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DYNAMICS OF A DAMPING OSCILLATOR WITH IMPACT AND IMPULSIVE EXCITATION

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Abstract There exist many types of external excitations that make the damping oscillator with impact have complex dynamics. In this study, both external impulsive excitation and impact are considered to construct a vibro-impact system. The fixed time pulse (impulsive excitation) and the state pulse (impact) lead to the complex and interesting dynamics. The conditions of the existence and stability of four kinds of periodic solutions are investigated, and the bifurcations of period-$(1, 0)$ and period-$(1, 1)$ solutions are analytically studied. Numerical simulations on periodic solutions and bifurcation diagrams are shown by the illustrative example.

Keywords Damping oscillator, impulsive excitation, impact, periodic solution, bifurcation.

MSC(2010) 34C10, 37C20, 37M20.

1. Introduction

A vibro-impact system, where a vibrator collides with one or more rigid walls, exists in a wide variety of engineering applications [9–11,13,18], particularly in mechanisms and machines with clearances or gaps [1, 20]. The trajectories of such systems in phase spaces have discontinuities caused by impacts. As an important type of non-smooth systems [3], it has been studied extensively in the past several decades. The strong nonlinearity, which is proposed because of the presence of impact, leads to complex dynamics of vibro-impact systems. Shaw and Rand [22] dealt with the saddle-node bifurcation in an inverted pendulum with rigid barriers. Hopf bifurcation was proved in a two-degree-of-freedom vibro-impact system by using the Poincaré map [15, 19]. The Melnikov method was used to explore the homoclinic bifurcation [6]. Period-doubling and saddle-node bifurcations were discussed in a vibro-impact system [21]. Besides these traditional bifurcations, there exist quite a few new non-classical bifurcations in vibro-impact systems. Periodic motions and grazing in a harmonically forced, piecewise, linear oscillator with impacts were demonstrated [16]. A piecewise linear second order equation was considered in [4] which is topologically equivalent to the sine-Gordon equation. The system is subjected to a time harmonic disturbance and the behavior of the periodic solutions

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was examined. To reduce C-bifurcation to the ordinary types of bifurcations, a visco-elastic model of impact was proposed and analyzed through a regular approach [14]. Shen and Du [23] discussed double impact periodic orbits in an inverted pendulum with harmonic excitation.

On the other hand, excitation plays a critical role in the dynamics of vibro-impact systems. One of the most important dynamical problems is the analysis of stability and bifurcations of periodic motions. In connect with excitation systems, this problem can be handled through experimental, analytical, numerical and approximate methods etc. For example, a two-dimensional linear oscillator with harmonic excitation was studied and periodic solutions were discussed by an analytical method [24]. A three-degree-of-freedom vibro-impact with harmonic excitation was considered, and double Neimark-Sacker bifurcation was presented by numerical simulations [7]. Vibration control capability of a combined tuned absorber and impact damper, under a random excitation, was performed numerically and experimentally [5]. Stochastic and chaotic response of a vibration system with random excitation was investigated [8]. Gan and Lei [12] discussed the global stochastic dynamics of a kind of vibro-impact oscillator under the multiple harmonic and bounded noisy excitations.

In contrast with harmonic excitation, there is not much work on impulsive excitation and its applications. Lenci and Rega [17] considered impulsive excitation in an impact inverted pendulum and investigated single impact periodic solution and non-classical bifurcation. In this study, we consider impulsive excitation of the same form as described by Bainov and Simeonov [2].

The paper is organized as follows. In the next section, impact and impulsive excitation are considered in a damping oscillator. In Section 3, we discuss the existence of four types of periodic solutions. Bifurcation analysis of two kinds of periodic solutions is presented in Section 4. In Section 5, numerical simulations on periodic solutions and bifurcation diagrams are shown by an illustrative example.

2. Model Description

Consider a linear damping oscillator.

$$x'' + 2\alpha x' + \beta^2 x = f(t),$$

where 2α $(\alpha > 0)$ and β^2 are the damping coefficient and the rigidity coefficient, respectively, and $f(t)$ is an external excitation.

It is notable that the external harmonic force $f(t) = F\cos(\omega t)$ or $f(t) = F\sin(\omega t + \tau)$, and the external random excitation $f(t) = F\sin\varphi(t)$, where $\varphi'(t) = \Omega + \xi(t)$, were usually used to build impact systems in physical phenomena. In [17], Lenci and Rega considered an impulsive excitation $f(t) = [-\delta(t-iT)+\delta(t-iT-a)]$, where $\delta(t)$ is the Dirac delta function and a is the time distance between the first and the second impulse.

In [2], to describe impulsive excitation precisely, it takes

$$x(t)_+ = x(t)_-, \quad x'(t)_+ = x'(t)_- + h, \quad t = nT \ \ (n \in \mathbf{N}),$$

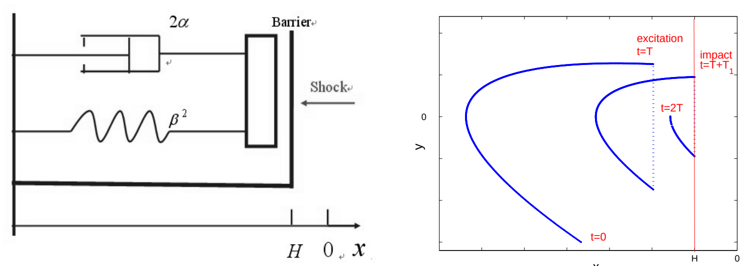
as the external impulsive excitation. At $t = nT$, the damping oscillator is subject to an external impulsive excitation or a shock effect, and its velocity $x'(t)$ obtains a constant increment h. Then the system which describes the work of damping oscillator with impulsive excitation and impact takes the form

$$\begin{cases} x'' + 2\alpha x' + \beta^2 x = 0, & t \neq nT, x < H, \\ x_+ = x_-, \ x'_+ = -\gamma x'_-, & x = H, \\ x_+ = x_-, \ \ x'_+ = x'_- + h, & t = nT, \end{cases} \tag{2.1}$$

where $n \in \mathbf{N}$, T is the time between two consecutive impulsive excitations, H represents the distance from the system's static equilibrium position to the single rigid barrier, and γ is the restitution factor to be a known parameter of impact losses, whereas subscripts "minus" and "plus" refer to values of response velocity just before and after the instantaneous impact or shock, respectively. System (2.1) is shown schematically in Figure 1(a), where the damping oscillator is supposed to be on the left side of the rigid barrier. Furthermore, we suppose $\gamma = 1$, $h < 0$, $H < 0$, and $\beta^2 > \alpha^2$ in this system. Let $y = x'$, then system (2.1) is equivalent to

$$\begin{cases} \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\beta^2 & -2\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, & t \neq nT, \quad x < H, \\ \begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_- \\ y_- \end{pmatrix}, & x = H, \\ \begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_- \\ y_- \end{pmatrix} + \begin{pmatrix} 0 \\ h \end{pmatrix}, & t = nT, \end{cases} \tag{2.2}$$

for which the phase portrait is shown in Figure 1(b). In the following, the values of α, β, H, and T are fixed. Our goal is to investigate the complex dynamics of system (2.2) by viewing h as a parameter.



(a) (b)

Figure 1. (a) Schematic representation of a damping oscillator with impulsive excitation and impact; (b) phase portrait of system (2.2).

3. Periodic Solutions

The normalized (at $t = 0$) fundamental matrix of the first equation of system (2.2) is

$$\Phi(t) = \begin{pmatrix} \frac{\cos\sqrt{\beta^2-\alpha^2}t}{e^{\alpha t}} + \frac{\alpha \sin\sqrt{\beta^2-\alpha^2}t}{\sqrt{\beta^2-\alpha^2}e^{\alpha t}} & \frac{\sin\sqrt{\beta^2-\alpha^2}t}{\sqrt{\beta^2-\alpha^2}e^{\alpha t}} \\ \frac{-\beta^2 \sin\sqrt{\beta^2-\alpha^2}t}{\sqrt{\beta^2-\alpha^2}e^{\alpha t}} & \frac{\cos\sqrt{\beta^2-\alpha^2}t}{e^{\alpha t}} - \frac{\alpha \sin\sqrt{\beta^2-\alpha^2}t}{\sqrt{\beta^2-\alpha^2}e^{\alpha t}} \end{pmatrix}. \tag{3.1}$$

The matrix $\Phi(t)$ satisfies the group property $\Phi(t_1)\Phi(t_2) = \Phi(t_1 + t_2)$. To discuss the existence and stability of periodic solutions of system (2.2), we let the period-$(m,\ n)$ solution be the periodic solution to system (2.2), which denotes the periodic orbit of period mT of system (2.2) that is subject to external impulsive excitation m times and impacts at $x = H$ n times per period. Assume that $(x(t), y(t))$ is a period-$(m,\ n)$ orbit, then there is a sequence $T_1,\ T_2,\ \cdots,\ T_n$ with $0 < T_1 < T_2 < \cdots < T_n \leq mT$ such that $x(T_k) = H$ for $k = 1,\ 2,\ \cdots,\ n$ and $x(t) < H$ for $t \in (0,\ mT) - \{T_1,\ T_2, \cdots,\ T_n\}$.

3.1. Period-$(1,0)$ Solution

In the case of period-$(1,0)$ solution, there is no impact but one impulsive excitation per period T. So we first discuss the condition for the existence of period-$(1,0)$ solution.

Set the initial point of system (2.2) to be $A(x_A, y_k)$, where $x_A < H$ and $y_k < 0$. The trajectory originating from the initial point A reaches the point $B(x_B, y_B)$ at $t = T$. We suppose $x_B = x_A < H$, then the trajectory jumps to the point $C(x_C, y_{k+1})$ for the effect of external excitation. It follows from system (2.2) that

$$\begin{aligned} x_B &= \varphi_{11}(T)x_A + \varphi_{12}(T)y_k, & y_B &= \varphi_{21}(T)x_A + \varphi_{22}(T)y_k, \\ x_C &= x_B = x_A < H, & y_{k+1} &= y_B + h, \end{aligned}$$

where $\varphi_{ij}(t)$, $i, j = 1,\ 2$, are the entries of the matrix $\Phi(t)$ defined by (3.1). So a discrete map is obtained

$$y_{k+1} = \left(\frac{\varphi_{12}(T)\varphi_{21}(T)}{1 - \varphi_{11}(T)} + \varphi_{22}(T)\right) y_k + h. \tag{3.2}$$

A direct calculation shows that the fixed point of map (3.2) is

$$y_0 = \left(1 - \varphi_{22}(T) - \frac{\varphi_{12}(T)\varphi_{21}(T)}{1 - \varphi_{11}(T)}\right)^{-1} h,$$

and the abscissa value of point A is

$$x_A = \left(-\varphi_{21} + \frac{(1 - \varphi_{11}(T))(1 - \varphi_{22}(T))}{\varphi_{12}(T)}\right)^{-1} h.$$

Furthermore, the calculation also indicates if there holds

$$\lambda_1 = \frac{\varphi_{12}(T)\varphi_{21}(T)}{1 - \varphi_{11}(T)} + \varphi_{22}(T) \in (-1,\ 1), \tag{3.3}$$

then the fixed point y_0 of map (3.2) is stable.

Under the condition of $y_k < 0$, $x_A < H$ and (3.3), the fixed point y_0 of map (3.2) corresponds to a stable period-$(1,0)$ solution of system (2.2). It follows that

$$\left(1 - \varphi_{22}(T) - \frac{\varphi_{12}(T)\varphi_{21}(T)}{1 - \varphi_{11}(T)}\right)^{-1} h < 0 \tag{3.4}$$

and

$$\left(-\varphi_{21}(T) + \frac{(1 - \varphi_{11}(T))(1 - \varphi_{22}(T))}{\varphi_{12}(T)}\right)^{-1} h < H. \tag{3.5}$$

In view of $h < 0$, condition (3.4) holds under condition (3.3). Hence, system (2.2) has a stable period-$(1, 0)$ solution without impacts under conditions (3.3) and (3.5). This stable period-$(1, 0)$ solution is also unique for the fixed point y_0 of map (3.2), which is stated in the following proposition.

Proposition 3.1. *Suppose that conditions* (3.3) *and* (3.5) *hold. Then system* (2.2) *has a unique stable period-*$(1, 0)$ *solution without impact.*

3.2. Period-$(1, 1)$ Solution

Set the initial point of system (2.2) to be $A_k(x_k, y_k)$, where $x_k < H$. The trajectory originating from the initial point A_k reaches the line $x = H$ at the point $A_{k0}(x_{k0},\ y_{k0})$ for $t = T_1$, where $0 < T_1 < T$. Then impact occurs and the trajectory jumps to the point $A_{k1}(x_{k1},\ y_{k1})$. The trajectory originating from the initial point A_{k1} reaches the point $A_{k2}(x_{k2},\ y_{k2})$ at $T_2 = T - T_1$, then jumps to $A_{k+1}(x_{k+1},\ y_{k+1})$ because of the effect of external impulsive excitation. It follows from (2.2) that

$$
\begin{aligned}
&x_{k0} = H = \varphi_{11}(T_1)x_k + \varphi_{12}(T_1)y_k, && y_{k0} = \varphi_{21}(T_1)x_k + \varphi_{22}(T_1)y_k,\\
&x_{k1} = x_{k0} = H, && y_{k1} = -y_{k0},\\
&x_{k2} = \varphi_{11}(T - T_1)x_{k1} + \varphi_{12}(T - T_1)y_{k1}, && y_{k2} = \varphi_{21}(T - T_1)x_{k1} + \varphi_{22}(T - T_1)y_{k1},\\
&x_{k+1} = x_{k2}, && y_{k+1} = y_{k2} + h.
\end{aligned}
$$

Thus we derive a corresponding discrete map

$$
\begin{cases}
x_{k+1} = \varphi_{11}(T_2)H - \varphi_{12}(T_2)(\varphi_{21}(T_1)x_k + \varphi_{22}(T_1)y_k),\\
y_{k+1} = \varphi_{21}(T_2)H - \varphi_{22}(T_2)(\varphi_{21}(T_1)x_k + \varphi_{22}(T_1)y_k) + h.
\end{cases}
\tag{3.6}
$$

Apparently, we see that the fixed point (x_0, y_0) of map (3.6) is actually associated with a period-$(1, 1)$ solution of system (2.2). Assume $1 + \varphi_{22}(T) \neq 0$. A direct calculation leads to the fixed point (x_0, y_0) of map (3.6)) as follows

$$
\left(\frac{\begin{vmatrix} \varphi_{11}(T_2)H & \varphi_{12}(T_2)\varphi_{22}(T_1) \\ \varphi_{21}(T_2)H + h & 1 + \varphi_{22}(T_2)\varphi_{22}(T_1) \end{vmatrix}}{1 + \varphi_{22}(T)},\ \frac{\begin{vmatrix} 1 + \varphi_{12}(T_2)\varphi_{21}(T_1) & \varphi_{11}(T_2)H \\ \varphi_{22}(T_2)\varphi_{21}(T_1) & \varphi_{21}(T_2)H + h \end{vmatrix}}{1 + \varphi_{22}(T)} \right),
\tag{3.7}
$$

and the associated characteristic polynomial is given by

$$
\begin{vmatrix} \lambda + \varphi_{12}(T_2)\varphi_{21}(T_1) & \varphi_{12}(T_2)\varphi_{22}(T_1) \\ \varphi_{22}(T_2)\varphi_{21}(T_1) & \lambda + \varphi_{22}(T_2)\varphi_{22}(T_1) \end{vmatrix},
$$

where $\lambda_2 = 0$ and $\lambda_3 = -\varphi_{22}(T)$. When

$$
-1 < \frac{\cos\sqrt{\beta^2 - \alpha^2}T}{e^{\alpha T}} - \frac{\alpha \sin\sqrt{\beta^2 - \alpha^2}T}{\sqrt{\beta^2 - \alpha^2}e^{\alpha T}} < 1,
\tag{3.8}
$$

where $1 + \varphi_{22}(T) \neq 0$ and $|\lambda_3| < 1$, the fixed point (x_0, y_0) is stable.

In view of $x_k < H$,

$$
\frac{\begin{vmatrix} \varphi_{11}(T_2)H & \varphi_{12}(T_2)\varphi_{22}(T_1) \\ \varphi_{21}(T_2)H + h & 1 + \varphi_{22}(T_2)\varphi_{22}(T_1) \end{vmatrix}}{1 + \varphi_{22}(T)} < H,
\tag{3.9}
$$

where $T_2 = T - T_1$ and T_1 is the positive root of equation

$$\varphi_{11}(T_1)x_0 + \varphi_{12}(T_1)y_0 = H. \tag{3.10}$$

Thus, the following proposition about the existence of period-$(1,1)$ solution is obtained.

Proposition 3.2. *Suppose that equation* (3.10) *has a root* T_1 *such that* $T_1 \in (0,\ T)$, *and conditions* (3.8) *and* (3.9) *hold, then system* (2.2) *has a stable period-*$(1,1)$ *solution.*

3.3. Period-$(2,1)$ and Period-$(3,2)$ Solutions

We first consider the period-$(2,1)$ solution, which is subject to external impulsive excitation twice and impacts at $x = H$ once per period. Set the initial point of system (2.2) to be $A_k(x_k, y_k)$, where $x_k \le H$. The trajectory originating from the initial point A_k reaches the point $A_{k0}(x_{k0},\ y_{k0})$ at $t = T$, and $x(t) < H$ for $t \in [0,\ T]$. External impulsive excitation encounters and the trajectory jumps to the point $A_{k1}(x_{k1},\ y_{k1})$. The trajectory originating from the initial point A_{k1} reaches the line $x = H$ at the point $A_{k2}(x_{k2},\ y_{k2})$ as $t = T + T_1$, where $0 < T_1 < T$. Then impact occurs and the trajectory jumps to the point $A_{k3}(x_{k3},\ y_{k3})$. The trajectory originating from the initial point A_{k3} reaches the point $A_{k4}(x_{k4},\ y_{k4})$ at $t = 2T$, and then jumps to $A_{k+1}(x_{k+1},\ y_{k+1})$ because of the effect of external excitation. It followed from system (2.2) that

$$\begin{aligned}
&x_{k0} = \varphi_{11}(T)x_k + \varphi_{12}(T)y_k, && y_{k0} = \varphi_{21}(T)x_k + \varphi_{22}(T)y_k,\\
&x_{k1} = x_{k0}, && y_{k1} = y_{k0} + h,\\
&x_{k2} = H = \varphi_{11}(T_1)x_{k1} + \varphi_{12}(T_1)y_{k1}, && y_{k0} = \varphi_{21}(T_1)x_{k1} + \varphi_{22}(T_1)y_{k1},\\
&x_{k3} = x_{k2} = H, && y_{k3} = -y_{k2},\\
&x_{k4} = \varphi_{11}(T - T_1)H + \varphi_{12}(T - T_1)y_{k3}, && y_{k4} = \varphi_{21}(T - T_1)H + \varphi_{22}(T - T_1)y_{k3},\\
&x_{k+1} = x_{k4}, && y_{k+1} = y_{k4} + h,
\end{aligned}$$

and then we obtain a discrete map

$$\begin{cases} x_{k+1} =\varphi_{11}(T - T_1)H - \varphi_{12}(T - T_1)((\varphi_{21}(T_1)\varphi_{11}(T)x_k + \varphi_{12}(T)y_k) \\ \qquad\quad + \varphi_{22}(T_1)(\varphi_{21}(T)x_k + \varphi_{22}(T)y_k + h)), \\ y_{k+1} =\varphi_{22}(T - T_1)H - \varphi_{21}(T - T_1)(\varphi_{21}(T_1)(\varphi_{11}(T)x_k + \varphi_{12}(T)y_k) \\ \qquad\quad + \varphi_{22}(T_1)(\varphi_{21}(T)x_k + \varphi_{22}(T)y_k + h)) + h. \end{cases} \tag{3.11}$$

While considering the period-$(3,2)$ solution, which is subject to external impulsive excitation three times and impacts at $x = H$ twice per period, by using a similar discussion we get the discrete map

$$\begin{cases} x_{k+1} = \varphi_{11}(T - T_2)H - \varphi_{12}(T - T_2)(\varphi_{21}(T_1)\tilde{x} + \varphi_{22}(T_1)\tilde{y}), \\ y_{k+1} = \varphi_{22}(T - T_2)H - \varphi_{21}(T - T_2)(\varphi_{21}(T_1)\tilde{x} + \varphi_{22}(T_1)\tilde{y}) + h, \end{cases} \tag{3.12}$$

where

$$\begin{cases} \tilde{x} =\varphi_{11}(T - T_1)H - \varphi_{12}(T - T_1)((\varphi_{21}(T_1)\varphi_{11}(T)x_k + \varphi_{12}(T)y_k) \\ \qquad + \varphi_{22}(T_1)(\varphi_{21}(T)x_k + \varphi_{22}(T)y_k + h)), \\ \tilde{y} =\varphi_{22}(T - T_1)H - \varphi_{21}(T - T_1)(\varphi_{21}(T_1)(\varphi_{11}(T)x_k + \varphi_{12}(T)y_k) \\ \qquad + \varphi_{22}(T_1)(\varphi_{21}(T)x_k + \varphi_{22}(T)y_k + h)) + h. \end{cases}$$

Note that the fixed point of map (3.11) corresponds to a period-$(2,1)$ solution of system (2.2) while the fixed point of map (3.12) corresponds to a period-$(3,2)$ solution of system (2.2). So maps (3.11) and (3.12) can be used to discuss the existence of periodic solutions of system (2.2).

4. Bifurcation Analysis

4.1. Bifurcation of Period-$(1,0)$ Solution

As described in subsection 3.1, system (2.2) has a unique stable period-$(1,0)$ solution under conditions (3.3) and (3.5). The fixed point of map (3.2) is

$$y_0 = \left(1 - \varphi_{22}(T) - \frac{\varphi_{12}(T)\varphi_{21}(T)}{1-\varphi_{11}(T)}\right)^{-1} h,$$

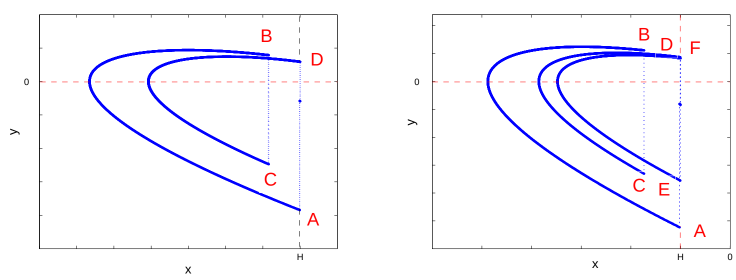
and the associated characteristic value of this fixed point is

$$\lambda_1 = \frac{\varphi_{12}(T)\varphi_{21}(T)}{1-\varphi_{11}(T)} + \varphi_{22}(T).$$

A direct calculation shows that under condition (3.3) the fixed point of map (3.2) is stable when $h < 0$. From (3.5), the critical value of h is

$$h_0 = \left(-\varphi_{21}(T) + \frac{(1-\varphi_{11}(T))(1-\varphi_{22}(T))}{\varphi_{12}(T)}\right) H, \tag{4.1}$$

and bifurcation occurs at $h = h_0$.



(a) (b)

Figure 2. (a) Period-$(2,1)$ solution of system (2.2); (b) period-$(3,1)$ solution of system (2.2).

Suppose that system (2.2) has a period-$(2,1)$ solution at $h = h_0$, which is shown in Figure 2(a). The trajectory originating from the initial point $A(H,\ y_A)$ reaches the point $B(x_B,\ y_B)$ at $t = T$, jumps to the point $C(x_C,\ y_C)$ for the effect of excitation, reaches the point $D(H,\ y_D)$ at $t = 2T$, and then jumps to the point $A(H,\ y_A)$, where $y_A = -y_D + h$. It follows from system (2.2) that

$$\begin{aligned}
&x_B = \varphi_{11}(T)H + \varphi_{12}(T)y_A, && y_B = \varphi_{21}(T)H + \varphi_{22}(T)y_A,\\
&x_C = x_B < H, && y_C = y_B + h,\\
&x_D = H = \varphi_{11}(T)x_C + \varphi_{12}(T)y_C, && y_D = \varphi_{21}(T)x_C + \varphi_{22}(T)y_C,\\
&x_A = x_D = H, && y_A = -y_D + h.
\end{aligned}$$

Then we have

$$
\begin{aligned}
&H = \varphi_{11}(T)(\varphi_{11}(T)H + \varphi_{12}(T)y_A) + \varphi_{12}(T)(\varphi_{21}(T)H + \varphi_{22}(T)y_A + h),\\
&y_A = -(\varphi_{21}(T)(\varphi_{11}(T)H + \varphi_{12}(T)y_A) + \varphi_{22}(T)(\varphi_{21}(T)H + \varphi_{22}(T)y_A + h)) + h.
\end{aligned}
$$

That is

$$
\begin{cases}
H = \varphi_{11}(2T)H + \varphi_{12}(2T)y_A + \varphi_{12}(T)h,\\
y_A = -(\varphi_{21}(2T)H + \varphi_{22}(2T)y_A + \varphi_{22}(T)h) + h,
\end{cases}
$$

and

$$
h = \frac{(1-\varphi_{11}(2T))(1+\varphi_{22}(2T)) + \varphi_{21}(2T)\varphi_{12}(2T)}{\varphi_{12}(T)(1+\varphi_{22}(2T)) + \varphi_{12}(2T)(1-\varphi_{22}(T))} H =: h_1. \tag{4.2}
$$

If $h_0 = h_1$, where h_0 is given in (4.1), then we have

$$
\begin{aligned}
&-\varphi_{21}(T) + \frac{(1-\varphi_{11}(T))(1-\varphi_{22}(T))}{\varphi_{12}(T)}\\
=&\frac{(1-\varphi_{11}(2T))(1+\varphi_{22}(2T)) + \varphi_{21}(2T)\varphi_{12}(2T)}{\varphi_{12}(T)(1+\varphi_{22}(2T)) + \varphi_{12}(2T)(1-\varphi_{22}(T))},
\end{aligned} \tag{4.3}
$$

and a period-$(2,1)$ solution bifurcates from the period-$(1,0)$ solution at $h = h_0$. So we obtain the following result.

Proposition 4.1. *Suppose that condition* (4.3) *holds. Then a period-*$(2,1)$ *solution bifurcates from a period-*$(1,0)$ *solution of system* (2.2) *at* $h = h_0$*, where* h_0 *is given in* (4.1).

Suppose that system (2.2) has a period-$(3,1)$ solution at $h = h_0$, which is shown in Figure 2(b). The trajectory originating from the initial point $A(H,\ y_A)$ reaches the point $B(x_B,\ y_B)$ at $t = T$, and jumps to the point $C(x_C,\ y_C)$ because of the effect of excitation. It reaches the point $D(x_D,\ y_D)$ at $t = 2T$, jumps to the point $E(x_E,\ y_E)$, reaches the point $F(H,\ y_F)$ at $t = 3T$, and then jumps to the point $A(H,\ y_A)$, where $y_A = -y_F + h$. It follows from system (2.2) that

$$
\begin{aligned}
&x_B = \varphi_{11}(T)H + \varphi_{12}(T)y_A, && y_B = \varphi_{21}(T)H + \varphi_{22}(T)y_A,\\
&x_C = x_B < H, && y_C = y_B + h,\\
&x_D = \varphi_{11}(T)x_C + \varphi_{12}(T)y_C, && y_D = \varphi_{21}(T)x_C + \varphi_{22}(T)y_C,\\
&x_E = x_D < H, && y_E = y_D + h,\\
&x_F = H = \varphi_{11}(T)x_E + \varphi_{12}(T)y_E, && y_F = \varphi_{21}(T)x_E + \varphi_{22}(T)y_E,\\
&x_F = x_A = H, && y_A = -y_F + h.
\end{aligned}
$$

Thus we get

$$
\begin{cases}
H = \varphi_{11}(3T)H + \varphi_{12}(3T)y_A + \varphi_{12}(2T)h + \varphi_{12}(T)h,\\
y_A = -(\varphi_{21}(3T)H + \varphi_{22}(3T)y_A + \varphi_{22}(2T)h + \varphi_{22}(T)h) + h,
\end{cases}
$$

and

$$
h = \frac{(1-\varphi_{11}(3T))(1+\varphi_{22}(3T)) + \varphi_{21}(3T)\varphi_{12}(3T)}{(\varphi_{12}(2T)+\varphi_{12}(T))(1+\varphi_{22}(3T)) + \varphi_{12}(3T)(1-\varphi_{22}(2T)-\varphi_{22}(T))} H =: h_2. \tag{4.4}
$$

Let $h_0 = h_2$, where h_0 is the same as (4.1). Then we obtain

$$\begin{aligned}&\frac{(1-\varphi_{11}(T))(1-\varphi_{22}(T))}{\varphi_{12}(T)} - \varphi_{21}(T)\\ =&\frac{(1-\varphi_{11}(3T))(1+\varphi_{22}(3T))+\varphi_{21}(3T)\varphi_{12}(3T)}{(\varphi_{12}(2T)+\varphi_{12}(T))(1+\varphi_{22}(3T))+\varphi_{12}(3T)(1-\varphi_{22}(2T)-\varphi_{22}(T))},\end{aligned} \tag{4.5}$$

and a period-$(3,1)$ solution bifurcates from the period-$(1,0)$ solution at $h = h_0$. Hence, we have the following result.

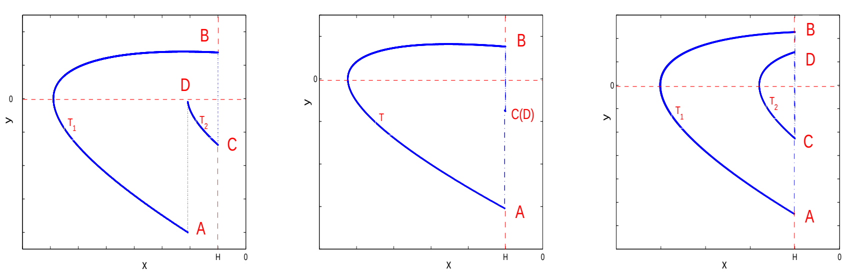
Proposition 4.2. *Suppose that condition* (4.5) *holds. Then a period-*$(3,1)$ *solution bifurcates from a period-*$(1,0)$ *solution of system* (2.2) *at* $h = h_0$*, where* h_0 *is given in* (4.1).

4.2. Bifurcation of Period-$(1,1)$ Solution

A period-$(1,1)$ solution of system (2.2) corresponds to a fixed point of map (3.6), indicated as in (3.7). The multipliers of the fixed point are $\lambda_2 = 0$ and $\lambda_3 = -\varphi_{22}(T)$, which have nothing to do with h. Figure 3(a) shows a period-$(1,1)$ solution of system (2.2), where

$$A(x_0,\ y_0)\xrightarrow{T_1}B(H,\ y_B)\xrightarrow{impact}C(H,\ -y_B)\xrightarrow{T_2}D(x_D,\ y_D)\xrightarrow{excitation}A(x_0,\ y_0).$$

The location of the point D varies as the parameter h changes. To discuss the bifurcation of the period-$(1,1)$ solution, the limit location of point D is analyzed.



(a) (b) (c)

Figure 3. Three period-$(1,1)$ solutions of system (2.2), where $T_1 + T_2 = T$.

In Figure 3(b), $T_2 = 0$ and the point D reaches the point C on the line $x = H$. The trajectory originating from the initial point $A(H,\ y_A)$ reaches the point $B(H,\ y_B)$ at $t = T$. The external impulsive excitation and impact make the trajectory jump to the point $A(H,\ y_A)$, where $y_A = -y_B + h$. It follows from system (2.2) that

$$x_B = H = \varphi_{11}(T)H + \varphi_{12}(T)y_A, \quad y_B = \varphi_{21}(T)H + \varphi_{22}(T)y_A, \tag{4.6}$$

$$y_A = -y_B + h. \tag{4.7}$$

Then we have

$$h = \varphi_{21}(T)H + \frac{(1-\varphi_{11}(T))(1+\varphi_{22}(T))}{\varphi_{12}(T)}H =: h_3. \tag{4.8}$$

In Figure 3(c), $T_2 \neq 0$ and the point D is above the point C on the line $x = H$. The trajectory originating from the initial point $A(H,\ y_A)$ reaches the point $B(H,\ y_B)$ at $t = T_1$, jumps to the point $C(H,\ y_C)$ for the effect of impact, where $y_C = -y_B$. It reaches the point $D(H,\ y_D)$ at $t = T$, and jumps to the point $A(H,\ y_A)$ for the effect of impulsive excitation, where $y_A = y_D + h$. It follows from system (2.2) that

$$\begin{aligned}
&x_B = H = \varphi_{11}(T_1)H + \varphi_{12}(T_1)y_A, && y_B = \varphi_{21}(T_1)H + \varphi_{22}(T_1)y_A,\\
&x_C = H, && y_C = -y_B,\\
&x_D = H = \varphi_{11}(T_2)H + \varphi_{12}(T_2)y_C, && y_D = \varphi_{21}(T_2)H + \varphi_{22}(T_2)y_C,\\
&x_A = H, && y_A = y_D + h.
\end{aligned}$$

So we have

$$y_A = \frac{1-\varphi_{11}(T_1)}{\varphi_{12}(T_1)}H, \quad y_C = -y_B = -\varphi_{21}(T_1)H - \frac{(1-\varphi_{11}(T_1))\varphi_{22}(T_1)}{\varphi_{12}(T_1)}H,$$

$$\begin{cases} H = \varphi_{11}(T_2)H + \varphi_{12}(T_2)\left(-\varphi_{21}(T_1)H - \frac{(1-\varphi_{11}(T_1))\varphi_{22}(T_1)}{\varphi_{12}(T_1)}H\right), \\ y_A = \varphi_{21}(T_2)H + \varphi_{22}(T_2)\left(-\varphi_{21}(T_1)H - \frac{(1-\varphi_{11}(T_1))\varphi_{22}(T_1)}{\varphi_{12}(T_1)}H\right) + h, \end{cases}$$

and

$$h = (\varphi_{21}(T_1)\varphi_{22}(T_2) - \varphi_{21}(T_2) + \frac{(1-\varphi_{11}(T_1))(1+\varphi_{22}(T_1)\varphi_{22}(T_2))}{\varphi_{12}(T_1)})H =: h_4, \tag{4.9}$$

where $T_1 + T_2 = T$ and T_1 is the solution of equation

$$\varphi_{11}(T-T_1) + \varphi_{12}(T-T_1)\left(-\varphi_{21}(T_1) - \frac{(1-\varphi_{11}(T_1))\varphi_{22}(T_1)}{\varphi_{12}(T_1)}\right) = 1. \tag{4.10}$$

The values of the bifurcation parameter h at the bifurcation points of period-$(1,1)$ solution are calculated and shown in (4.8) and (4.9). It follows from Proposition 3.2 that the period-$(1,1)$ solution of system (2.2) is stable for $h \in (h_3,\ h_4)$ or $h \in (h_4,\ h_3)$. So we have

Proposition 4.3. *The period-$(1,1)$ solution in Proposition 3.2 is stable for $h \in (h_3,\ h_4)$ or $h \in (h_4,\ h_3)$, where h_3 and h_4 are given in (4.8) and (4.9), respectively. Furthermore, bifurcation occurs at $h = h_3$ and $h = h_4$.*

5. Numerical Results

Consider

$$\begin{cases} \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -0.2 & -0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, & t \neq 5n,\ x < -0.1, \\ \begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_- \\ y_- \end{pmatrix}, & x = -0.1, \\ \begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_- \\ y_- \end{pmatrix} + \begin{pmatrix} 0 \\ h \end{pmatrix}, & t = 5n,\ n \in \mathbf{N}. \end{cases} \tag{5.1}$$

Let $\alpha = 0.4$, $\beta^2 = 0.2$, $T = 5$ and $H = -0.1$. The normalized fundamental matrix is

$$\Phi(t) = \begin{pmatrix} e^{-0.4t}(\cos 0.2t + 2\sin 0.2t) & 5e^{-0.4t}\sin 0.2t \\ -e^{-0.4t}\sin 0.2t & e^{-0.4t}(\cos 0.2t - 2\sin 0.2t) \end{pmatrix}.$$

Set $h=-0.2$, $\varphi_{22}(T)+\frac{\varphi_{12}(T)\varphi_{21}(T)}{1-\varphi_{11}(T)}\approx -0.2474\in(-1,\ 1)$ and

$$\left(-\varphi_{21}(T)+\frac{(1-\varphi_{11}(T))(1-\varphi_{22}(T))}{\varphi_{12}(T)}\right)^{-1}h\approx -0.1306<H=-0.1.$$

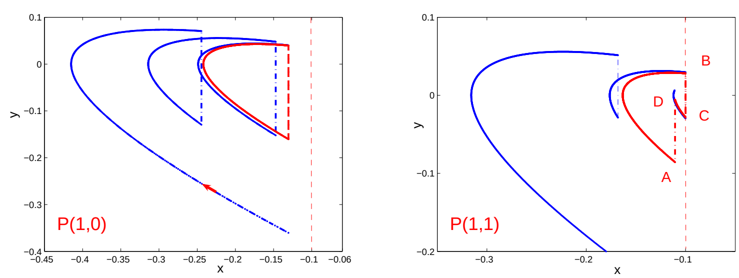
Assume that conditions (3.3) and (3.5) hold. It follows from Proposition 3.1 that system (5.1) has a unique stable period-$(1,0)$ solution, which is shown in Figure 4(a).

Set $h=-0.08$. After calculations, we find that the fixed point of map (3.6) with is $A(-0.1105,-0.0859)$, $T_1\approx 4.342$ and $T_2=T-T_1\approx 0.658$. Since $x_A=-0.1105<H=-0.1$, condition (3.9) holds. Furthermore, we see that $\varphi_{22}(T)\approx -0.1546\in(-1,\ 1)$ and condition (3.8) holds. It is observed from Figure 4(b) that system (5.1) has a stable period-$(1,1)$ solution

$$\begin{aligned}A(-0.1105,-0.0859)&\xrightarrow{T_1}B(-0.1,0.0282)\xrightarrow{impact}C(-0.1,-0.0282)\\&\xrightarrow{T_2}D(-0.1105,-0.0059)\xrightarrow{excitation}A(-0.1105,\ -0.0859),\end{aligned}$$

which is in agreement with Proposition 3.2.

Set $h=-0.12$ in system (5.1). It is observed from Figure 5 that system (5.1) has a period-$(2,1)$ solution. The damping oscillator is subject to the impulsive excitation twice and collides with the rigid wall once every period 10. Set $h=-0.106$. It is shown from Figure 6 that system (5.1) has a period-$(3,2)$ solution, where the damping oscillator is subject to impulsive excitation three times of and collides with the rigid wall twice every period 15.

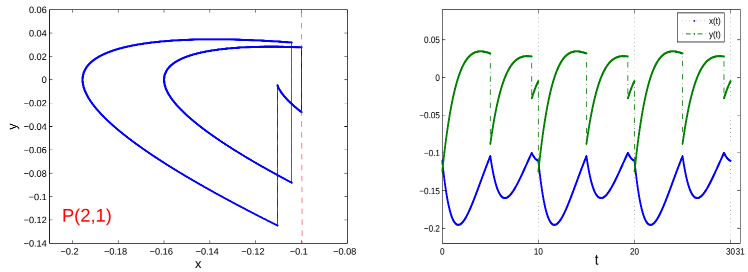


(a) (b)

Figure 4. (a) A period-(1, 0) solution (red) of system (5.1) with $h=-0.2$, and the trajectory (blue) originating from the initial point (-0.13, -0.37) tending to the period-(1, 0) solution; (b) a period-(1, 1) solution (red) of system (5.1) with $h=-0.08$, and the trajectory (blue) originating from the initial point (-0.18, -0.2) tending to the period-(1, 1) solution.

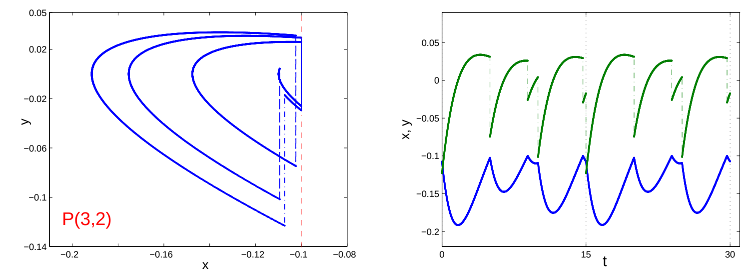
It is illustrated in Figure 4(b), Figure 5, and Figure 6 that system (5.1) has a solution with period-(1, 1), with period-(3, 2), and with period-(2, 1) in the cases of $h=\tilde{h}_1=-0.08$, $h=\tilde{h}_3=-0.106$ and $h=\tilde{h}_2=-0.12$, respectively, where $\tilde{h}_2<\tilde{h}_3<\tilde{h}_1$.

Now we consider the bifurcation of the period-$(1,0)$ solution. A straightforward



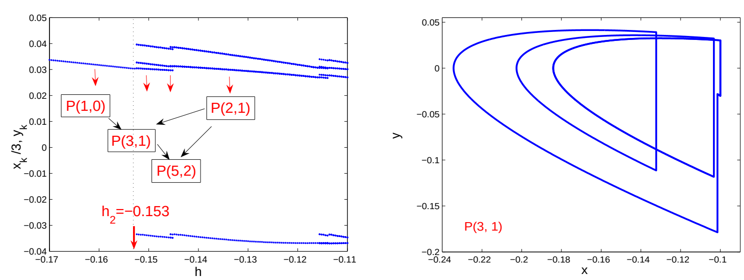
(a) (b)

Figure 5. For system (5.1) with $h = -0.12$: (a) the period-(2, 1) solution; (b) the time series of x and y.



(a) (b)

Figure 6. For system (5.1) with $h = -0.106$: (a) the period-(3, 2) solution, (b) the time series of x and y.



(a) (b)

Figure 7. (a) The bifurcation diagram of system (5.1) for $h \in (-0.17, -0.11)$; (b) a period-(3, 1) solution of system (5.1) for $h = -0.151$.

calculation gives

$$
\begin{aligned}
&\frac{(1-\varphi_{11}(T))(1-\varphi_{22}(T))}{\varphi_{12}(T)} - \varphi_{21}(T) \\
=&\frac{(1-e^{-2}(cos1+2sin1))(1-e^{-2}(cos1-2sin1))}{5e^{-2}sin1} + e^{-2}\sin 1 \\
\approx&1.53,
\end{aligned}
$$

and

$$\frac{(1-\varphi_{11}(3T))(1+\varphi_{22}(3T))+\varphi_{21}(3T)\varphi_{12}(3T)}{(\varphi_{12}(2T)+\varphi_{12}(T))(1+\varphi_{22}(3T))+\varphi_{12}(3T)(1-\varphi_{22}(2T)-\varphi_{22}(T))}\approx 1.53,$$

where $T=5$. It is easy to see that condition (4.5) holds and $h_0=h_2=-0.153$. It is demonstrated from Figure 7(a) that system (5.1) has a stable period-(1, 0) solution for $h<-0.153$ and a period-(3, 1) solution bifurcates from this period-(1, 0) solution at $h=h_0\approx-0.153$, which is in agreement with Proposition 4.2. The phase portrait of a period-(3, 1) solution for $h=-0.151$ is shown in Figure 7(b).

It follows from (4.8) that

$$\begin{aligned} h_3 &= \big(-e^{-2}\sin 1+\tfrac{(1-5e^{-2}(0.2\cos 1+0.4\sin 1))(1+5e^{-2}(0.2\cos 1-0.4\sin 1))}{5e^{-2}\sin 1}\big)\times(-0.1)\\ &\approx -0.09241. \end{aligned}$$

Similarly, it follows from (4.9) and (4.10) that

$$h_4\approx-0.06899,\quad T_1\approx 3.2453.$$

By viewing h as a parameter, the bifurcation diagram of periodic solutions of (5.1) for $h\in(-0.12,\ -0.04)$ is presented in Figure 8. The bifurcation diagram of the period-(m, n) solutions is divided into two parts for the fact that there exist a fixed time pulse and a state pulse in system (5.1). Assume that $(x(t),y(t))$ is a period-(m, n) orbit for some parameter h_0, the m points $(h_0,\ x(T))$, $(h_0,\ x(2T))$, $\cdots$, and $(h_0,x(mT))$ are plotted in the lower half of the bifurcation diagram. Set the impact surface $x=-0.1$ as the Poincaré section. The trajectory reaches the Poincaré section at $(-0.1,y(T_k))$, where $k=1,\ 2,\ \cdots,\ n$, $x(T_k)=-0.1$, and $0<T_1<T_2<\cdots<T_n\leq mT$. The n points $(h_0,\ y(T_1))$, $(h_0,\ y(T_2))$, $\cdots$, and $(h_0,y(T_n))$ are plotted in the upper half of the bifurcation diagram.

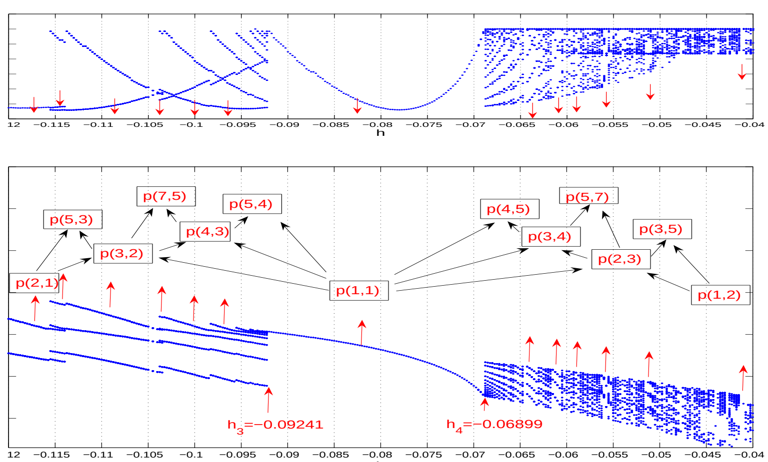
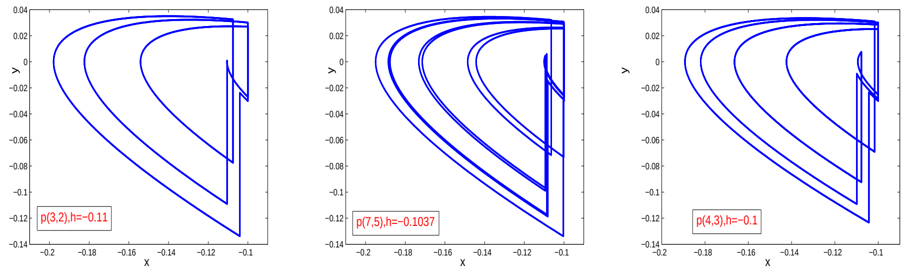


Figure 8. Bifurcation diagrams of system (5.1) with respect to the parameter h.

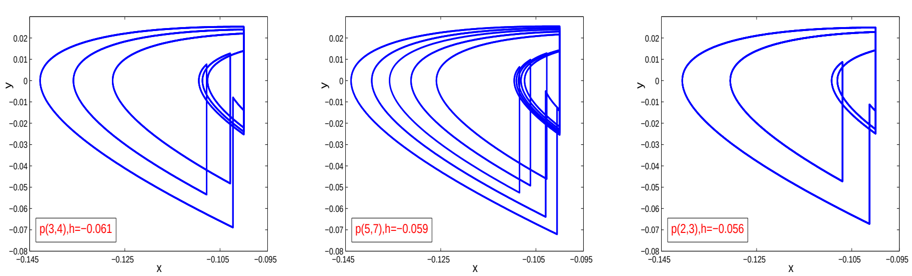
System (5.1) has a stable period-(1, 1) solution for $h\in(-0.09241,\ -0.06899)$, and bifurcation occurs at $h=h_3=-0.09241$ and $h=h_4=-0.06899$, which agrees well with Proposition 4.3.



(a) (b) (c)

Figure 9. Periodic solutions of system (5.1): (a) the period-(3, 2) solution with $h = -0.11$; (b) the period-(7, 5) solution with $h = -0.1037$; (c) the period-(4, 3) solution with $h = -0.1$.

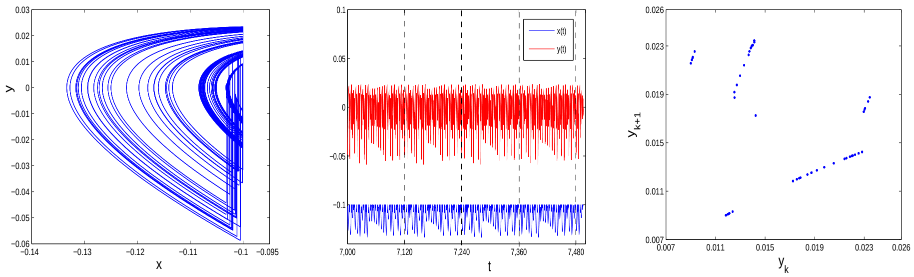
Figure 8 shows that system (5.1) has a stable period-(3, 2) for $h \in (-0.104, -0.01138)$ and a stable period-(2, 1) solution for $h \in (-0.12, -0.01155)$. It is demonstrated that the period-(3, 2) solution locates between the period-(1, 1) and the period-(2, 1) solutions. It is also seen that the period-(4, 3) solution locates between the period-(3, 2) and the period-(1, 1) solutions, the period-(5, 3) solution locates between the period-(2, 1) and the period-(3, 2) solutions, the period-(5, 4) solution locates between the period-(4, 3) and the period-(1, 1) solutions, and the period-(7, 5) solution locates between the period-(3, 2) and the period-(4, 3) solutions (See Figure 9).



(a) (b) (c)

Figure 10. Periodic solutions of system (5.1): (a) the period-(3, 4) solution with $h = -0.061$; (b) the period-(5, 7) solution with $h = -0.059$; (c) the period-(2, 3) solution with $h = -0.056$.

Compared with the case of $h \in (-0.12, -0.069)$, the bifurcation of periodic solution of system (5.1) looks more complicated for $h \in (-0.069, -0.04)$. It is illustrated in Figure 8 that there exist many periodic windows in the bifurcation diagram. For example, the windows of period-(4, 5), period-(3, 4), period-(5, 7), period-(2, 3), period-(3, 5), and period-(1, 2) solutions appear near $h = -0.064$, $h = -0.061$, $h = -0.059$, $h = -0.056$, $h = -0.0515$, $h = -0.041$, respectively. The period-(3, 4), period-(5, 7), and period-(2, 3) solutions are shown in Figure 10. It follows from Figure 8, Figure 9, and Figure 10 that the period-$(m_1 + m_2,\ n_1 + n_2)$ solution locates between the period-$(m_1,\ n_1)$ and the period-$(m_2,\ n_2)$ solutions. Set $h = -0.0458$, the phase portrait of system (5.1) is shown in Figure 11(a). It is seen from Figure 11(a) that there exists a period-120 orbit in (5.1). The orbit reaches the Poincaré section $x = -0.1$ at $(-0.1,\ y_k)$ and the portrait of $(y_k,\ y_{k+1})$ is shown



(a) (b) (c)

Figure 11. For system (5.1) with $h = -0.0458$ and $t \in (7000,\ 7500)$: (a) the phase portrait, (b) the time series of x and y, (c) the portrait of $(y_k,\ y_{k+1})$.

in Figure 11(c).

The dynamics of a damping oscillator with impact and impulsive excitation was studied in this paper. It was seen that the fixed time pulse and the state pulse lead to the complex and interesting dynamics of system (2.2). The conditions for the existence of four kinds of periodic solutions were particularly investigated. The bifurcations of period-(1, 0) and period-(1, 1) solutions were analytically studied. Numerical results were provided to show the relationship of periodic solutions, that is, the period-(m_1+m_2, n_1+n_2) solution is between a period-(m_1, n_1) and a period-(m_2, n_2) solution.

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