^{+\infty} \bigcup_{i=0}^{n} U(k+1,k)U(k,k-(i-1))W_{k-(i-1)}^{n-1-(i-1)} \\
&\supset U(k+1,k) \bigcup_{n=1}^{+\infty} \bigcup_{i=1}^{n} U(k,k-(i-1))W_{k-(i-1)}^{n-1-(i-1)} \\
&= U(k+1,k) \bigcup_{n=1}^{+\infty} \bigcup_{i=0}^{n-1} U(k,k-i)W_{k-i}^{n-1-i} \\
&= U(k+1,k) \bigcup_{n=0}^{+\infty} \bigcup_{i=0}^{n} U(k,k-i)W_{k-i}^{n-i} \\
&= U(k+1,k)M(k).
\end{aligned}$$

Consequently, for all $n \in \mathbb{Z}$, the family $\{M(n)|n \in \mathbb{Z}\}$ is positively semi-invariant.

By Lemma 3.2, we know that $\mathcal{D}(n)$ is a family of positively semi-invariant compact sets. Let $\mathcal{M}(n) = \mathcal{D}(n) \cup M(n)$, we claim that $\{\mathcal{M}(n)|n \in \mathbb{Z}\}$ is a pullback exponential attractor of $\{U(n,m)|n \ge m\}$;

(Compactness) For any sequence $x_k \in \mathcal{M}(n)$, if there exists a subsequence $x_{k_j} \in M(n)$, since the process $\{U(n,m)|n \ge m\}$ is pullback $\omega{-}\mathcal{D}{-}$limit compact, by lemma 2.3, x_{k_j} exists subsequence convergent in $\mathcal{D}(n)$;if there exists a subsequence $x_{k_j} \in \mathcal{D}(n)$, $\mathcal{D}(n)$ is compact, so we get x_{k_j} exists subsequence convergent in $\mathcal{D}(n)$; that is, for any sequence in $\mathcal{M}(n)$, there exists subsequence convergent in $\mathcal{M}(n)$.

(Positively semi-invariant) Since $U(n+1,n)M(n) \subset M(n+1)$, $U(n+1,n)\mathcal{D}(n) \subset \mathcal{D}(n+1)$, we get that $U(n+1,n)\mathcal{M}(n) = U(n+1,n)\{M(n) \cup \mathcal{D}(n)\} \subset M(n+1) \cup \mathcal{D}(n+1) = \mathcal{M}(n+1)$ for all $n \in \mathbb{Z}$.

(Having an uniformly bounded of fractal dimension) Let $\varepsilon > 0$, there exists $N = [\frac{\ln \varepsilon}{\ln \theta}]$ ($[\frac{\ln \varepsilon}{\ln \theta}]$ is an integer part of the number $\frac{\ln \varepsilon}{\ln \theta}$) such that $\theta^j < \varepsilon$ for all $j \geq N$.

$$M(k) = \bigcup_{n=0}^{+\infty} \bigcup_{i=0}^{n} U(k,k-i)W_{k-i}^{n-i} \subset E_0 \bigcup E_1 \bigcup E_2,$$

here

$$E_0 = \bigcup_{i \leq N, n \leq N} U(k,k-i)W_{k-i}^{n-i},$$
$$E_1 = \bigcup_{i \leq N, n > N} U(k,k-i)W_{k-i}^{n-i},$$
$$E_2 = \bigcup_{i > N, n > N} U(k,k-i)W_{k-i}^{n-i}.$$

Let $i \leq N, n > N$, by the assumption (H1), we get

$$\begin{aligned} &W_{k-i}^{n-i} \subset U(k-i,k-i-(n-i))B(k-i-(n-i)) = U(k-i,k-n)B(k-n) \\ =&U(k-i,k-N)U(k-N,k-n)B(k-n) \subset U(k-i,k-N)B(k-N), \end{aligned}$$

which implies

$$U(k,k-i)W_{k-i}^{n-i} \subset U(k-N)B(k-N), \forall i \leq N, n > N.$$

For any $i > N$,

$$\begin{aligned} U(k,k-i)W_{k-i}^{n-i} &= U(k,k-N)U(k-N,k-i)W_{k-i}^{n-i} \\ &\subset U(k,k-N)B(k-N). \end{aligned}$$

$\mathcal{D}(k) = U(k,k-N)\mathcal{D}(k-N)$, and consequently

$$\begin{aligned} \mathcal{D}(k) \cup E_1 \cup E_2 \subset \overline{U(k,k-N)B(k-N)} &\subset \overline{\bigcup_{i=1}^{N_k^N} B(x_{k,k-N}^i, \theta^N)} \\ &\subset \bigcup_{i=1}^{N_k^N} B(x_{k,k-N}^i, \varepsilon) = \bigcup_{x \in W_k^N} B(x,\varepsilon). \end{aligned}$$

$\mathcal{M}(k) = \mathcal{D}(k) \cup E_0 \cup E_1 \cup E_2 \subset E_0 \bigcup \bigcup_{x \in W_k^N} B(x,\varepsilon)$, $W_k^N \subset E_0$. The number of points in E_0 is $\sum_{n=0}^{N}(K + KM + KM^2 + \cdots + kM^n)$ at most. Hence, we can estimate the number of $\varepsilon-$ball in X needed to cover $\mathcal{M}(k)$ by

$$\begin{aligned} N_\varepsilon^X(\mathcal{M}(k)) &\leq \sum_{n=0}^{N}(K + KM + \cdots + kM^n) \\ &= \begin{cases} \frac{K(N+1)(N+2)}{2}, & M = 1, \\ \frac{KN(1-M)-KM(1-M^{N+1})}{(1-M)^2} \leq \frac{KM^{N+2}}{(1-M)^2}, & M > 1. \end{cases} \end{aligned}$$

For the fractal dimension of the sets $\mathcal{M}(k)$, $N = [\dfrac{\ln\varepsilon}{\ln\theta}]$, we conclued

$$\dim_f^X(\mathcal{M}(k)) \le \lim_{\varepsilon\to 0^+}\sup\frac{\ln N_\varepsilon^X(\mathcal{M}(k))}{\ln\frac{1}{\varepsilon}} \le \begin{cases} \lim\limits_{\varepsilon\to 0^+}\dfrac{\ln\frac{K(N+1)(N+2)}{2}}{\ln\frac{1}{\varepsilon}} = 0, & M = 1, \\ \lim\limits_{\varepsilon\to 0^+}\dfrac{\ln(KM^N/(1-M)^2)}{\ln\frac{1}{\varepsilon}} = \frac{\ln M}{\ln\frac{1}{\theta}}, & M > 1. \end{cases}$$

Consequently, the fractal dimension of the $\mathcal{M}(k)$ is uniformly bounded by the same value $\dfrac{\ln M}{\ln\frac{1}{\theta}}$.

(Pullback exponential attraction) Let $B \in B(X)$, by the assumption (H3), there exists $T \in \mathbb{Z}^+$, for all $n \ge m$, $U(n, m-T)B \subset B(n)$. We get that for any $k > l+T$, $U(k,l)B = U(k,T+l)U(T+l,(T+l)-T)B \subset U(k,T+l)B(T+l)$. By the assumption (H2), we get

$$U(k,l)B \subset U(k,T+l)B(T+l) \subset \bigcup_{i=1}^{N_k^{k-(T+l)}} B(x_i, \theta^{k-(T+l)}), x_i \in W_k^{k-(T+l)} \subset \mathcal{M}(k).$$

Then

$$dist(U(k,l)B, \mathcal{M}(k)) \le dist(U(k,l)B, W_k^{k-(T+l)}) \le \theta^{k-(T+l)} = \nu e^{-\lambda(k-l)}.$$

Here $\nu = e^{T\ln\frac{1}{\theta}}$, $\lambda = \ln\frac{1}{\theta}$.

This show that $\{\mathcal{M}(k)|k \in \mathbb{Z}\}$ is a pullback exponential attractors for the process $\{U(n,m)|n \ge m\}$ in X. □

3.2. Pullback exponential attractors for continuous processes in Banach space

Using the results of previous section we now construct pullback exponential attractors for time continuous process $\{U(t,\tau)|t \ge \tau\}$ in X. Moreover, we assume that the process $\{U(t,\tau)|t \ge \tau\}$ satisfies the following properties:

(A1) For the process $\{U(t,\tau)|t \ge \tau\}$ there exists a family of bounded absorbing sets $\{B(t)|t \in \mathbb{R}\}$ and it is positively semi-invariant for the process $\{U(t,\tau)|t \ge \tau\}$, that is

$$U(t,\tau)B(\tau) \subset B(t) \quad \text{for all } t \ge \tau.$$

(A2) There exists $\widetilde{T} > 0$, the process $\{U(t,\tau)|t \ge \tau\}$ is Lipschitz continuous within the absorbing sets; that is for all $k \in \mathbb{Z}$ and $t, \tau \in [k\widetilde{T}, (k+1)\widetilde{T}]$, there exists a constant $\lambda > 0$ such that

$$||U(t,\tau)u - U(t,\tau)v|| \le \lambda||u-v|| \quad \text{for all} \quad u, v \in B(k\widetilde{T}).$$

(A3) $\forall B \in B(X)$, $\exists T > 0, \forall t \ge \tau$, $U(t, \tau - T)B \subset B(t)$.

(A4) Let $\overline{U}(n,m) = U(n\widetilde{T}, m\widetilde{T})$, we get that $\overline{U}(n,m)|n \ge m\}$ is a discrete process in X, and the assumption (H2) holds true for the process $\{\overline{U}(n,m)|n \ge m\}$ in $\{B(n\widetilde{T})|n \in \mathbb{Z}\}$.

Theorem 3.3. *Let $\{U(t,\tau)|t \ge \tau\}$ be a time continuous process in X and the assumptions (A1)-(A4) be satisfied. Then there exists a pullback exponential attractor $\{\mathcal{M}(t)|t \in \mathbb{R}\}$ for the process $\{U(t,\tau)|t \ge \tau\}$.*

Proof. The assumptions of (A1)-(A4), we know that the discrete process $\{\overline{U}(n,m)|\ n \geq m\}$ satisfies the assumption (H1)-(H3), Theorem 3.2 implies the existence of a pullback exponential attractor $\{\overline{\mathcal{M}}(n)|n \in \mathcal{Z}\}$ for the discrete process $\{\overline{U}(n,m)|n \geq m\}$.

To obtain a pullback exponential attractor for the continuous time process we define

$$\mathcal{M}(t) = U(t, k\widetilde{T})\overline{\mathcal{M}}(k), t \in [k\widetilde{T}, (k+1)\widetilde{T}) \text{ for all } t \in \mathbb{R}.$$

Due to the Lipschitz continuity of the process, the sets $\mathcal{M}(t)$ are compact in X, and the fractal dimension of the sets $\mathcal{M}(t)$ are uniformly bounded.

Let $t, s \in \mathbb{R}$, and $t \geq s$, then there exist $k, l \in \mathbb{Z}$, $k \geq l$ and $t_1, s_1 \in [0, \widetilde{T})$ such that $t = k\widetilde{T} + t_1,\ s = l\widetilde{T} + s_1$. If $k \geq l+1$, we obtain

$$\begin{aligned}
U(t,s)\mathcal{M}(s) &= U(k\widetilde{T} + t_1, l\widetilde{T} + s_1)U(l\widetilde{T} + s_1, l\widetilde{T})\overline{\mathcal{M}}(l) \\
&= U(k\widetilde{T} + t_1, l\widetilde{T})\overline{\mathcal{M}}(l) \\
&= U(k\widetilde{T} + t_1, (k-1)\widetilde{T})U((k-1)\widetilde{T}, l\widetilde{T})\overline{\mathcal{M}}(l) \\
&\subset U(k\widetilde{T} + t_1, (k-1)\widetilde{T})\overline{\mathcal{M}}(k-1) \\
&= \mathcal{M}(t).
\end{aligned}$$

If $k = l$, we have

$$\begin{aligned}
U(t,s)\mathcal{M}(s) &= U(k\widetilde{T} + t_1, k\widetilde{T} + s_1)U(k\widetilde{T} + s_1, k\widetilde{T})\overline{\mathcal{M}}(k) \\
&= U(k\widetilde{T} + t_1, k\widetilde{T})\overline{\mathcal{M}}(k) \\
&= \mathcal{M}(t).
\end{aligned}$$

We get the semi-invariant of the sets $\{\mathcal{M}(t)|t \in \mathbb{R}\}$.

Due to the continuity of the process and the assumption (A2), we get that for any $B \in B(X)$, the set $D = U((l+1)\widetilde{T}, l\widetilde{T} + s_1)B$ is bounded for $s_1 \in [0, \widetilde{T}]$ and for any $l \in \mathbb{Z}$. Since the discrete process $\{\overline{U}(n,m)|n \geq m\}$ exists pullback exponential attractor, there exist $\mu_0, \lambda_0 > 0$ such that

$$dist(\overline{U}(n,m)D, \overline{\mathcal{M}}(n)) \leq \mu_0 e^{-\lambda_0(n-m)}.$$

By the assumption (A3), we get that there exists $T > 0$, for any $s \leq k\widetilde{T} - T$, $U(k\widetilde{T}, s)B \subset B(k\widetilde{T})$ and $\mathcal{M}(k\widetilde{T}) \subset \overline{B(k\widetilde{T})}$. By the Lipschitz continuity of the process, we obtain

$$\begin{aligned}
dist(U(t,s)B, \mathcal{M}(t)) &\leq dist(U(k\widetilde{T} + t_1, k\widetilde{T})U(k\widetilde{T}, s)B, U(k\widetilde{T} + t_1, k\widetilde{T})\mathcal{M}(k\widetilde{T})) \\
&\leq \lambda dist(U(k\widetilde{T}, s)B, \mathcal{M}(k\widetilde{T})) \\
&= \lambda dist(U(k\widetilde{T}, l\widetilde{T} + s_1)B, \mathcal{M}(k\widetilde{T})) \\
&= \lambda dist(U(k\widetilde{T}, (l+1)\widetilde{T})U((l+1)\widetilde{T}, l\widetilde{T} + s_1)B, \mathcal{M}(k\widetilde{T})) \\
&= \lambda dist(\overline{U}(k, l+1)D, \overline{\mathcal{M}}(k)) \leq \lambda\mu_0 e^{-\lambda_0(k-l-1)} \\
&= \lambda\mu_0 e^{\lambda_0(\frac{1+t_1-s_1}{\widetilde{T}})} e^{-\frac{\lambda_0}{\widetilde{T}}(t-s)} \\
&= \mu e^{-\lambda(t-s)},
\end{aligned}$$

This show that $\{\mathcal{M}(t)|t \in \mathbb{R}\}$ is a pullback exponential attractor for the process $\{U(t,\tau)|t \geq \tau\}$ in X. $\square$

3.3. Pullback exponential attractors for continuous process in uniformly convex Banach space

We now present a method to verify the existence of pullback exponential attractors for the time continuous process.

Let X be an uniformly convex Banach space, i.e., for all $\varepsilon > 0$, there exists $\delta > 0$ such that, given $x, y \in X$, $||x|| \le 1, ||y|| \le 1, ||x - y|| > \varepsilon$, then $\frac{||x+y||}{2} < 1 - \delta$. Requiring a space to be uniformly convex is not a severe restriction in application, since this property is satisfied by all Hilbert space, the L^p space with $1 < p < \infty$, and most Sobolev space $W^{k,p}$ with $1 < p < \infty$.

Theorem 3.4. *Let X be an uniformly convex Banach space, $\{U(t,\tau)|t \ge \tau\}$ be a time continuous process in X. Then the process $\{U(t,\tau)|t \ge \tau\}$ exists a pullback exponential attractor in X if the following conditions hold true:*

(i) There exists an uniformly bounded absorbing $B \subset X$, that is, for any $t \ge \tau$ and $D \in B(X)$, there exists $T_0 > 0$ such that

$$U(t, \tau - s)D \subset B, \quad \forall s \le T_0.$$

(ii) There exist $0 < \delta < 1, 0 < \theta < 1 - \delta, T_1 > 0$, and a finite dimension subspace $X_1 \subset X$, such that

$$\|U(t,\tau)u_1 - U(t,\tau)u_2\| \le l\|u_1 - u_2\|, l > 0, \forall t, \tau \in [kT_1, (k+1)T_1],\ \forall k \in \mathbb{Z}, \tag{3.1}$$

$$\|(I - P_m)(U(t, t - T_1)u_1 - U(t, t - T_1)u_2)\| \le \delta\|u_1 - u_2\|, \tag{3.2}$$

$$\|(I - P_m) \bigcup_{s \le T_1} U(t, \tau - s)u\| \le \theta, \quad \forall t \ge \tau, \tag{3.3}$$

for all $u, u_1, u_2 \in B$ and $t \in \mathbb{R}$, where δ is independent on the choice of t, and $\|\cdot\|$ is the norm in X, and $P_m : X \to X_1$ is a bounded projector, m is the dimension of X_1.

Next, we will prove that the process $\{U(t,\tau)|t \ge \tau\}$ satisfy all the conditions of (A1)-(A4).

Proof. By the assumption (i), for the bounded absorbing set B, there exists $T' > T_1$ such that $U(t, t - T')B \subset B$ for all $t \in \mathbb{R}$. Let $B(t) = \bigcup_{\tau \le T'} U(t, t - \tau)B$, and consequently $B(t) \subset B$ for all $t \in \mathbb{R}$. For any $t, s \in \mathbb{R}$ and $t \ge s$, then

$$\begin{aligned} U(t,s)B(s) &= U(t,s) \bigcup_{\tau \le T'} U(s, s - \tau)B = \bigcup_{\tau \le T'} U(t, s - \tau)B \\ &= \bigcup_{\tau \le T'} U(t, t - [\tau + (t - s)])B = \bigcup_{\tau \le T' + (t-s)} U(t, t - \tau)B \\ &\subset \bigcup_{\tau \le T'} U(t, t - \tau)B = B(t), \end{aligned}$$

which imply that the process $\{U(t,\tau)|t \ge \tau\}$ exists a family of bounded absorbing sets $\{B(t)|t \in \mathbb{R}\}$ and it is positively semi-invariant. We obtain that (A1)holds true.

By the assumption (i) and (ii), it is easy to get that (A2) and (A3) hold.

For $n, m \in \mathbb{Z}$ and $n \geq m$, let $\overline{U}(n,m) = U(nT_1, mT_1)$. We will prove that the discrete process $\{\overline{U}(n,m)|n \geq m\}$ satisfies the condition (H2), i.e.,(A4) holds for the process $\{U(t,\tau)|t \geq \tau\}$.

Let $D(n) = B(nT_1)$, then $\{D(n)|n \in \mathbb{Z}\}$ be a family of bounded absorbing sets of the discrete process $\{\overline{U}(n,m)|n \geq m\}$ and $D(n) \subset B$. From the assumption (i) we know that there exists $R > 0$ such that $\mathrm{diam} D(n) \leq 2R$, $\forall n \in \mathbb{Z}$, where $\mathrm{diam} D(n)$ denotes the diameter of $D(n)$.

$$
\begin{aligned}
D(n) &= \bigcup_{\tau \leq T'} U(nT_1, nT_1 - \tau)B \\
&= \bigcup_{\tau \leq T'} P_m U(nT_1, nT_1 - \tau)B + \bigcup_{\tau \leq T'} (I - P_m) U(nT_1, nT_1 - \tau)B.
\end{aligned}
$$

$\bigcup\limits_{\tau \leq T'} P_m U(nT_1, nT_1 - \tau)B$ is a bounded set in finite dimension space X_1, by Lemma 2.4, for $\delta > 0$, there exist balls of $\{B_i\}_{i=1}^K$ in X_1 with radius δ such that $\{B_i\}_{i=1}^K$ is a cover of $\bigcup\limits_{\tau \leq T'} P_m U(nT_1, nT_1 - \tau)B$. Let $D_i = B_i + \bigcup\limits_{\tau \leq T'} (I - P_m) U(nT_1, nT_1 - \tau)B$, $i = 1, 2, \cdots K$. By the assumption (3.3), we know that

$$
|| \bigcup_{\tau \leq T'} (I - P_m) U(nT_1, nT_1 - \tau)x|| \leq \theta, \quad \forall x \in B,
$$

and consequently $\{D_i\}_{i=1}^K$ is a cover of the set $D(n), \forall n \in \mathbb{Z}$ and $\mathrm{diam} D_i \leq \mathrm{diam} B_i + 2\theta \leq 2(\delta + \theta)$, which implies $D(n)$ exists a cover $\{D_i\}_{i=1}^K$ with radius $\delta + \theta$ for all $n \in \mathbb{Z}$.

From the proof above and the arbitrary of n, we know that $D(n-k)$ exists a cover $\{B_i\}_{i=1}^K$ with radius $\delta + \theta$ for all $n \in \mathbb{Z}, k \in \mathbb{N}$. Let $D_i^{n-k} = B_i \cap D(n-k)$, therefore, $\{D_i^{n-k}\}_{i=1}^K$ is also a cover of $D(n-k)$ and $D_i^{n-k} \subset D(n-k) \subset B$. By virtue of (3.1), we get $\|P_m(\overline{U}(n-k+1, n-k)u_1 - \overline{U}(n-k+1, n-k)u_2)\| \leq l\|u_1 - u_2\| \leq 2l(\delta + \theta), \forall u_1, u_2 \in D_i^{n-k}$. By Lemma 2.4, $P_m(\overline{U}(n-k+1, n-k)D_i^{n-k}$ exists a cover $\{B_{n-k+1}^{ij}\}_{j=1}^{N_{n-k+1}}$ with the balls of the radius $\theta(\delta + \theta)$ and $N_{n-k+1} \leq (1 + \frac{2l(\delta+\theta)}{\theta(\delta+\theta)})^m = (1 + \frac{2l}{\theta})^m$, hence $\{E_{n-k+1}^{ij} = B_{n-k+1}^{ij} + (I - P_m)\overline{U}(n-k+1, n-k)D_i^{n-k}, i = 1, 2, \cdots, K, j = 1, 2, \cdots N_{n-k+1}\}$ is a cover of $\overline{U}(n-k+1, n-k)D(n-k)$. By (3.2), $\mathrm{diam}(I - P_m)\overline{U}(n-k+1, n-k)D_i^{n-k} \leq 2\delta(\delta + \theta)$, which implies that $\mathrm{diam} E_{n-k+1}^{ij} \leq 2(\theta(\delta + \theta) + \delta(\delta + \theta)) = 2(\delta + \theta)^2$ and $\{E_{n-k+1}^{ij} |\ i = 1, 2, \cdots, K, j = 1, 2, \cdots N_{n-k+1}\}$ is a cover of $\overline{U}(n-k+1, n-k)D(n-k)$ with radius $(\delta + \theta)^2$ and $N(\overline{U}(n-k+1, n-k)D(n-k), (\delta + \theta)^2) \leq K(1 + \frac{2l}{\theta})^m$. Let $D_{n-k+1}^{ij} = E_{n-k+1}^{ij} \cap D(n-k+1)$, By virtue of (3.1), we obtain

$$
\begin{aligned}
&\|P_m(\overline{U}(n-k+2, n-k+1)u_1 - \overline{U}(n-k+2, n-k+1)u_2)\| \\
\leq &l\|u_1 - u_2\| \leq 2l(\delta + \theta)^2, \forall u_1, u_2 \in D_{n-k+1}^{ij}.
\end{aligned}
$$

By Lemma 2.4, $P_m\overline{U}(n-k+2, n-k+1)D_{n-k+1}^{ij}$ exists a cover $\{B_{n-k+2}^{ijk}\}_{k=1}^{N-k+2}$ with the balls of the radius $\theta(\delta + \theta)^2$ and $N_{n-k+2} \leq (1 + \frac{2l(\delta+\theta)^2}{\theta(\delta+\theta)^2})^m = (1 + \frac{2l}{\theta})^m$, Let $D_{n-k+2}^{ijk} = B_{n-k+2}^{ijk} + (I - P_m)\overline{U}(n-k+2, n-k+1)D_{n-k+1}^{ij}, i = 1, 2, \cdots, K; j = 1, 2, \cdots, N_{n-k+1}; k = 1, 2, \cdots, N_{n-k+2}$. By (3.2), we know that $\mathrm{diam}(I - P_m)\overline{U}(n-k+2, n-k+1)D_{n-k+1}^{ij} \leq 2\delta(\delta + \theta)^2$, which imply that $\mathrm{diam} D_{n-k+2}^{ijk} \leq 2(\delta + \theta)^3$

and $\{D^{ijk}_{n-k+2}|\ i=1,2,\cdots,K; j=1,2,\cdots,N_{n-k+1}; k=1,2,\cdots,N_{n-k+2}\}$ is cover of $\overline{U}(n-k+2,n-k+1)D(n-k+1)$ with radius $(\delta+\theta)^3$ and $N(\overline{U}(n-k+2,n-k+1)D(n-k+1),(\delta+\theta)^3)\leq 2(1+\frac{2l}{\theta})^{2m}$. After iterations, we get there exist at most $K(1+\frac{2l}{\theta})^{km}$ balls in X covering $\overline{U}(n,n-k)D(n-k)$ with radius $(\delta+\theta)^{k+1}$. By the assumption of (ii) we know that $\delta+\theta<1$, and consequently that the discrete process $\{\overline{U}(n,m)|n\geq m\}$ satisfing the condition (H2). □

4. The existence of pullback exponential attractors for non-autonomous reaction diffusion equation

In this section we will apply our theory developed in section 3 to obtain the pullback exponential attractors for non-autonomous reaction diffusion equation.

We consider the following non-autonomous differential equation

$$\begin{cases} u_t-\triangle u+f(u)=g(t), & x\in\Omega,\\ u|_{\partial\Omega}=0,\\ u(\tau)=u_\tau. \end{cases} \tag{4.1}$$

Where $f\in C^1(\mathbb{R},\mathbb{R}),\ g(\cdot)\in L^2_{loc}(\mathbb{R},L^2(\Omega))$, Ω is a bounded open subset of $\mathbb{R}^n$ and there exist $p\geq 2,\ c_i>0,\ i=1,...,5, l>0$ such that

$$c_1|t|^p-c_2\leq f(t)t\leq c_3|t|^p+c_4, \tag{4.2}$$

$$f'(t)\geq -l,\ |f'(t)|\leq c_5(1+|t|^{p-2}) \tag{4.3}$$

for all $s,t\in\mathbb{R}$.

Denote $H=L^2(\Omega)$ with norm $|\cdot|$ and scalar product $(\cdot)$, $H^1_0(\Omega)$ with norm $\|\cdot\|$, $|\cdot|_k$ denote the norm of $L^k(\Omega)$, c denote constants which may change from line to line and even in the same line.

Suppose that the function $g(t)$ is normal([16]) in $L^2_{loc}(\mathbb{R};H)$ that is, for any $\varepsilon>0$, there exists $\eta>0$ such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+\eta}|g(s)|^2ds<\varepsilon. \tag{4.4}$$

Lemma 4.1 ([1, 13, 22])**.** *Let the assumption (4.2) and (4.3) hold and $g(t)$ be translation bounded in $L^2_{loc}(\mathbb{R},H)$, that is $\sup_{t\in\mathbb{R}}\int_t^{t+1}|g(s)|^2ds<c$. Then for any initial data $u_\tau\in H$ and any $T\geq\tau$, there exists a unique solution u for (4.1) which satifies*

$$u\in L^2(\tau,T;H^1_0)\cap L^p(\tau,T;L^p(\Omega)).$$

If furthermore, $u_\tau\in H^1_0$, then

$$u\in C([\tau,T);H^1_0)\cap L^2(\tau,T;H^2(\Omega)).$$

By Lemma 4.1, we can define the process $\{U(t,\tau)|t\geq\tau\}$ as follows:

$$U(t,\tau)u_\tau: H\times[\tau,+\infty)\to H^1_0(\Omega).$$

Theorem 4.1 ([19,21]). *If $g(t)$ is normal in $L^2_{loc}(\mathbb{R};H)$, $f(t)$ satisfies conditions (4.2) and (4.3) where $2 \leq p < \infty (n \leq 2)$, $2 \leq p \leq \frac{n}{n-2} + 1 (n \geq 3)$, then the process $U(t,\tau)$ corresponding to problem (4.1) possesses an uniformly pullback absorbing set D and a pullback attractor $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ in H^1_0.*

We set $A = -\triangle$, since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \leq ..., \quad \lambda_j \to +\infty, \text{ as } j \to \infty,$$

and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $H^1_0(\Omega)$ which are orthogonal in H such that

$$Ae_j = \lambda_j e_j, \quad \forall j \in N.$$

Let $H_m = span\{e_1, e_2, ..., e_m\}$ in H and $P : H \to H_m$ is a orthogonal projector. For any $u \in H$ we write

$$u = Pu + (I - P)u \triangleq u_1 + u_2.$$

Theorem 4.2. *Assume that $g(t)$ and $f(t)$ satisfies conditions of Theorem 4.1 and D is the uniformly pullback absorbing set in H^1_0 corresponding to problem (4.1). Then the process $\{U(t,\tau)|t \geq \tau\}$ possesses a pullback exponential attractor in H^1_0.*

Proof. By Theorem 4.1, there exists $T_0 > 0$ such that $U(t, \tau - T_0)D \subset D$ for any $t \geq \tau$. Let $B = \bigcup_{t\in\mathbb{R}} \bigcup_{\tau \leq T_0} U(t, t-\tau)D$, we obtain that B is also a uniformly pullback bounded absorbing set in $H^1_0(\Omega)$ and $U(t,\tau)B \subset B$ for any $t \geq \tau$.

We set $u_1(t) = u_1(t,\tau)u_{1\tau}$ and $u_2(t) = u_2(t,\tau)u_{2\tau}$ to be solutions associated with equation (4.1) with initial data $u_{1\tau}, u_{2\tau} \in B$, since B is the uniformly pullback absorbing set in H^1_0, so there exists $M > 0$, such that $\|u_{i\tau}\|^2 \leq M, i = 1,2$.

Let $w(t) = u_1(t) - u_2(t)$, by (4.1) we get

$$w_t - \triangle w + f(u_1(t)) - f(u_2(t)) = 0. \tag{4.5}$$

Taking inner product of (4.5) with $-\triangle w$ in H, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + |\triangle w|^2 + (f(u_1) - f(u_2), -\triangle w) = 0.$$

Taking into account (4.3) and Hölder inequality, it is immediate to see that

$$\begin{aligned}|(f(u_1) - f(u_2), -\triangle w)| &\leq \int_\Omega |f(u_1) - f(u_2)||\triangle w| dx \\ &\leq \frac{1}{2}|\triangle w|^2 + \frac{1}{2}\int_\Omega |f(u_1) - f(u_2)|^2 dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |f(u_1) - f(u_2)|^2 dx &= \int_\Omega |f'(u_1 + \theta(u_2 - u_1))|^2 |u_1 - u_2|^2 dx \\ &\leq c\int_\Omega (1 + |u_1|^{p-2} + |u_2|^{p-2})^2 |u_1 - u_2|^2 dx \\ &\leq c(\int_\Omega (1 + |u_1|^{2(p-1)} + |u_2|^{2(p-1)} dx)^{\frac{p-2}{p-1}} \\ &\quad (\int_\Omega |u_1 - u_2|^{2(p-1)})^{\frac{1}{p-1}} \\ &\leq c(1 + |u_1|^{2(p-2)}_{2(p-1)} + |u_2|^{2(p-2)}_{2(p-1)})|w|^2_{2(p-1)}.\end{aligned}$$

Since $2 \leq p < \infty (n \leq 2)$, $2 \leq p \leq \frac{n}{n-2} + 1 (n \geq 3)$, using Sobolev embedding theorem, and

$$\int_\Omega |f(u_1) - f(u_2)|^2 dx \leq c(1 + \|u_1\|^{2(p-2)} + \|u_2\|^{2(p-2)})\|w\|^2 \leq c\|w\|^2, \tag{4.6}$$

we get

$$\frac{d}{dt}\|w\|^2 \leq c\|w\|^2,$$

hence

$$\|w(t)\|^2 \leq \|w(\tau)\|^2 e^{c(t-\tau)}. \tag{4.7}$$

Let $w = w_1 + w_2$, w_1 be the project in PH. Taking inner product of (4.5) with $-\triangle w_2$ in H, we have

$$\frac{1}{2}\frac{d}{dt}\|w_2\|^2 + |\triangle w_2|^2 + (f(u_1) - f(u_2), -\triangle w_2) = 0. \tag{4.8}$$

$$\begin{aligned} |(f(u_1) - f(u_2), -\triangle w_2)| &\leq \int_\Omega |f(u_1) - f(u_2)||\triangle w_2| dx \\ &\leq \frac{1}{2}|\triangle w_2|^2 + \frac{1}{2}\int_\Omega |f(u_1) - f(u_2)|^2 dx. \end{aligned}$$

Taking into (4.6) account, we obtain

$$\frac{d}{dt}\|w_2\|^2 + |\triangle w_2|^2 \leq c||w\|^2.$$

Using the Poincaré inequality $\lambda_n\|w_2\|^2 \leq |\Delta w_2|^2$, it is immediate that

$$\frac{d}{dt}\|w_2\|^2 + \lambda_n\|w_2\|^2 \leq c\|w\|^2,$$

by Gronwall's lemma, we have

$$\|w_2(t)\|^2 \leq e^{-\lambda_n(t-\tau)}\|w(\tau)\|^2 + ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|w(s)\|^2 ds. \tag{4.9}$$

Using (4.7), we get

$$e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|w(s)\|^2 ds \leq e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s} e^{c(s-\tau)}\|w(\tau)\|^2 ds \leq \frac{e^{c(t-\tau)}}{\lambda_n}\|w(\tau)\|^2,$$

hence

$$\|w_2(t)\|^2 \leq e^{-\lambda_n(t-\tau)}\|w(\tau)\|^2 + c_0\frac{e^{c(t-\tau)}}{\lambda_n}\|w(\tau)\|^2. \tag{4.10}$$

Let $u(t) = u_1(t) + u_2(t)$, u_1 be the project in PH. Taking inner product of (4.1) with $-\Delta u_2$, we get

$$\frac{1}{2}\frac{d}{dt}\|u_2(t)\|^2 + |\triangle u_2|^2 + (f(u), -\triangle u_2) = (g(t), -\triangle u_2).$$

Since

$$|(g(t), -\Delta u_2)| \leq |g(t)|^2 + \frac{1}{4}|\Delta u_2|^2$$

and

$$|(f(u),-\Delta u_2)| \le \int_\Omega |f(u)|^2 dx + \frac{1}{4}|\Delta u_2|^2,$$

using (4.2) and Sobolev embedding theorem, we have

$$\int_\Omega |f(u)|^2 dx \le c(1+\|u\|^{2(p-1)}) \le c,$$

and by Poincar$\acute{e}$ inequality $\lambda_n\|u_2\|^2 \le |\Delta u_2|^2$, we have

$$\frac{d}{dt}\|u_2(t)\|^2 + \lambda_n\|u_2\|^2 \le c + 2|g(t)|^2.$$

By Gronwall's lemma, we get

$$\|u_2(t)\|^2 \le e^{-\lambda_n(t-\tau)}\|u\|^2 + ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}(1+|g(s|^2)ds,$$

By (4.4), we obtain that there exists $c>0$ such that

$$\int_t^{t+1} |g(s)|^2 ds \le c, \quad \forall t \in \mathbb{R},$$

and for any $\varepsilon > 0$, there exits $\eta > 0$, such that

$$\int_{t-\eta}^t |g(s)|^2 ds < \frac{\varepsilon}{3},$$

we obtain

$$\begin{aligned}
e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}|g(s)|^2 ds =& \int_\tau^t e^{-\lambda_n(t-s)}|g(s)|^2 ds \\
\le& \int_{t-\eta}^t e^{-\lambda_n(t-s)}|g(s)|^2 ds + \int_{t-\eta-1}^{t-\eta} e^{-\lambda_n(t-s)}|g(s)|^2 ds \\
&+ \int_{t-\eta-2}^{t-\eta-1} e^{-\lambda_n(t-s)}|g(s)|^2 ds + \cdots \\
\le& \frac{\varepsilon}{3} + |g|_b^2 e^{-\lambda_n \eta}(1+e^{-\lambda_n}+e^{-2\lambda_n}+\cdots) \\
\le& \frac{\varepsilon}{3} + |g|_b^2 \frac{e^{-\lambda_n\eta}}{1-e^{-\lambda_n}},
\end{aligned}$$

and

$$\int_\tau^t e^{-\lambda_n(t-s)} ds \le \frac{1}{\lambda_n},$$

we get

$$\|u_2(t)\|^2 \le e^{-\lambda_n(t-\tau)}\|u\|^2 + c(\frac{1}{\lambda_n} + \frac{\varepsilon}{3} + \frac{e^{-\lambda_n\eta}}{1-e^{-\lambda_n}}). \tag{4.11}$$

Let $T_1 = t-\tau = 1$, by (4.7), we get

$$\|U(t,\tau)u_{1\tau} - U(t,\tau)u_{2\tau}\| \le e^c\|u_{1\tau} - u_{2\tau}\|. \tag{4.12}$$

Since $\lambda_n \to +\infty$, for $0 < \varepsilon < 1$, from (4.10) and (4.11), there exist $m \in \mathbb{N}^+$, $0 < \delta < 1$, $0 < \theta < 1$ and $\delta + \theta < 1$ such that

$$\|w_2(t)\| = \|(I - P_m)(U(t,\tau)u_{1\tau} - U(t,\tau)u_{2\tau})\| \leq \delta\|u_{1\tau} - u_{2\tau}\|, \tag{4.13}$$

$$\|u_2(t)\| \leq \|(I - P_m)\bigcup_{s\leq 1}(U(t,\tau - s)u\| \leq \theta. \tag{4.14}$$

By Lemma 4.1, we know that the process is continuous for time in B, and, applying inequality (4.12), (4.13) and (4.14), we know that the process $\{U(t,\tau)|t \geq \tau\}$ generated by (4.1) satisfies all conditions of Theorem 3.4. $\square$

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