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RANDOM ATTRACTOR FOR NON-AUTONOMOUS STOCHASTIC STRONGLY DAMPED WAVE EQUATION ON UNBOUNDED DOMAINS*

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Abstract In this paper we study the asymptotic dynamics for the non-autonomous stochastic strongly damped wave equation driven by additive noise defined on unbounded domains. First we introduce a continuous cocycle for the equation and then investigate the existence and uniqueness of tempered random attractors which pullback attract all tempered random sets.

Keywords Stochastic strongly damped wave equation, unbounded domains, random attractor.

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1. Introduction

Consider the following non-autonomous stochastic strongly damped wave equation with additive noise defined in the entire space $\mathbb{R}^n$ $(n \in \mathbb{N})$:

$$u_{tt} - \alpha\Delta u_t - \Delta u + u_t + \lambda u + f(x,u) = g(x,t) + h(x)\frac{dW}{dt}, \tag{1.1}$$

with the initial value conditions

$$u(\tau,x) = u_0(x),\ u_t(\tau,x) = u_1(x),\quad x \in \mathbb{R}^n, \tag{1.2}$$

where Δ is the Laplacian with respect to the variable $x \in \mathbb{R}^n$ with $1 \leqslant n \leqslant 3$; $u = u(t,x)$ is a real function of $x \in \mathbb{R}^n$ and $t \geqslant \tau$, $\tau \in \mathbb{R}$; $\alpha > 0$ is the strong damping coefficient; λ is a positive dissipative coefficient; f is a nonlinearity satisfying certain growth and dissipative conditions; $g(x,\cdot)$ and h are given functions in $L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ and $H^2(\mathbb{R})$, respectively; $W(t)$ is a two-sided real-valued Wiener process on a probability space.

Eq.(1.1) can model a random perturbation of strongly damped wave equation. In applications, the unknown u naturally represents the displacement of the body relative to a fixed reference configuration. There have been a lot of profound results on the dynamics of a variety of systems related to Eq.(1.1). For example, the

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asymptotical behavior of solutions for deterministic strongly damped wave equation has been studied by many authors (see [4, 17, 19–21, 24, 27, 36–38, 40–42], etc.). For autonomous stochastic wave equation, the asymptotical behavior of solutions have been studied by several authors (see [8, 13–15, 18, 22, 23, 28–32, 39, 43]). Recently, Wang [33] studied the non-autonomous stochastic damped wave equations on unbounded domain. So far as we know, there were no results on random attractors for non-autonomous stochastic strongly damped wave equation (1.1) on unbounded domains. The case of g depending on time is of great physical interest. It is therefore important to investigate the existence of attractors for Eq.(1.1) when g is dependent on time.

The goal of the present paper is to study random attractors of non-autonomous stochastic equation (1.1). In this case, we need to deal with the deterministic perturbations as well as the stochastic perturbations. Since the behavior of stochastic and deterministic perturbations is quite different, it is better to use two separate parametric spaces to take care of these perturbations: one is for deterministic perturbations and the other is for stochastic perturbations.

Random attractors for non-autonomous stochastic PDEs have been investigated in [9,12] in bounded domains and in [2,33–35] on unbounded domains. In the present paper, by applying the abstract result in [35], we will prove the stochastic strongly damped wave equation (1.1) has tempered random attractors in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

In general, the existence of global random attractor depends on some kind compactness (see, e.g., [5–7, 16]). To prove the existence of random attractors for (1.1) in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, we must establish the pullback asymptotic compactness of solutions. Since Sobolev embeddings are not compact on $\mathbb{R}^n$, we cannot get the desired asymptotic compactness directly from the regularity of solutions. The noncompactness of embeddings on $\mathbb{R}^n$ is a major obstacle for proving the existence of random attractors for (1.1). We here overcome the difficulty by using uniform estimates on the tails of solutions outside a bounded ball in $\mathbb{R}^n$ and decomposing the solutions in a bounded domain in terms of eigenfunctions of negative Laplacian as in [28, 32].

This paper is organized as follows. In the next section, we recall a sufficient and necessary criterion for existence of pullback attractors for cocycle or nonautonomous random dynamical systems. In Section 3, we define a continuous cocycle for Eq.(1.1) in $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Then we derive all necessary uniform estimates of solutions in Section 4. Finally, in Section 5, we prove the existence and uniqueness of tempered random attractor for the non-autonomous stochastic strongly damped wave equation.

Throughout this paper, we use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. The norms of $L^p(\mathbb{R}^n)$ and a Banach space X are generally written as $\|\cdot\|_p$ and $\|\cdot\|_X$, respectively. The letters c and c_i $(i = 1, 2, ...)$ are used to denote positive constants which do not depend on ε in the context.

2. Preliminaries

In this section, we recall some known results from [35] regarding pullback attractors for non-autonomous random dynamical systems. All results given in this section are not original and they are presented here just for the reader's convenience. The theory of pullback attractors for autonomous random dynamical systems can be found in [1, 5–7, 10, 16].

Assume X is a separable Banach space. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$ (see [1]). There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathcal{P})$ which is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega, t \in \mathbb{R}. \tag{2.1}$$

We often say that $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system.

Definition 2.1. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (1)-(4) are satisfied:

(1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(2) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;

(3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$

(4) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Hereafter, we assume Φ is a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$, and $\mathcal{D}$ is the collection of all tempered families of nonempty bounded subsets of X. Remember that a family $\mathcal{D} = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ of nonempty bounded subsets of X is said to be tempered if there exists $x_0 \in X$ such that for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the following holds:

$$\lim_{t\to-\infty} e^{ct} d(D(\tau + t, \theta_t\omega), x_0) = 0. \tag{2.2}$$

Given $D \in \mathcal{D}$, the family $\Omega(D) = \{\Omega(D, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called the Ω-limit set of D where

$$\Omega(D, \tau, \omega) = \bigcap_{r\geqslant 0} \overline{\bigcup_{t\geqslant r} \Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega))}. \tag{2.3}$$

The cocycle Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X \tag{2.4}$$

whenever $t_n \to \infty$, and $x_n \in D(\tau - t_n, \theta_{-t_n}\omega)$ with $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.2. A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $D \in \mathcal{D}$, there exists $T = T(D, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \quad \text{for all } t \geq T. \tag{2.5}$$

If, in addition, $K(\tau, \omega)$ is closed in X and is measurable in ω with respect to $\mathcal{F}$, then K is called a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 2.3. A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (1)-(3) are fulfilled: for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(1) $\mathcal{A}(\tau, \omega)$ is compact in X and is measurable in ω with respect to $\mathcal{F}$.

(2) $\mathcal{A}$ is invariant, that is,

$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau+t,\theta_t\omega). \tag{2.6}$$

(3) For every $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t\to\infty} d_H(\Phi(t,\tau-t,\theta_{-t}\omega, D(\tau-t,\theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0. \tag{2.7}$$

where d_H is the Hausdorff semi-distance given by $d_H(F,G) = \sup_{u\in F}\inf_{v\in G}\|u - v\|_X$, for any $F, G \subset X$.

As in the deterministic case, random complete solutions can be used to characterized the structure of a $\mathcal{D}$-pullback attractor. The definition of such solutions are given below.

Definition 2.4. A mapping $\psi : \mathbb{R}\times\mathbb{R}\times\Omega \to X$ is called a random complete solution of Φ if for every $t \in \mathbb{R}^+, s, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t,\tau+s,\theta_s\omega,\psi(s,\tau,\omega)) = \psi(t+s,\tau,\omega). \tag{2.8}$$

If, in addition, there exists a tempered family $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ such that $\psi(t,\tau,\omega)$ belongs to $D(\tau+t,\theta_t\omega)$ for every $t \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then ψ is called a tempered random complete solution of Φ.

We borrow the following result from [35] on $\mathcal{D}$-pullback attractors for non-autonomous random dynamical systems. Similar results can be found in [1, 5–7, 10, 16] for autonomous random dynamical systems.

Proposition 2.1. *Suppose Φ is $\mathcal{D}$-pullback asymptotically compact in X and has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$. Then Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ which is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\mathcal{A}(\tau,\omega) = \Omega(K,\tau,\omega) = \bigcup_{D\in\mathcal{D}} \Omega(D,\tau,\omega) \tag{2.9}$$

$$= \{\psi(0,\tau,\omega) : \psi \textit{ is a tempered random complete solution of } \Phi\ \}. \tag{2.10}$$

3. Stochastic strongly damped wave equation on $\mathbb{R}^n$

In this section, we outline the basic setting of (1.1)-(1.2) and show that it generates a continuous cocycle in $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$.

Let $\xi = u_t + \delta u$ where δ is a small positive constant whose value will be determined later. Then applying this transformation to (1.1)-(1.2) we find that

$$\begin{cases} \dfrac{du}{dt} = \xi - \delta u, \\ \dfrac{d\xi}{dt} = \alpha\Delta\xi + (1-\alpha\delta)\Delta u + (\delta-1)\xi + (\delta-\lambda-\delta^2)u \\ \qquad\qquad - f(x,u) + g(x,t) + h(x)\dfrac{dW}{dt}, \end{cases} \tag{3.1}$$

with the initial value conditions

$$u(\tau,x) = u_0(x), \quad \xi(\tau,x) = \xi_0(x), \tag{3.2}$$

where $\xi_0(x) = u_1(x) + \delta u_0(x)$, $x \in \mathbb{R}^n$.

Denote by $F(x,u) = \int_0^u f(x,s)ds$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$. Throughout this paper, we assume that the nonlinear function f satisfies the following conditions, for every $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$,

$$| f(x,u) | \leqslant c_1 | u |^\gamma + \phi_1(x), \quad \phi_1(x) \in L^2(\mathbb{R}^n), \tag{3.3}$$

$$f(x,u)u - c_2 F(x,u) \geqslant \phi_2(x), \quad \phi_2(x) \in L^1(\mathbb{R}^n), \tag{3.4}$$

$$F(x,u) \geqslant c_3 | u |^{\gamma+1} - \phi_3(x), \quad \phi_3(x) \in L^1(\mathbb{R}^n), \tag{3.5}$$

$$| f_u(x,u) | \leqslant c_4 | u |^{\gamma-1} + \phi_4(x), \quad \phi_4(x) \in H^1(\mathbb{R}^n), \tag{3.6}$$

where $c_i (i = 1,2,3,4)$ are positive constants, $\gamma \geqslant 1$ for $n = 1,2$ and $\gamma \in [1,3)$ for $n = 3$. The restriction for $n \leqslant 3$ is needed when deriving uniform estimates on solutions, see, e.g., Lemma 4.4. We also need the following condition on g: there exists a positive constants σ such that

$$\int_{-\infty}^{\tau} e^{\sigma s} \|g(\cdot,s)\|^2 ds < \infty, \quad \forall\, \tau \in \mathbb{R}, \tag{3.7}$$

which implies that

$$\lim_{r\to\infty} \int_{-\infty}^{\tau} \int_{|x|\geqslant r} e^{\sigma s} |g(x,s)|^2 dx ds = 0, \quad \forall\, \tau \in \mathbb{R}, \tag{3.8}$$

where $|\cdot|$ denotes the absolute value of real number in $\mathbb{R}$.

For our purpose, it is convenient to convert the problem (3.1)-(3.2) (or (1.1)-(1.2)) into a deterministic system with a random parameter, and then show that it generates a cocycle over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

We identify $\omega(t)$ with $W(t)$, i.e., $\omega(t) = W(t) = W(t,\omega)$, $t \in \mathbb{R}$. Consider Ornstein-Uhlenbeck equation $dy + ydt = dW(t)$, and Ornstein-Uhlenbeck process $y(\theta_t\omega) = -\int_{-\infty}^0 e^s(\theta_t\omega)(s)ds$, $t \in \mathbb{R}$. From [3, 11], it is known that the random variable $|y(\omega)|$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega} \subset \Omega$ of full $\mathcal{P}$ measure such that $y(\theta_t\omega)$ is continuous in t for every $\omega \in \widetilde{\Omega}$. Put

$$z(\theta_t\omega) = z(x,\theta_t\omega) = h(x)y(\theta_t\omega), \tag{3.9}$$

which solves

$$dz + zdt = hdW. \tag{3.10}$$

Lemma 3.1 (See [26]). *For any $\epsilon > 0$, there exists a tempered random variable $k : \Omega \mapsto \mathbb{R}^+$, such that for all $t \in \mathbb{R}$, $\omega \in \Omega$,*

$$\|z(\theta_t\omega)\| \leqslant e^{\epsilon|t|}k(\omega)\|h\|,$$
$$\|\nabla z(\theta_t\omega)\| \leqslant e^{\epsilon|t|}k(\omega)\|\nabla h\|,$$
$$\|\Delta z(\theta_t\omega)\| \leqslant e^{\epsilon|t|}k(\omega)\|\Delta h\|,$$

where $k(\omega)$ satisfies

$$e^{-\epsilon|t|}k(\omega) \leqslant k(\theta_t\omega) \leqslant e^{\epsilon|t|}k(\omega).$$

To define a cocycle for problem (3.1)-(3.2), we let

$$v(t,\tau,\omega)=\xi(t,\tau,\omega)-z(\theta_t\omega),$$

then (3.1)-(3.2) can be rewritten as the equivalent system with random coefficients but without white noise

$$\begin{cases} \dfrac{du}{dt}=v-\delta u+z(\theta_t\omega), \\ \dfrac{dv}{dt}=\alpha\Delta v+(1-\alpha\delta)\Delta u+(\delta-1)v+(\delta-\lambda-\delta^2)u \\ \qquad\quad +\alpha\Delta z(\theta_t\omega)+\delta z(\theta_t\omega)-f(x,u)+g(x,t), \end{cases} \tag{3.11}$$

with the initial value conditions

$$u(\tau,\tau,x)=u_0(x),\quad v(\tau,\tau,x)=v_0(x), \tag{3.12}$$

where $v_0(x)=\xi_0(x)-z(\theta_\tau\omega)$, $x\in\mathbb{R}^n$.

We will consider (3.11)-(3.12) for $\omega\in\widetilde{\Omega}$ and write $\widetilde{\Omega}$ as Ω from now on.

Let $E(U)=H^1(U)\times L^2(U), U\subseteq\mathbb{R}^n$, endowed with the usual norm

$$\|Y\|_{H^1(U)\times L^2(U)}=\big(\|\nabla u\|^2+\|u\|^2+\|v\|^2\big)^{\frac12}\quad\text{for}\quad Y=(u,v)^\top\in E(U), \tag{3.13}$$

where $\top$ stands for the transposition. We define a new norm $\|\cdot\|_{E(U)}$ by

$$\|Y\|_{E(U)}=\big(\|v\|^2+\big(\delta^2+\lambda-\delta\big)\|u\|^2+\big(1-\alpha\delta\big)\|\nabla u\|^2\big)^{\frac12}, \tag{3.14}$$

for $Y=(u,v)^\top\in E(U)$. It is easy to check that $\|\cdot\|_{E(U)}$ is equivalent to the usual norm $\|\cdot\|_{H^1(U)\times L^2(U)}$ in (3.13).

By the classical theory concerning the existence and uniqueness of the solutions [25, 27, 28], we have the following Lemma.

Lemma 3.2. *Put $\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0)=(u(t+\tau,\tau,\theta_{-\tau}\omega,u_0),v(t+\tau,\tau,\theta_{-\tau}\omega,v_0))^\top$, where $\varphi_0=(u_0,v_0)^\top$, and let (3.3)-(3.6) hold. Then for every $\omega\in\Omega$, $\tau\in\mathbb{R}$ and $\varphi_0\in E(\mathbb{R}^n)$, problem (3.11)-(3.12) has a unique $(\mathcal{F},\ \mathcal{B}(H^1(\mathbb{R}^n))\times\mathcal{B}(L^2(\mathbb{R}^n)))$–measurable solution $\varphi(\cdot,\tau,\omega,\varphi_0)\in C([\tau,\infty),E(\mathbb{R}^n))$ with $\varphi(\tau,\tau,\omega,\varphi_0)=\varphi_0$, $\varphi(t,\tau,\omega,\varphi_0)\in E(\mathbb{R}^n)$ being continuous in φ_0 with respect to the usual norm of $E(\mathbb{R}^n)$ for each $t>\tau$. Moreover, for every $(t,\tau,\omega,\varphi_0)\in\mathbb{R}^+\times\mathbb{R}\times\Omega\times E(\mathbb{R}^n)$, the mapping*

$$\Phi(t,\tau,\omega,\varphi_0)=\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0) \tag{3.15}$$

generates a continuous cocycle from $\mathbb{R}^+\times\mathbb{R}\times\Omega\times E(\mathbb{R}^n)$ to $E(\mathbb{R}^n)$ over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathcal{P},\{\theta_t\}_{t\in\mathbb{R}})$.

Introducing the homeomorphism $P(\theta_t\omega)\big(u,v\big)^\top=\big(u,\ v+z(\theta_t\omega)\big)^\top,\big(u,v\big)^\top\in E(\mathbb{R}^n)$ with an inverse homeomorphism $P^{-1}(\theta_t\omega)\big(u,v\big)^\top=\big(u,\ v-z(\theta_t\omega)\big)^\top$. Then, the transformation

$$\widetilde{\Phi}(t,\tau,\omega,(u_0,\xi_0))=P(\theta_t\omega)\Phi(t,\tau,\omega,(u_0,v_0))P^{-1}(\theta_t\omega) \tag{3.16}$$

generates a continuous cocycle with (3.1)-(3.2) over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathcal{P},\{\theta_t\}_{t\in\mathbb{R}})$.

Note that these two continuous cocycles are equivalent. By (3.16), it is easy to check that $\widetilde{\Phi}$ has a random attractor provided Φ possesses a random attractor. Then, we only need to consider the continuous cocycle Φ.

4. Uniform estimates of solutions

In this section, we derive uniform estimates on the solutions of the non-autonomous stochastic strongly damped wave equations (3.11)-(3.12) defined on $\mathbb{R}^n$ when $t \to \infty$. These estimates are necessary for proving the existence of $\mathcal{D}$-pullback attractor for the equations. In particular, we will show that the tails of the solutions for large space variables are uniformly small when time is sufficiently large.

Let $\delta \in (0,1)$ be small enough such that

$$\delta^2 + \lambda - \delta > 0, \quad 1 - \alpha\delta > 0,$$

and define σ appearing in (3.7) by

$$\sigma = \min\{1-\delta, \delta, \frac{c_2\delta}{2}\}, \tag{4.1}$$

where c_2 is the positive constant in (3.4).

Lemma 4.1. *Assume that $h \in H^2(\mathbb{R})$ and (3.3)-(3.7) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D) > 0$ such that for all $t \geqslant T$,*

$$\begin{aligned}
&\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{E(\mathbb{R}^n)} + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2 ds \\
&+ \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2 ds \\
&+ \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2 ds \\
&+ \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2 ds \\
< \; & c + c\int_{-\infty}^{0} e^{\sigma s}\big(\|\nabla z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|^{\gamma+1}_{H^1}\big) ds \\
&+ c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\|g(x,s)\|^2 ds,
\end{aligned} \tag{4.2}$$

where $\varphi_0 = (u_0, v_0)^\top \in D(\tau-t,\theta_{-t}\omega)$ and c is a positive constant depending on λ, σ, α and δ, but independent of τ, ω and D.

Proof. Taking the inner product of the second equation of (3.11) with v in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2 = &-\alpha\|\nabla v\|^2 - (1-\delta)\|v\|^2 - (\delta^2+\lambda-\delta)\big(u,v\big) + \big(1-\alpha\delta\big)\big(\Delta u, v\big) \\
&+ \big(\alpha\Delta z(\theta_t\omega), v\big) + \big(\delta z(\theta_t\omega), v\big) + \big(g(x,t), v\big) - \big(f(x,u), v\big).
\end{aligned} \tag{4.3}$$

By the first equation of (3.11), we have

$$v = \frac{du}{dt} + \delta u - z(\theta_t\omega). \tag{4.4}$$

Then substituting the above v into the third, fourth and last terms on the left-hand side of (4.3), we find that

$$\begin{aligned}
&\big(u,v\big)\\
&=\Big(u,\frac{du}{dt}+\delta u-z(\theta_t\omega)\Big)\\
&=\frac{1}{2}\frac{d}{dt}\|u\|^2+\delta\|u\|^2-\big(u,z(\theta_t\omega)\big)\\
&\geqslant\frac{1}{2}\frac{d}{dt}\|u\|^2+\delta\|u\|^2-\|z(\theta_t\omega)\|\cdot\|u\|\\
&\geqslant\frac{1}{2}\frac{d}{dt}\|u\|^2+\frac{3\delta}{4}\|u\|^2-\frac{1}{\delta}\|z(\theta_t\omega)\|^2,
\end{aligned}\tag{4.5}$$

$$\begin{aligned}
&\big(\Delta u,v\big)\\
&=-\big(\nabla u,\nabla v\big)\\
&=-\Big(\nabla u,\nabla\big(\frac{du}{dt}+\delta u-z(\theta_t\omega)\big)\Big)\\
&=-\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2-\delta\|\nabla u\|^2+\big(\nabla u,\nabla z(\theta_t\omega)\big)\\
&\leqslant-\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2-\delta\|\nabla u\|^2+\|\nabla z(\theta_t\omega)\|\cdot\|\nabla u\|\\
&\leqslant-\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2-\frac{3\delta}{4}\|\nabla u\|^2+\frac{1}{\delta}\|\nabla z(\theta_t\omega)\|^2,
\end{aligned}\tag{4.6}$$

$$\begin{aligned}
&\big(f(x,u),v\big)\\
&=\Big(f(x,u),\frac{du}{dt}+\delta u-z(\theta_t\omega)\Big)\\
&=\frac{d}{dt}\int_{\mathbb{R}^n}F(x,u)dx+\delta\big(f(x,u),u\big)-\big(f(x,u),z(\theta_t\omega)\big).
\end{aligned}\tag{4.7}$$

From condition (3.4) we get

$$\big(f(x,u),u\big)\geqslant c_2\int_{\mathbb{R}^n}F(x,u)dx+\int_{\mathbb{R}^n}\phi_2(x)dx.\tag{4.8}$$

By conditions (3.3) and (3.5) we have

$$\begin{aligned}
&\big(f(x,u),z(\theta_t\omega)\big)\\
&\leqslant\int_{\mathbb{R}^n}\Big(c_1|u|^\gamma+\phi_1(x)\Big)|z(\theta_t\omega)|dx\\
&\leqslant\|\phi_1(x)\|\cdot\|z(\theta_t\omega)\|+c_1\Big(\int_{\mathbb{R}^n}|u|^{\gamma+1}dx\Big)^{\frac{\gamma}{\gamma+1}}\cdot\|z(\theta_t\omega)\|_{\gamma+1}\\
&\leqslant\|\phi_1(x)\|\cdot\|z(\theta_t\omega)\|+c_1\Big(\int_{\mathbb{R}^n}\big(F(x,u)+\phi_3(x)\big)dx\Big)^{\frac{\gamma}{\gamma+1}}\cdot\|z(\theta_t\omega)\|_{\gamma+1}\\
&\leqslant\frac{1}{2}\|\phi_1(x)\|^2+\frac{1}{2}\|z(\theta_t\omega)\|^2+\frac{\delta c_2}{2}\int_{\mathbb{R}^n}F(x,u)dx\\
&\quad+\frac{\delta c_2}{2}\int_{\mathbb{R}^n}\phi_3(x)dx+c\|z(\theta_t\omega)\|_{H^1}^{\gamma+1}.
\end{aligned}\tag{4.9}$$

Using the Cauchy-Schwartz inequality and the Young inequality, we have

$$\big(\alpha\Delta z(\theta_t\omega), v\big) = -\alpha\big(\nabla z(\theta_t\omega), \nabla v\big) \leqslant \frac{\alpha}{2}\|\nabla z(\theta_t\omega)\|^2 + \frac{\alpha}{2}\|\nabla v\|^2, \tag{4.10}$$

$$\big(\delta z(\theta_t\omega), v\big) \leqslant \delta\|z(\theta_t\omega)\| \cdot \|v\| \leqslant \frac{2\delta^2}{1-\delta}\|z(\theta_t\omega)\|^2 + \frac{1-\delta}{8}\|v\|^2, \tag{4.11}$$

$$\big(g(x,t), v\big) \leqslant \|g(x,t)\| \cdot \|v\| \leqslant \frac{2}{1-\delta}\|g(x,t)\|^2 + \frac{1-\delta}{8}\|v\|^2. \tag{4.12}$$

By (4.5)-(4.12), it follows from (4.3) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|v\|^2 + \big(\delta^2+\lambda-\delta\big)\|u\|^2 + \big(1-\alpha\delta\big)\|\nabla u\|^2 + 2\int_{\mathbb{R}^n} F(x,u)dx\Big) \\ \leqslant & -\frac{3(1-\delta)}{4}\|v\|^2 - \frac{3\delta\big(\delta^2+\lambda-\delta\big)}{4}\|u\|^2 - \frac{3\delta\big(1-\alpha\delta\big)}{4}\|\nabla u\|^2 \\ & - \frac{\delta c_2}{2}\int_{\mathbb{R}^n} F(x,u)dx - \frac{\alpha}{2}\|\nabla v\|^2 + c\big(1 + \|\nabla z(\theta_t\omega)\|^2 \\ & + \|z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|_{H^1}^{\gamma+1}\big) + \frac{2}{1-\delta}\|g(x,t)\|^2. \end{aligned} \tag{4.13}$$

Recall the new norm $\|\cdot\|_{E(U)}$ in (3.14). By (4.1) we obtain from (4.13) that

$$\begin{aligned} &\frac{d}{dt}\Big(\|\varphi\|_{E(\mathbb{R}^n)}^2 + 2\int_{\mathbb{R}^n} F(x,u)dx\Big) + \sigma\Big(\|\varphi\|_{E(\mathbb{R}^n)}^2 + 2\int_{\mathbb{R}^n} F(x,u)dx\Big) \\ & + \frac{1-\delta}{2}\|v\|^2 + \frac{\delta\big(\delta^2+\lambda-\delta\big)}{2}\|u\|^2 + \frac{\delta\big(1-\alpha\delta\big)}{2}\|\nabla u\|^2 + \alpha\|\nabla v\|^2 \\ \leqslant & \; c\big(1 + \|\nabla z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|_{H^1}^{\gamma+1}\big) + \frac{4}{1-\delta}\|g(x,t)\|^2. \end{aligned} \tag{4.14}$$

Multiplying (4.14) by $e^{\sigma t}$ and then integrating over $(\tau-t,\tau)$, we have

$$\begin{aligned} & e^{\sigma\tau}\Big(\|\varphi(\tau,\tau-t,\omega,\varphi_0)\|_{E(\mathbb{R}^n)}^2 + 2\int_{\mathbb{R}^n} F(x,u(\tau,\tau-t,\omega,u_0))dx\Big) \\ & + \frac{1-\delta}{2}\int_{\tau-t}^{\tau} e^{\sigma s}\|v(s,\tau-t,\omega,v_0)\|^2 ds \\ & + \frac{\delta(\delta^2+\lambda-\delta)}{2}\int_{\tau-t}^{\tau} e^{\sigma s}\|u(s,\tau-t,\omega,u_0)\|^2 ds \\ & + \frac{\delta(1-\alpha\delta)}{2}\int_{\tau-t}^{\tau} e^{\sigma s}\|\nabla u(s,\tau-t,\omega,u_0)\|^2 ds \\ & + \alpha\int_{\tau-t}^{\tau} e^{\sigma s}\|\nabla v(s,\tau-t,\omega,v_0)\|^2 ds \\ \leqslant & \; e^{\sigma(\tau-t)}\Big(\|\varphi_0\|_{E(\mathbb{R}^n)}^2 + 2\int_{\mathbb{R}^n} F(x,u_0)dx\Big) \\ & + c\int_{\tau-t}^{\tau} e^{\sigma s}\big(1 + \|\nabla z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|_{H^1}^{\gamma+1}\big)ds \\ & + \frac{4}{1-\delta}\int_{\tau-t}^{\tau} e^{\sigma s}\|g(x,s)\|^2 ds. \end{aligned} \tag{4.15}$$

Substituting ω by $\theta_{-\tau}\omega$, then we have from (4.15) that

$$
\begin{aligned}
&\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{E(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))dx\\
&\quad+\frac{1-\delta}{2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2ds\\
&\quad+\frac{\delta(\delta^2+\lambda-\delta)}{2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds\\
&\quad+\frac{\delta(1-\alpha\delta)}{2}\int_{\tau-t}^{\tau}e^{\sigma s}\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds\\
&\quad+\alpha\int_{\tau-t}^{\tau}e^{\sigma s}\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2ds\\
&\leqslant e^{-\sigma t}\Big(\|\varphi_0\|^2_{E(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx\Big)\\
&\quad+c\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\big(1+\|\nabla z(\theta_{s-\tau}\omega)\|^2+\|z(\theta_{s-\tau}\omega)\|^2+\|z(\theta_{s-\tau}\omega)\|^{\gamma+1}_{H^1}\big)ds\\
&\quad+\frac{4}{1-\delta}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\|g(x,s)\|^2ds\\
&\leqslant e^{-\sigma t}\Big(\|\varphi_0\|^2_{E(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx\Big)\\
&\quad+c\int_{-t}^{0}e^{\sigma s}\big(1+\|\nabla z(\theta_s\omega)\|^2+\|z(\theta_s\omega)\|^2+\|z(\theta_s\omega)\|^{\gamma+1}_{H^1}\big)ds\\
&\quad+\frac{4}{1-\delta}\int_{-\infty}^{\tau}e^{\sigma(s-\tau)}\|g(x,s)\|^2ds.
\end{aligned}
\tag{4.16}
$$

By lemma 3.1 with $\epsilon=\frac{\sigma}{2(\gamma+1)}$, we have that

$$
\begin{aligned}
&\int_{-t}^{0}e^{\sigma s}\big(\|\nabla z(\theta_s\omega)\|^2+\|z(\theta_s\omega)\|^2+\|z(\theta_s\omega)\|^{\gamma+1}_{H^1}\big)ds\\
&\leqslant\int_{-\infty}^{0}e^{\sigma s}\big(\|\nabla z(\theta_s\omega)\|^2+\|z(\theta_s\omega)\|^2+\|z(\theta_s\omega)\|^{\gamma+1}_{H^1}\big)ds\\
&\leqslant\int_{-\infty}^{0}e^{\frac{\sigma s}{2}}\Big(k^2(\omega)\big(\|\nabla h\|^2+\|h\|^2\big)+k^{\gamma+1}(\omega)\big(\|\nabla h\|^{\gamma+1}+\|h\|^{\gamma+1}\big)\Big)ds\\
&<+\infty.
\end{aligned}
\tag{4.17}
$$

Note that (3.3) and (3.4) imply that

$$
\int_{\mathbb{R}^n}F(x,u_0)dx\leqslant c\big(1+\|u_0\|^2+\|u_0\|^{\gamma+1}_{H_1}\big).
\tag{4.18}
$$

Due to $\varphi_0=(u_0,v_0)^\top\in D(\tau-t,\theta_{-t}\omega)$ and $D\in\mathcal{D}$, we get from (4.18) that

$$
\lim_{t\to+\infty}e^{-\sigma t}\Big(\|\varphi_0\|^2_{E(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx\Big)=0.
\tag{4.19}
$$

Therefore, there exists $T=T(\tau,\omega,D)>0$ such that $e^{-\sigma t}\big(\|\varphi_0\|^2_{E(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}F(x,u_0)dx\big)\leqslant 1$ for all $t\geqslant T$. Thus the lemma follows from (3.7), (4.16) and (4.17). □

Lemma 4.2. *Assume that $h \in H^2(\mathbb{R})$ and (3.3)-(3.7) hold. Then there exists a random ball $\{A(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ centered at 0 with random radius*

$$\begin{aligned}\varrho(\tau,\omega) = c + c\int_{-\infty}^{0} e^{\sigma s}\big(\|\nabla z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|_{H^1}^{\gamma+1}\big)ds \\ + c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\|g(x,s)\|^2 ds\end{aligned}$$

such that $\{A(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a closed measurable $\mathcal{D}$-pullback absorbing set for the continuous cocycle associated with problem (3.11)-(3.12) in $\mathcal{D}$, that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D) > 0$, such that for all $t \geqslant T$,

$$\Phi\big(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)\big) \subseteq A(\tau,\omega). \tag{4.20}$$

Proof. This is an immediate consequence of (3.15) and Lemma 4.1. □

Choose a smooth function ρ, such that $0 \leqslant \rho(s) \leqslant 1$ for $s \in \mathbb{R}$, and

$$\rho(s) = \begin{cases} 0, & 0 \leqslant |s| \leqslant 1, \\ 1, & |s| \geqslant 2, \end{cases} \tag{4.21}$$

and there exist constants μ_1, μ_2, such that $|\rho'(s)| \leqslant \mu_1$, $|\rho''(s)| \leqslant \mu_2$ for $s \in \mathbb{R}$.

Given $r \geqslant 1$, denote by $\mathbb{H}_r = \{x \in \mathbb{R}^n : |x| < r\}$ and $\mathbb{R}^n \backslash \mathbb{H}_r$ the complement of $\mathbb{H}_r$. To prove asymptotic compactness of solution on $\mathbb{R}^n$, we prove the following lemma.

Lemma 4.3. *Assume that $h \in H^2(\mathbb{R})$ and (3.3)-(3.7) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exist $T = T(\tau,\omega,D,\varepsilon) > 0$ and $\widetilde{R} = \widetilde{R}(\tau,\omega,\varepsilon) \geqslant 1$, such that for all $t \geqslant T$, $r \geqslant \widetilde{R}$,*

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|^2_{E(\mathbb{R}^n \backslash \mathbb{H}_r)} \leqslant \varepsilon, \tag{4.22}$$

where $\varphi_0 = (u_0, v_0)^\top \in D(\tau - t, \theta_{-t}\omega)$.

Proof. We first consider the random equations (3.11)-(3.12). Then taking the inner product of the second equation of (3.11) with $\rho\big(\frac{|x|^2}{r^2}\big)v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho\Big(\frac{|x|^2}{r^2}\Big)|v|^2 dx \\ &= \alpha\int_{\mathbb{R}^n} (\Delta v)\rho\Big(\frac{|x|^2}{r^2}\Big)v dx - (1-\delta)\int_{\mathbb{R}^n} \rho\Big(\frac{|x|^2}{r^2}\Big)|v|^2 dx \\ &\quad - (\delta^2 + \lambda - \delta)\int_{\mathbb{R}^n} \rho\Big(\frac{|x|^2}{r^2}\Big)uv dx + (1 - \alpha\delta)\int_{\mathbb{R}^n} (\Delta u)\rho\Big(\frac{|x|^2}{r^2}\Big)v dx \\ &\quad + \alpha\int_{\mathbb{R}^n} (\Delta z(\theta_t\omega))\rho\Big(\frac{|x|^2}{r^2}\Big)v dx + \delta\int_{\mathbb{R}^n} \rho\Big(\frac{|x|^2}{r^2}\Big)z(\theta_t\omega)v dx \\ &\quad + \int_{\mathbb{R}^n} \rho\Big(\frac{|x|^2}{r^2}\Big)g(x,t)v dx - \int_{\mathbb{R}^n} \rho\Big(\frac{|x|^2}{r^2}\Big)f(x,u)v dx.\end{aligned} \tag{4.23}$$

Substituting v in (4.4) into the third, fourth and last terms on the left-hand side of (4.23), we get that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho\Big(\frac{|x|^2}{r^2}\Big)uvdx\\
=&\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)u\big(\frac{du}{dt}+\delta u-z(\theta_t\omega)\big)dx\\
=&\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)\Big(\frac{1}{2}\frac{d}{dt}u^2+\delta u^2-z(\theta_t\omega)u\Big)dx\\
\geqslant&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|u|^2dx+\frac{\delta}{2}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|u|^2dx\\
&-\frac{1}{2\delta}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|z(\theta_t\omega)|^2dx,
\end{aligned}\tag{4.24}$$

$$\begin{aligned}
&\int_{\mathbb{R}^n}(\Delta u)\rho\big(\frac{|x|^2}{r^2}\big)vdx\\
=&\int_{\mathbb{R}^n}(\Delta u)\rho\big(\frac{|x|^2}{r^2}\big)\big(\frac{du}{dt}+\delta u-z(\theta_t\omega)\big)dx\\
=&-\int_{\mathbb{R}^n}(\nabla u)\nabla\Big(\rho\big(\frac{|x|^2}{r^2}\big)\big(\frac{du}{dt}+\delta u-z(\theta_t\omega)\big)\Big)dx\\
=&-\int_{\mathbb{R}^n}(\nabla u)\Big(\frac{2x}{r^2}\rho'\big(\frac{|x|^2}{r^2}\big)\big(\frac{du}{dt}+\delta u-z(\theta_t\omega)\big)\\
&+\rho\big(\frac{|x|^2}{r^2}\big)\nabla\big(\frac{du}{dt}+\delta u-z(\theta_t\omega)\big)\Big)dx\\
\leqslant&\int_{r<x<\sqrt{2}r}\frac{2\mu_1 x}{r^2}|(\nabla u)v|dx-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla u|^2dx\\
&-\delta\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla u|^2dx-\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla u|\cdot|\nabla z(\theta_t\omega)|dx\\
\leqslant&\int_{\mathbb{R}^n}\frac{2\sqrt{2}\mu_1}{r}|(\nabla u)v|dx-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla u|^2dx\\
&-\delta\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla u|^2dx+\frac{\delta}{2}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla u|^2dx\\
&+\frac{1}{2\delta}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla z(\theta_t\omega)|^2dx\\
\leqslant&\frac{\sqrt{2}\mu_1}{r}\big(\|\nabla u\|^2+\|v\|^2\big)-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla u|^2dx\\
&-\frac{\delta}{2}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla u|^2dx+\frac{1}{2\delta}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)|\nabla z(\theta_t\omega)|^2dx,
\end{aligned}\tag{4.25}$$

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho\Big(\frac{|x|^2}{r^2}\Big)f(x,u)vdx\\
=&\int_{\mathbb{R}^n}\rho\Big(\frac{|x|^2}{r^2}\Big)f(x,u)\big(\frac{du}{dt}+\delta u-z(\theta_t\omega)\big)dx\\
=&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)F(x,u)dx+\delta\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)f(x,u)udx
\end{aligned}\tag{4.26}$$

$$- \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) f(x,u) z(\theta_t\omega) dx.$$

From condition (3.4), we find

$$\begin{aligned} &\delta \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) f(x,u) u dx \\ &\geqslant c_2\delta \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) F(x,u) dx + \delta \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) \phi_2(x) dx. \end{aligned} \tag{4.27}$$

By conditions (3.3) and (3.5), we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) f(x,u) z(\theta_t\omega) dx \\ \leqslant& \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) \Big(c_1|u|^\gamma + \phi_1(x)\Big) |z(\theta_t\omega)| dx \\ \leqslant& \frac{1}{2} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) |\phi_1(x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) |z(\theta_t\omega)|^2 dx \\ &+ c \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) |z(\theta_t\omega)|^{\gamma+1} dx + \frac{c_2\delta}{2} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) \big(F(x,u) + \phi_3(x)\big) dx. \end{aligned} \tag{4.28}$$

By the Cauchy-Schwartz inequality and the Young inequality, we obtain

$$\begin{aligned} &\alpha \int_{\mathbb{R}^n} (\Delta v) \rho(\frac{|x|^2}{r^2}) v dx \\ =& -\alpha \int_{\mathbb{R}^n} (\nabla v) \nabla\Big(\rho(\frac{|x|^2}{r^2}) v\Big) dx \\ =& -\alpha \int_{\mathbb{R}^n} (\nabla v) \Big(\frac{2x}{r^2} \rho'(\frac{|x|^2}{r^2}) v + \rho(\frac{|x|^2}{r^2}) \nabla v\Big) dx \\ \leqslant& \int_{r<x<\sqrt{2}r} \frac{2\alpha\mu_1 x}{r^2} |(\nabla v) v| dx - \alpha \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) |\nabla v|^2 dx \\ \leqslant& \frac{\sqrt{2}\alpha\mu_1}{r} \left(\|\nabla v\|^2 + \|v\|^2\right) - \alpha \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) |\nabla v|^2 dx, \end{aligned} \tag{4.29}$$

$$\begin{aligned} &\alpha \int_{\mathbb{R}^n} \big(\Delta z(\theta_t\omega)\big) \rho(\frac{|x|^2}{r^2}) v dx \\ =& -\alpha \int_{\mathbb{R}^n} \big(\nabla z(\theta_t\omega)\big) \nabla\Big(\rho(\frac{|x|^2}{r^2}) v\Big) dx \\ =& -\alpha \int_{\mathbb{R}^n} \big(\nabla z(\theta_t\omega)\big) \Big(\frac{2x}{r^2} \rho'(\frac{|x|^2}{r^2}) v + \rho(\frac{|x|^2}{r^2}) \nabla v\Big) dx \\ \leqslant& \int_{r<x<\sqrt{2}r} \frac{2\alpha\mu_1 x}{r^2} |\big(\nabla z(\theta_t\omega)\big) v| dx - \alpha \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) (|\nabla v|)(|\nabla z(\theta_t\omega)|) dx \\ \leqslant& \frac{\sqrt{2}\alpha\mu_1}{r} \left(\|\nabla z(\theta_t\omega)\|^2 + \|v\|^2\right) + \alpha \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) |\nabla v|^2 dx \\ &+ \frac{\alpha}{4} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2}) |\nabla z(\theta_t\omega)|^2 dx, \end{aligned} \tag{4.30}$$

$$\begin{aligned}&\delta\int_{\mathbb{R}^n}\rho\Big(\frac{|x|^2}{r^2}\Big)z(\theta_t\omega)vdx\\ \leqslant&\frac{\delta^2}{1-\delta}\int_{\mathbb{R}^n}\rho\Big(\frac{|x|^2}{r^2}\Big)|z(\theta_t\omega)|^2dx+\frac{1-\delta}{4}\int_{\mathbb{R}^n}\rho\Big(\frac{|x|^2}{r^2}\Big)|v|^2dx,\end{aligned}\tag{4.31}$$

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho\Big(\frac{|x|^2}{r^2}\Big)g(x,t)vdx\\ \leqslant&\ \frac{\delta^2}{1-\delta}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})|g(x,t)|^2dx+\frac{1-\delta}{4}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})|v|^2dx.\end{aligned}\tag{4.32}$$

Then it follows from (4.24)-(4.32)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})\Big(|v|^2+\big(\delta^2+\lambda-\delta\big)|u|^2+(1-\alpha\delta)|\nabla u|^2+2F(x,u)\Big)dx\\ \leqslant&\ \frac{\sqrt{2}\alpha\mu_1}{r}\big(\|\nabla v\|^2+2\|v\|^2+\|\nabla z(\theta_t\omega)\|^2\big)\\ &+(1-\alpha\delta)\frac{\sqrt{2}\mu_1}{r}\big(\|\nabla u\|^2+\|v\|^2\big)-\frac{1-\delta}{2}\int_{\mathbb{R}^n}\rho\Big(\frac{|x|^2}{r^2}\Big)|v|^2dx\\ &-\frac{\delta(\delta^2+\lambda-\delta)}{2}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})|u|^2dx-\frac{\delta(1-\alpha\delta)}{2}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})|\nabla u|^2dx\\ &-\frac{c_2\delta}{2}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})F(x,u)dx+c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})\big(1+|\nabla z(\theta_t\omega)|^2+|z(\theta_t\omega)|^2\\ &+|z(\theta_t\omega)|^{\gamma+1}+|g(x,t)|^2\big)dx.\end{aligned}\tag{4.33}$$

Let

$$X=|v|^2+\big(\delta^2+\lambda-\delta\big)|u|^2+(1-\alpha\delta)|\nabla u|^2.\tag{4.34}$$

Then, by (4.1) we have from (4.33) and (4.34) that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})\big(X+2F(x,u)\big)dx+\sigma\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})\big(X+2F(x,u)\big)dx\\ \leqslant&\ \frac{c}{r}\big(\|\nabla v\|^2+\|v\|^2+\|\nabla u\|^2+\|\nabla z(\theta_t\omega)\|^2\big)\\ &+c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})\Big(1+|\nabla z(\theta_t\omega)|^2+|z(\theta_t\omega)|^2\\ &+|z(\theta_t\omega)|^{\gamma+1}+|g(x,t)|^2\Big)dx.\end{aligned}\tag{4.35}$$

Multiplying (4.35) by $e^{\sigma t}$ and then integrating over $(\tau-t,\tau)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})\Big(X(\tau,\tau-t,\omega,X_0)+2F(x,u(\tau,\tau-t,\omega,u_0))\Big)dx\\ \leqslant&\ e^{-\sigma t}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{r^2})\big(X_0+2F(x,u_0)\big)dx\\ &+\frac{c}{r}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\Big(\|\nabla v(s,\tau-t,\omega,v_0)\|^2+\|v(s,\tau-t,\omega,v_0)\|^2\\ &+\|\nabla u(s,\tau-t,\omega,u_0)\|^2+\|\nabla z(\theta_s\omega)\|^2\Big)ds\end{aligned}\tag{4.36}$$

$$+ c\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \int_{\mathbb{R}^n} \rho\big(\frac{|x|^2}{r^2}\big)\Big(1+|\nabla z(\theta_s\omega)|^2+|z(\theta_s\omega)|^2$$
$$+|z(\theta_s\omega)|^{\gamma+1}+|g(x,s)|^2\Big)dxds.$$

By replacing ω by $\theta_{-\tau}\omega$, it then follows from (4.36) that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho\big(\frac{|x|^2}{r^2}\big)\Big(X(\tau,\tau-t,\theta_{-\tau}\omega,X_0)+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))\Big)dx\\
&\leqslant e^{-\sigma t}\int_{\mathbb{R}^n} \rho\big(\frac{|x|^2}{r^2}\big)\big(X_0+2F(x,u_0)\big)dx\\
&\quad+\frac{c}{r}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\Big(\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\\
&\quad+\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla z(\theta_{s-\tau}\omega)\|^2\Big)ds\\
&\quad+c\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\int_{\mathbb{R}^n} \rho\big(\frac{|x|^2}{r^2}\big)\Big(1+|\nabla z(\theta_{s-\tau}\omega)|^2+|z(\theta_{s-\tau}\omega)|^2\\
&\quad+|z(\theta_{s-\tau}\omega)|^{\gamma+1}+|g(x,s)|^2\Big)dxds\\
&\leqslant e^{-\sigma t}\int_{\mathbb{R}^n} \rho\big(\frac{|x|^2}{r^2}\big)\big(X_0+2F(x,u_0)\big)dx\\
&\quad+\frac{c}{r}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\Big(\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\\
&\quad+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\Big)ds\\
&\quad+\frac{c}{r}\int_{-t}^{0} e^{\sigma s}\|\nabla z(\theta_s\omega)\|^2ds+c\int_{\tau-t}^{\tau} e^{\sigma s}\int_{|x|\geqslant r}|g(x,s)|^2dxds\\
&\quad+c\int_{-t}^{0} e^{\sigma s}\int_{|x|\geqslant r}\Big(1+|\nabla z(\theta_s\omega)|^2+|z(\theta_s\omega)|^2+|z(\theta_s\omega)|^{\gamma+1}\Big)dxds\\
&\leqslant e^{-\sigma t}\int_{\mathbb{R}^n} \rho\big(\frac{|x|^2}{r^2}\big)\big(X_0+2F(x,u_0)\big)dx\\
&\quad+\frac{c}{r}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\Big(\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\\
&\quad+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\Big)ds\\
&\quad+\frac{c}{r}\int_{-\infty}^{0} e^{\sigma s}\|\nabla z(\theta_s\omega)\|^2ds+c\int_{-\infty}^{\tau} e^{\sigma s}\int_{|x|\geqslant r}|g(x,s)|^2dxds\\
&\quad+c\int_{-\infty}^{0} e^{\sigma s}\int_{|x|\geqslant r}\Big(1+|\nabla z(\theta_s\omega)|^2+|z(\theta_s\omega)|^2+|z(\theta_s\omega)|^{\gamma+1}\Big)dxds.
\end{aligned}\tag{4.37}$$

In what follows, we estimate the terms on the right-hand side of (4.37). Due to $\varphi_0 \in D(\tau-t,\theta_{-t}\omega)\in\mathcal{D}$ and (4.18), we have that, there exists $\widetilde{T}_1=\widetilde{T}_1(\tau,\varepsilon,\omega,D)>0$, such that for all $t>\widetilde{T}_1$,

$$e^{-\sigma t}\int_{\mathbb{R}^n} \rho\big(\frac{|x|^2}{r^2}\big)\big(X_0+2F(x,u_0)\big)dx\leqslant\varepsilon. \tag{4.38}$$

By Lemma 4.1, there are $\widetilde{T}_2 = \widetilde{T}_2(\tau, \varepsilon, \omega, D) > 0$ and $\widetilde{R}_1 = \widetilde{R}_1(\varepsilon, \omega, D) > 1$, such that for all $t > \widetilde{T}_2$ and $r > \widetilde{R}_1$,

$$\begin{aligned}&\frac{c}{r}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\Big(\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2 + \|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\\&\quad + \|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\Big)ds\\&\leqslant \varepsilon.\end{aligned} \tag{4.39}$$

By Lemma 3.1 with $\epsilon = \frac{\sigma}{2(\gamma+1)}$, there are $\widetilde{T}_3 = \widetilde{T}_3(\varepsilon, \omega) > 0$ and $\widetilde{R}_2 = \widetilde{R}_2(\varepsilon, \omega) > 1$, such that for all $t > \widetilde{T}_3$ and $r > \widetilde{R}_2$,

$$\begin{aligned}&c\int_{-\infty}^{0} e^{\sigma s}\int_{|x|\geqslant r}\Big(1 + |\nabla z(\theta_s\omega)|^2 + |z(\theta_s\omega)|^2 + |z(\theta_s\omega)|^{\gamma+1}\Big)dxds\\&\quad + \frac{c}{r}\int_{-\infty}^{0} e^{\sigma s}\|\nabla z(\theta_s\omega)\|^2 ds\\&\leqslant \varepsilon.\end{aligned} \tag{4.40}$$

By condition (3.8), there is $\widetilde{R}_3 = \widetilde{R}_3(\tau, \varepsilon) > 1$, such that for all $r > \widetilde{R}_3$,

$$c\int_{-\infty}^{\tau} e^{\sigma s}\int_{|x|\geqslant r}|g(x,s)|^2 dxds \leqslant \varepsilon. \tag{4.41}$$

Letting $\widetilde{T} = \max\{\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3\}$, $\widetilde{R} = \max\{\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3\}$, then combining (4.38)-(4.41), we have for all $t > \widetilde{T}$ and $r > \widetilde{R}$,

$$\int_{\mathbb{R}^n}\rho\big(\frac{|x|^2}{r^2}\big)\Big(X(\tau,\tau-t,\theta_{-\tau}\omega,X_0) + 2F(x,(\tau,\tau-t,\theta_{-\tau}\omega,u_0))\Big)dx \leqslant 4\varepsilon, \tag{4.42}$$

which implies

$$\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{E(\mathbb{R}^n\setminus\mathbb{H}_r)} \leqslant 4\varepsilon. \tag{4.43}$$

Then we complete the proof. □

Let $\widehat{\rho} = 1 - \rho$ with ρ given by (4.21). Fix $r \geqslant 1$ and set

$$\begin{cases}\widehat{u}(t,\tau,\omega,\widehat{u}_0) = \widehat{\rho}\big(\frac{|x|^2}{r^2}\big)u(t,\tau,\omega,u_0),\\ \widehat{v}(t,\tau,\omega,\widehat{v}_0) = \widehat{\rho}\big(\frac{|x|^2}{r^2}\big)v(t,\tau,\omega,v_0),\end{cases} \tag{4.44}$$

then $\widehat{\varphi}(t,\tau,\omega,\widehat{\varphi}_0) = (\widehat{u}(t,\tau,\omega,\widehat{u}_0), \widehat{v}(t,\tau,\omega,\widehat{v}_0))^\top$ is the solution of problem (3.11)-(3.12) on the bounded domain $\mathbb{H}_{2r}$, where $\widehat{\varphi}_0 = \widehat{\rho}\big(\frac{|x|^2}{r^2}\big)\varphi_0 \in E(\mathbb{H}_{2r})$.

Multiplying (3.11) by $\widehat{\rho}(\frac{|x|^2}{r^2})$ and using (4.44) we find that

$$\begin{cases} \dfrac{d\widehat{u}}{dt} = \widehat{v} - \delta\widehat{u} + \widehat{\rho}(\dfrac{|x|^2}{r^2})z(\theta_t\omega), \\ \dfrac{d\widehat{v}}{dt} = \alpha\Delta\widehat{v} + (\delta-1)\widehat{v} + (1-\alpha\delta)\Delta\widehat{u} + (\delta-\lambda-\delta^2)\widehat{u} \\ \qquad + \alpha\widehat{\rho}(\dfrac{|x|^2}{r^2})\Delta z(\theta_t\omega) + \delta\widehat{\rho}(\dfrac{|x|^2}{r^2})z(\theta_t\omega) + \widehat{\rho}(\dfrac{|x|^2}{r^2})g(x,t) \\ \qquad - \widehat{\rho}(\dfrac{|x|^2}{r^2})f(u,t) - \alpha v\Delta\widehat{\rho}(\dfrac{|x|^2}{r^2}) - 2\alpha\nabla v\nabla\widehat{\rho}(\dfrac{|x|^2}{r^2}) \\ \qquad - (1-\alpha\delta)u\Delta\widehat{\rho}(\dfrac{|x|^2}{r^2}) - 2(1-\alpha\delta)\nabla u\nabla\widehat{\rho}(\dfrac{|x|^2}{r^2}). \end{cases} \tag{4.45}$$

Considering the eigenvalue problem

$$-\Delta\widehat{u} = \lambda\widehat{u} \ \text{ in } \ \mathbb{H}_{2r}, \quad \text{with} \ \ \widehat{u} = 0 \ \text{ on } \ \partial\mathbb{H}_{2r}. \tag{4.46}$$

The problem (4.46) has a family of eigenfunctions $\{e_i\}_{i\in\mathbb{N}}$ with the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$:

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots, \ \lambda_i \to +\infty(i \to +\infty),$$

such that $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{H}_{2r})$. Given n, let $X_n = \operatorname{span}\{e_1, \cdots, e_n\}$ and $P_n : L^2(\mathbb{H}_{2r}) \to X_n$ be the projection operator.

Lemma 4.4. *Assume that $h \in H^2(\mathbb{R})$ and (3.3)-(3.7) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exist $\widehat{T} = \widehat{T}(\tau,\omega,D,\varepsilon) > 0$, $\widehat{R} = \widehat{R}(\tau,\omega,\varepsilon) \geqslant 1$ and $N = N(\tau,\omega,\varepsilon) > 0$, such that for all $t \geqslant \widehat{T}$, $r \geqslant \widehat{R}$ and $n \geqslant N$,*

$$\|(I-P_n)\widehat{\varphi}(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{\varphi}_0)\|^2_{E(\mathbb{H}_{2r})} \leqslant \varepsilon, \tag{4.47}$$

where $\widehat{\varphi}_0 = \widehat{\rho}(\frac{|x|^2}{r^2})\varphi_0, \varphi_0 = (u_0,v_0)^\top \in D(\tau-t,\theta_{-t}\omega)$.

Proof. Let $\widehat{u}_{n,1} = P_n\widehat{u}$, $\widehat{u}_{n,2} = (I-P_n)\widehat{u}$, $\widehat{v}_{n,1} = P_n\widehat{v}$ and $\widehat{v}_{n,2} = (I-P_n)\widehat{v}$. Applying $I-P_n$ to the first equation of (4.45), we obtain

$$\widehat{v}_{n,2} = \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - (I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega). \tag{4.48}$$

Then applying $I-P_n$ to the second equation of (4.45) and taking the inner product of the resulting equation with $\widehat{v}_{n,2}$ in $L^2(\mathbb{H}_{2r})$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\widehat{v}_{n,2}\|^2 \\ =& -\alpha\|\nabla\widehat{v}_{n,2}\|^2 + (\delta-1)\|\widehat{v}_{n,2}\|^2 + (\delta-\lambda-\delta^2)\big(\widehat{u}_{n,2},\ \widehat{v}_{n,2}\big) \\ &+ (1-\alpha\delta)\big(\Delta\widehat{u}_{n,2},\widehat{v}_{n,2}\big) + \alpha\big(\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega),\widehat{v}_{n,2}\big) \\ &+ \delta\big(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega),\ \widehat{v}_{n,2}\big) + \big(\widehat{\rho}(\frac{|x|^2}{r^2})g(x,t),\ \widehat{v}_{n,2}\big) \\ &- \big(\widehat{\rho}(\frac{|x|^2}{r^2})f(x,u),\ \widehat{v}_{n,2}\big) - \alpha\big(v\Delta\widehat{\rho}(\frac{|x|^2}{r^2}) + 2\nabla v\nabla\widehat{\rho}(\frac{|x|^2}{r^2}),\ \widehat{v}_{n,2}\big) \\ &- (1-\alpha\delta)\big(u\Delta\widehat{\rho}(\frac{|x|^2}{r^2}) + 2\nabla u\nabla\widehat{\rho}(\frac{|x|^2}{r^2}),\ \widehat{v}_{n,2}\big). \end{aligned} \tag{4.49}$$

Substituting $\widehat{v}_{n,2}$ in (4.48) into the third, fourth and last terms on the left-hand side of (4.49), we obtain

$$
\begin{aligned}
&(\widehat{u}_{n,2},\ \widehat{v}_{n,2})\\
&= \Big(\widehat{u}_{n,2},\ \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\Big)\\
&\geqslant \frac{1}{2}\frac{d}{dt}\|\widehat{u}_{n,2}\|^2 + \delta\|\widehat{u}_{n,2}\|^2 - \|\widehat{u}_{n,2}\|\cdot\|(I-P_n)\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\|\\
&\geqslant \frac{1}{2}\frac{d}{dt}\|\widehat{u}_{n,2}\|^2 + \frac{\delta}{2}\|\widehat{u}_{n,2}\|^2 - \frac{1}{2\delta}\|(I-P_n)\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\|^2,
\end{aligned}
\tag{4.50}
$$

$$
\begin{aligned}
&(\Delta\widehat{u}_{n,2},\ \widehat{v}_{n,2})\\
&= -\Big(\nabla\widehat{u}_{n,2},\ \nabla\big(\frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\big)\Big)\\
&\leqslant -\frac{1}{2}\frac{d}{dt}\|\nabla\widehat{u}_{n,2}\|^2 - \delta\|\nabla\widehat{u}_{n,2}\|^2 + \|\nabla\widehat{u}_{n,2}\|\cdot\|(I-P_n)\nabla\big(\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\big)\|\\
&\leqslant -\frac{1}{2}\frac{d}{dt}\|\nabla\widehat{u}_{n,2}\|^2 - \frac{3\delta}{4}\|\nabla\widehat{u}_{n,2}\|^2 + \frac{1}{\delta}\|(I-P_n)\nabla\big(\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\big)\|^2,
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
&\Big(\widehat{\rho}\Big(\frac{|x|^2}{r^2}\Big)f(x,u),\widehat{v}_{n,2}\Big)\\
&= \Big(\widehat{\rho}\Big(\frac{|x|^2}{r^2}\Big)f(x,u),\frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\Big)\\
&= \frac{d}{dt}\Big(\widehat{\rho}\Big(\frac{|x|^2}{r^2}\Big)f(x,u),\widehat{u}_{n,2}\Big) - \Big(\widehat{\rho}\Big(\frac{|x|^2}{r^2}\Big)f_u(x,u)u_t,\widehat{u}_{n,2}\Big)\\
&\quad + \delta\Big(\widehat{\rho}\Big(\frac{|x|^2}{r^2}\Big)f(x,u),\widehat{u}_{n,2}\Big) - \Big(\widehat{\rho}\Big(\frac{|x|^2}{r^2}\Big)f(x,u),(I-P_n)\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\Big).
\end{aligned}
\tag{4.52}
$$

From condition (3.6), we find

$$
\begin{aligned}
&\Big(\widehat{\rho}\Big(\frac{|x|^2}{r^2}\Big)f_u(x,u)u_t,\widehat{u}_{n,2}\Big)\\
&\leqslant c\|\phi_4\|_6\cdot\|u_t\|\cdot\|\widehat{u}_{n,2}\|_3 + c\|u_t\|\cdot\|u\|_6^{\gamma-1}\cdot\|\widehat{u}_{n,2}\|_{\frac{6}{4-\gamma}}\\
&\leqslant c\|\phi_4\|_{H^1}\cdot\|u_t\|\cdot\|\widehat{u}_{n,2}\|^{\frac{1}{2}}\cdot\|\nabla\widehat{u}_{n,2}\|^{\frac{1}{2}}\\
&\quad + c\|u_t\|\cdot\|u\|_{H^1}^{\gamma-1}\cdot\|\widehat{u}_{n,2}\|^{\frac{3-\gamma}{2}}\cdot\|\nabla\widehat{u}_{n,2}\|^{\frac{\gamma-1}{2}}\\
&\leqslant c\lambda_{n+1}^{-\frac{1}{4}}\cdot\|u_t\|\cdot\|\nabla\widehat{u}_{n,2}\| + c\lambda_{n+1}^{\frac{\gamma-3}{4}}\|u_t\|\cdot\|u\|_{H^1}^{\gamma-1}\cdot\|\nabla\widehat{u}_{n,2}\|\\
&\leqslant \frac{\delta(1-\alpha\delta)}{4}\|\nabla\widehat{u}_{n,2}\|^2 + c\lambda_{n+1}^{-\frac{1}{2}}\|u_t\|^2 + c\lambda_{n+1}^{\frac{\gamma-3}{2}}\|u_t\|^2\cdot\|u\|_{H^1}^{2\gamma-2}.
\end{aligned}
\tag{4.53}
$$

By condition (3.3), we get

$$
\begin{aligned}
&\big(\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)f(x,u),(I-P_n)\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\big)\\
&\leqslant c\|u\|_{H^1}^{\gamma}\cdot\|(I-P_n)\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\| + c\|(I-P_n)\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)z(\theta_t\omega)\|.
\end{aligned}
\tag{4.54}
$$

By using the Cauchy-Schwartz inequality and the Young inequality, we have

$$
\begin{aligned}
&\alpha\big(\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega),\ \widehat{v}_{n,2}\big)\\
\leqslant{}& \alpha\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega)\|\cdot\|\widehat{v}_{n,2}\| \\
\leqslant{}& \frac{7\alpha^2}{2(1-\delta)}\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega)\|^2+\frac{1-\delta}{14}\|\widehat{v}_{n,2}\|^2,
\end{aligned}
\tag{4.55}
$$

$$
\begin{aligned}
&\delta\big(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega),\ \widehat{v}_{n,2}\big)\\
\leqslant{}& \delta\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\|\cdot\|\widehat{v}_{n,2}\| \\
\leqslant{}& \frac{7\delta^2}{2(1-\delta)}\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\|^2+\frac{1-\delta}{14}\|\widehat{v}_{n,2}\|^2,
\end{aligned}
\tag{4.56}
$$

$$
\begin{aligned}
&\big(\widehat{\rho}(\frac{|x|^2}{r^2})g(x,t),\ \widehat{v}_{n,2}\big)\\
\leqslant{}& \|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,t)\|\cdot\|\widehat{v}_{n,2}\| \\
\leqslant{}& \frac{7}{2(1-\delta)}\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,t)\|^2+\frac{1-\delta}{14}\|\widehat{v}_{n,2}\|^2,
\end{aligned}
\tag{4.57}
$$

$$
\begin{aligned}
&-\alpha\Big(v\Delta\widehat{\rho}(\frac{|x|^2}{r^2})+2\nabla v\nabla\widehat{\rho}(\frac{|x|^2}{r^2}),\ \widehat{v}_{n,2}\Big)\\
={}& -\alpha\Big(v\big(\frac{4x^2}{r^4}\widehat{\rho}\,''(\frac{|x|^2}{r^2})+\frac{2}{r^2}\widehat{\rho}\,'(\frac{|x|^2}{r^2})\big)+\frac{4x}{r^2}\nabla v\cdot\widehat{\rho}\,'(\frac{|x|^2}{r^2}),\ \widehat{v}_{n,2}\Big)\\
\leqslant{}& \frac{2\alpha(4\mu_2+\mu_1)}{r^2}\|v\|\cdot\|\widehat{v}_{n,2}\|+\frac{4\sqrt{2}\mu_1\alpha}{r}\|\nabla v\|\cdot\|\widehat{v}_{n,2}\|\\
\leqslant{}& \frac{7\alpha^2}{2(1-\delta)}\Big(\frac{(8\mu_2+2\mu_1)^2}{r^4}\|v\|^2+\frac{32\mu_1^2}{r^2}\|\nabla v\|^2\Big)+\frac{1-\delta}{7}\|\widehat{v}_{n,2}\|^2,
\end{aligned}
\tag{4.58}
$$

$$
\begin{aligned}
&-(1-\alpha\delta)\Big(u\Delta\widehat{\rho}(\frac{|x|^2}{r^2})+2\nabla u\nabla\widehat{\rho}(\frac{|x|^2}{r^2}),\ \widehat{v}_{n,2}\Big)\\
={}& -(1-\alpha\delta)\Big(u\big(\frac{4x^2}{r^4}\widehat{\rho}\,''(\frac{|x|^2}{r^2})+\frac{2}{r^2}\widehat{\rho}\,'(\frac{|x|^2}{r^2})\big)+\frac{4x}{r^2}\nabla u\cdot\widehat{\rho}\,'(\frac{|x|^2}{r^2}),\ \widehat{v}_{n,2}\Big)\\
\leqslant{}& \frac{2(1-\alpha\delta)(4\mu_2+\mu_1)}{r^2}\|u\|\cdot\|\widehat{v}_{n,2}\|+\frac{4\sqrt{2}\mu_1(1-\alpha\delta)}{r}\|\nabla u\|\cdot\|\widehat{v}_{n,2}\|\\
\leqslant{}& \frac{7(1-\alpha\delta)^2}{2(1-\delta)}\Big(\frac{(8\mu_2+2\mu_1)^2}{r^4}\|u\|^2+\frac{32\mu_1^2}{r^2}\|\nabla u\|^2\Big)+\frac{1-\delta}{7}\|\widehat{v}_{n,2}\|^2.
\end{aligned}
\tag{4.59}
$$

From (4.50)-(4.59) we can obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\widehat{v}_{n,2}\|^2+\big(\delta^2+\lambda-\delta\big)\|\widehat{u}_{n,2}\|^2+(1-\alpha\delta)\|\nabla\widehat{u}_{n,2}\|^2\\
&\quad+2\big(\widehat{\rho}(\frac{|x|^2}{r^2})f(x,u),\widehat{u}_{n,2}\big)\Big)\\
\leqslant{}& -\alpha\|\nabla\widehat{v}_{n,2}\|^2-\frac{1-\delta}{2}\|\widehat{v}_{n,2}\|^2-\frac{\delta}{2}\big(\delta^2+\lambda-\delta\big)\|\widehat{u}_{n,2}\|^2\\
&\quad-\frac{\delta}{2}(1-\alpha\delta)\|\nabla\widehat{u}_{n,2}\|^2-\delta\big(\widehat{\rho}(\frac{|x|^2}{r^2})f(x,u),\widehat{u}_{n,2}\big)
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
&+c\Big(\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\|^2+\|(I-P_n)\nabla\big(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\big)\|^2\\
&+\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega)\|^2+\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,t)\|^2\Big)\\
&+\frac{c}{r^4}\Big(\|u\|^2+\|v\|^2\Big)+\frac{c}{r^2}\Big(\|\nabla u\|^2+\|\nabla v\|^2\Big)+c\lambda_{n+1}^{-\frac{1}{2}}\|u_t\|^2\\
&+c\lambda_{n+1}^{\frac{\gamma-3}{2}}\|u_t\|^2\cdot\|u\|_{H^1}^{2\gamma-2}+c\|u\|_{H^1}^{\gamma}\cdot\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\|.
\end{aligned}
$$

Recalling the new norm $\|\cdot\|_{E(U)}$ in (3.14), we have from (4.1) and (4.60) that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\widehat{\varphi}_{n,2}\|^2_{E(\mathbb{H}_{2r})}+2\big(\widehat{\rho}(\frac{|x|^2}{r^2})f(x,u),\widehat{u}_{n,2}\big)\Big)\\
\leqslant&-\sigma\Big(\|\widehat{\varphi}_{n,2}\|^2_{E(\mathbb{H}_{2r})}+2\big(\widehat{\rho}(\frac{|x|^2}{r^2})f(x,u),\widehat{u}_{n,2}\big)\Big)\\
&+c\Big(\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\|^2+\|(I-P_n)\nabla\big(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\big)\|^2\\
&+\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega)\|^2+\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,t)\|^2\Big)\\
&+\frac{c}{r^4}\Big(\|u\|^2+\|v\|^2\Big)+\frac{c}{r^2}\Big(\|\nabla u\|^2+\|\nabla v\|^2\Big)+c\lambda_{n+1}^{-\frac{1}{2}}\|u_t\|^2\\
&+c\lambda_{n+1}^{\frac{\gamma-3}{2}}\|u_t\|^2\cdot\|u\|_{H^1}^{2\gamma-2}+c\|u\|_{H^1}^{\gamma}\cdot\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\|.
\end{aligned}
\tag{4.61}
$$

Since $1\leqslant\gamma<3$, $\lambda_n\to\infty$ and (3.9), there exist $\widehat{N}_1=\widehat{N}_1(\varepsilon)>0$ and $\widehat{R}_1=\widehat{R}_1(\varepsilon)>0$ such that for all $n>\widehat{N}_1$ and $r>\widehat{R}_1$,

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\widehat{\varphi}_{n,2}\|^2_{E(\mathbb{H}_{2r})}+2\big(\widehat{\rho}(\frac{|x|^2}{r^2})f(x,u),\widehat{u}_{n,2}\big)\Big)\\
\leqslant&-\sigma\Big(\|\widehat{\varphi}_{n,2}\|^2_{E(\mathbb{H}_{2r})}+2\big(\widehat{\rho}(\frac{|x|^2}{r^2})f(x,u),\widehat{u}_{n,2}\big)\Big)\\
&+c\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,t)\|^2+\frac{c}{r^4}\Big(\|u\|^2+\|v\|^2\Big)\\
&+\frac{c}{r^2}\Big(\|\nabla u\|^2+\|\nabla v\|^2\Big)+\epsilon\big(\|u_t\|^6+\|u\|^6_{H^1}+|y(\theta_t\omega)|^2\big).
\end{aligned}
\tag{4.62}
$$

Multiplying (4.62) by $e^{\sigma t}$ and then integrating over $(\tau-t,\tau)$, we have for all $n>\widehat{N}_1$ and $r>\widehat{R}_1$,

$$
\begin{aligned}
&\|\widehat{\varphi}_{n,2}(\tau,\tau-t,\omega,\widehat{\varphi}_{n,2,0})\|^2_{E(\mathbb{H}_{2r})}\\
&+2\big(\widehat{\rho}(\frac{|x|^2}{r^2})f(x,u(\tau,\tau-t,\omega,u_0)),\widehat{u}_{n,2}(\tau,\tau-t,\omega,\widehat{u}_{n,2,0})\big)\\
\leqslant&\, e^{-\sigma t}\Big(\|\widehat{\varphi}_{n,2,0}\|^2_{E(\mathbb{H}_{2r})}+2\big(\widehat{\rho}(\frac{|x|^2}{r^2})f(x,u_0),\widehat{u}_{n,2,0}\big)\Big)\\
&+c\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,s)\|^2ds\\
&+\frac{c}{r^4}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\Big(\|u(s,\tau-t,\omega,u_0)\|^2+\|v(s,\tau-t,\omega,v_0)\|^2\Big)ds\\
&+\frac{c}{r^2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\Big(\|\nabla u(s,\tau-t,\omega,u_0)\|^2+\|\nabla v(s,\tau-t,\omega,v_0)\|^2\Big)ds
\end{aligned}
\tag{4.63}
$$

$$+\,\varepsilon\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\Big(\|u_t(s,\tau-t,\omega,u_0)\|^6+\|u(s,\tau-t,\omega,u_0)\|_{H^1}^6$$
$$+\,|y(\theta_s\omega)|^2\Big)ds.$$

By substituting ω by $\theta_{-\tau}\omega$, we can get from (4.63) that,

$$\begin{aligned}
&\|\widehat{\varphi}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{\varphi}_{n,2,0})\|_{E(\mathbb{H}_{2r})}^2\\
&+2\big(\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)f(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)),\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{u}_{n,2,0})\big)\\
\leqslant\,& e^{-\sigma t}\Big(\|\widehat{\varphi}_{n,2,0}\|_{E(\mathbb{H}_{2r})}^2+2\big(\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)f(x,u_0),\widehat{u}_{n,2,0}\big)\Big)\\
&+c\int_{-\infty}^{\tau}e^{\sigma(s-\tau)}\|(I-P_n)\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)g(x,s)\|^2ds\\
&+\frac{c}{r^4}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\Big(\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\Big)ds\\
&+\frac{c}{r^2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\Big(\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\Big)ds\\
&+\varepsilon\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\Big(\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^6\\
&+\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|_{H^1}^6+|y(\theta_{s-\tau}\omega)|^2\Big)ds.
\end{aligned}\tag{4.64}$$

We next estimate each term on the right-hand side of (4.64). By condition (3.3), $\varphi_0\in D(\tau-t,\theta_{-t}\omega)$ and $D(\tau-t,\theta_{-t}\omega)\in\mathcal{D}$, there exist $\widehat{T}_1=\widehat{T}_1(\tau,\varepsilon,D,\omega)>0$ and $\widehat{R}_1=\widehat{R}_1(\tau,\varepsilon,\omega)>1$, such that if $t>\widehat{T}_1$ and $r>\widehat{R}_1$, then

$$e^{-\sigma t}\Big(\|\widehat{\varphi}_{n,2,0}\|_{E(\mathbb{H}_{2r})}^2+2\big(\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)f(x,u_0),\widehat{u}_{n,2,0}\big)\Big)\leqslant\varepsilon.\tag{4.65}$$

For the second term on the right-hand side of (4.64), by condition (3.7), there is $\widehat{N}=\widehat{N}(\tau,\varepsilon,\omega)>0$, such that for all $n>\widehat{N}$, then

$$c\int_{-\infty}^{\tau}e^{\sigma(s-\tau)}\|(I-P_n)\widehat{\rho}\big(\frac{|x|^2}{r^2}\big)g(x,s)\|^2ds\leqslant\varepsilon.\tag{4.66}$$

For the third and fourth terms on the right-hand side of (4.64), by Lemma 4.1, there exist $\widehat{T}_2=\widehat{T}_2(\tau,\varepsilon,D,\omega)>0$ and $\widehat{R}_2=\widehat{R}_2(\tau,\varepsilon,\omega)>1$, such that for all $t>\widehat{T}_2$ and $r>\widehat{R}_2$, we obtain

$$\begin{aligned}
&\frac{c}{r^4}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\Big(\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\Big)ds\\
&+\frac{c}{r^2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\Big(\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2\Big)ds\\
\leqslant\,&\varepsilon.
\end{aligned}\tag{4.67}$$

For the last term on the right-hand side of (4.64), by Lemma 4.1 and Lemma 3.1 with $\epsilon = \frac{\sigma}{12}$, there is $\widehat{T}_3 = \widehat{T}_3(\tau, \varepsilon, D, \omega) > 0$, such that for all $t > \widehat{T}_3$, we obtain

$$\begin{aligned}
&\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\Big(\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^6 \\
&\quad + \|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|_{H^1}^6 + |y(\theta_{s-\tau}\omega)|^2\Big)ds \\
\leqslant &\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\Big(\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^6 + \|v(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^6 \\
&\quad + \|z(\theta_{s-\tau}\omega)\|^6 \\
&\quad + \|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|_{H^1}^6 + |y(\theta_{s-\tau}\omega)|^2\Big)ds \\
< &\infty.
\end{aligned} \tag{4.68}$$

Let $\widehat{T} = \max\{\widehat{T}_1, \widehat{T}_2, \widehat{T}_3\}$, and $\widehat{R} = \max\{\widehat{R}_1, \widehat{R}_2\}$. Then, it follows from (4.65), (4.66), (4.67) and (4.68) that, for all $t > \widehat{T}$, $r > \widehat{R}$ and $n > \widehat{N}$,

$$\|\widehat{\varphi}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{\varphi}_{n,2,0})\|_{E(\mathbb{H}_{2r})}^2 \leqslant c\varepsilon, \tag{4.69}$$

which completes the proof. □

5. Random attractors

In this section, we prove the existence of $\mathcal{D}$-pullback attractors for the stochastic problem (3.11)-(3.12) in $E(\mathbb{R}^n)$. We are now ready to apply the lemmas in Section 4 to prove the asymptotic compactness of solutions in $E(\mathbb{R}^n)$.

Lemma 5.1. *Assume that $h \in H^2(\mathbb{R})$ and (3.3)-(3.7) hold. Then the solution of problem (3.11)-(3.12) is asymptotic compactness in $E(\mathbb{R}^n)$; that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the sequence $\{\varphi(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m})\}$ has a convergent subsequence in $E(\mathbb{R}^n)$ provided $t_m \to \infty$ and $\varphi_{0,m} \in B(\tau - t_m, \theta_{-t_m}\omega)$.*

Proof. We first let $t_m \to \infty$, $B \in \mathcal{D}$, and $\varphi_{0,m} \in B(\tau - t_m, \theta_{-t_m}\omega)$. By Lemma 4.1, $\{\varphi(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m})\}$ is bounded in $E(\mathbb{R}^n)$; that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $M_1 = M_1(\tau, \omega, B) > 0$ such that for all $m > M_1$,

$$\|\varphi(\tau,\tau-t_m,\theta_{-\tau}\omega,\varphi_{0,m})\|_{E(\mathbb{R}^n)}^2 \leqslant \varrho^2(\tau,\omega). \tag{5.1}$$

In addition, it follows from Lemma 4.3 that there exist $k_1 = k_1(\tau,\varepsilon,\omega) > 0$ and $\widehat{M_2} = \widehat{M_2}(\tau, B, \varepsilon, \omega) > 0$, such that for every $m \geqslant \widehat{M_2}$,

$$\|\varphi(\tau,\tau-t_m,\theta_{-\tau}\omega,\varphi_{0,m})\|_{E(\mathbb{R}^n\setminus\mathbb{H}_{k_1})}^2 \leqslant \varepsilon. \tag{5.2}$$

Next, by using Lemma 4.4, there are $N = N(\tau,\varepsilon,\omega) > 0$, $k_2 = k_2(\tau,\varepsilon,\omega) \geqslant k_1$ and $\widehat{M_3} = \widehat{M_3}(\tau, B, \varepsilon, \omega) > 0$, such that for every $m \geqslant \widehat{M_3}$,

$$\|(I-P_N)\widehat{\varphi}(\tau,\tau-t_m,\theta_{-\tau}\omega,\varphi_{0,m})\|_{E(\mathbb{H}_{2k_2})}^2 \leqslant \varepsilon. \tag{5.3}$$

Using (4.44) and (5.1), we find that $\{P_N\widehat{\varphi}(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m})\}$ is bounded in the finite-dimensional space $P_N E(\mathbb{H}_{2k_2})$, which associates with (5.3) implies that $\{\widehat{\varphi}(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m})\}$ is precompact in $H_0^1(\mathbb{H}_{2k_2}) \times L^2(\mathbb{H}_{2k_2})$.

Note that $\widehat{\rho}\big(\frac{|x|^2}{k_2^2}\big) = 1$ for $|x| \leqslant k_2$. Recalling (4.44), we find that $\{\varphi\big(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m}\big)\}$ is precompact in $E(\mathbb{H}_{k_2})$, which along with (5.2) shows that the precompactness of this sequence in $E(\mathbb{R}^n)$. This completes the proof. □

The main result of this section can now be stated as follows.

Theorem 5.1. *Assume that $h \in H^2(\mathbb{R})$ and (3.3)-(3.7) hold. Then the continuous cocycle Φ associated with problem (3.11)-(3.12) has a unique $\mathcal{D}$-pullback attractor $\mathscr{A} = \{\mathscr{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $E(\mathbb{R}^n)$.*

Proof. Notice that the continuous cocycle Φ has a closed measurable $\mathcal{D}$-pullback absorbing set by Lemma 4.2. On the other hand, by (3.15) and Lemma 5.1, the continuous cocycle Φ is asymptotically compact in $E(\mathbb{R}^n)$. Then, by Proposition 2.1, the continuous cocycle Φ associated with (3.11)-(3.12) has a unique $\mathcal{D}$-pullback random attractor in $E(\mathbb{R}^n)$. □

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