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A GLOBAL SUPERCONVERGENT L^∞-ERROR ESTIMATE OF MIXED FINITE ELEMENT METHODS FOR SEMILINEAR ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract In this paper, we discuss the superconvergence of mixed finite element methods for a semilinear elliptic control problem with an integral constraint. The state and co-state are approximated by the order $k = 1$ Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. Approximation of the optimal control of the continuous optimal control problem will be constructed by a projection of the discrete adjoint state. It is proved that this approximation has convergence order h^2 in L^∞-norm. Finally, a numerical example is given to demonstrate the theoretical results.

Keywords Semilinear elliptic equations, optimal control problems, superconvergence, mixed finite element methods, L^∞-error estimate.

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1. Introduction

It is well known that the finite element approximation plays an important role in the numerical treatment of optimal control problems. There have been extensive studies in convergence and superconvergence of finite element approximations for optimal control problems, see, for example, [1, 6, 12–14, 17, 18, 21–23]. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, for example, [10, 20].

Since 2006, Chen etc. have done some works on priori error estimates and superconvergence properties of mixed finite elements for optimal control problems [3–5, 7, 8, 16]. In [4], the author used the postprocessing projection operator, which was defined by Meyer & Rösch (see [21]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, the authors derived error estimates and superconvergence of mixed methods for convex optimal control problems in [5]. Hou & Chen [7] derived a superconvergent L^2-error estimates of RT1 mixed methods for semilinear elliptic optimal control problems. Next, in [15], Hou investigated the RT0 mixed finite element methods for a semilinear elliptic optimal control problem with a pointwise control constraint, he derive a superconvergence result for the control variable and L^∞-error estimates for all variables even for the divergence of the vector-valued functions. In [8], Chen & Hou considered the same

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problem as in [15], they derived the superconvergence for the vector-valued functions and a priori H^{-1}-error estimates for the control, the state and the co-state. As far as we know, there is no superconvergent L^∞-error estimates of RT mixed finite element method for semilinear elliptic optimal control problems in the literature.

The aim of this paper is to investigate the superconvergence property of mixed finite element approximation for a semilinear elliptic control problem with an integral constraint. Firstly, we derive the superconvergence property between average L^2 projection and the approximation of the control variable, the convergence order is h^2 instead of $h^{\frac{3}{2}}$ in [5], which is caused by the different admissible set. Then, after solving a fully discretized optimal control problem, a control $\hat{u}$ is calculated by the projection of the adjoint state z_h in a postprocessing step. Although the approximation of the discretized solution is only of order h in L^∞-norm, we will show that this postprocessing step improves the convergence order to h^2. Finally, we present a numerical experiment to demonstrate the practical side of the theoretical results about superconvergence.

We consider the following semilinear optimal control problems for the state variable y and the control u with an integral constraint:

$$\min_{u\in U_{ad}} \left\{ \frac{1}{2}\|y-y_d\|^2 + \frac{\nu}{2}\|u\|^2 \right\} \tag{1.1}$$

subject to the state equation

$$-\operatorname{div}(A(x)\mathbf{grad}y) + \phi(y) = u, \quad x \in \Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\operatorname{div}\boldsymbol{p} + \phi(y) = u, \quad \boldsymbol{p} = -A(x)\mathbf{grad}y, \quad x \in \Omega, \tag{1.3}$$

and the boundary condition

$$y = 0, \quad x \in \partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad} = \left\{ u \in L^\infty(\Omega) : \int_\Omega u dx \geq 0 \right\}. \tag{1.5}$$

We assume that the function $\phi(\cdot) \in W^{2,\infty}(-R,R) \cap H^3(-R,R)$ for any $R > 0$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi' \geq 0$. Moreover, we assume that $y_d \in W^{1,\infty}(\Omega)$ and ν is a fixed positive number. The coefficient $A(x) = (a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x) \in W^{2,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_*|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j, \quad \forall\ (\xi, x) \in \mathbb{R}^2 \times \bar{\Omega}, \quad c_* > 0.$$

Now, we recall a result from [1].

Lemma 1.1. *For every $p \geq 2$ and every function $g \in L^p(\Omega)$, the solution y of*

$$-\operatorname{div}(A\mathbf{grad}y) + \phi(y) = g \quad \text{in } \Omega, \ y|_{\partial\Omega} = 0, \tag{1.6}$$

belongs to $H_0^1(\Omega) \cap W^{2,p}(\Omega)$. Moreover, there exists a positive constant C such that

$$\|y\|_{W^{2,p}(\Omega)} \leq C\|g\|_{L^p(\Omega)}. \tag{1.7}$$

The plan of this paper is as follows. In Section 2, we construct the mixed finite element approximation scheme for the optimal control problem (1.1)-(1.4) and give its equivalent optimality conditions. The main results of this paper are stated in Section 3. In Section 3, we derive the superconvergence properties between the average L^2 projection and the approximation, as well as between the postprocessing solution and the exact control solution. In Section 4, we present a numerical example to demonstrate our theoretical results. In the last section, we briefly summarize the results obtained and some possible future extensions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition C denotes a general positive constant independent of h, where h is the spatial mesh-size for the control and state discretization.

2. Mixed methods for optimal control problems

In this section, we shall construct mixed finite element approximation scheme of the control problem (1.1)-(1.4). For sake of simplicity, we assume that the domain Ω is a convex polygon. Now, we introduce the co-state elliptic equation

$$-\operatorname{div}(A(x)\mathbf{grad} z) + \phi'(y)z = y - y_d, \quad x \in \Omega, \tag{2.1}$$

which can be written in the form of the first order system

$$\operatorname{div}\boldsymbol{q} + \phi'(y)z = y - y_d, \quad \boldsymbol{q} = -A(x)\mathbf{grad} z, \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$z = 0, \quad x \in \partial\Omega. \tag{2.3}$$

The domain Ω is said to be H^{s+2}-regular if the Dirichlet problem

$$-\operatorname{div}(A\mathbf{grad}\xi) + a_0\xi = F \quad \text{in } \Omega, \ \xi|_{\partial\Omega} = 0 \tag{2.4}$$

is uniquely solvable for $F \in L^2(\Omega)$ and if

$$\|\xi\|_{s+2} \le C\|F\|_s, \tag{2.5}$$

for all $F \in H^s(\Omega)$ and $a_0 \in L^2(\Omega)$.

Let

$$\boldsymbol{V} = H(\operatorname{div};\Omega) = \left\{\boldsymbol{v} \in (L^2(\Omega))^2, \operatorname{div}\boldsymbol{v} \in L^2(\Omega)\right\}, \quad W = L^2(\Omega). \tag{2.6}$$

We recast (1.1)-(1.4) as the following weak form: find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U_{ad}$ such that

$$\min_{u\in U_{ad}} \left\{\frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}\|u\|^2\right\}, \tag{2.7}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \operatorname{div}\boldsymbol{v}) = 0, \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.8}$$

$$(\operatorname{div}\boldsymbol{p}, w) + (\phi(y), w) = (u, w), \qquad \forall\, w \in W. \tag{2.9}$$

It follows from [20] that the optimal control problem (2.7)-(2.9) has a solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (2.7)-(2.9) if there is a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \text{div}\boldsymbol{v}) = 0, \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.10}$$
$$(\text{div}\boldsymbol{p}, w) + (\phi(y), w) = (u, w), \qquad \forall\, w \in W, \tag{2.11}$$
$$(A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (z, \text{div}\boldsymbol{v}) = 0, \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.12}$$
$$(\text{div}\boldsymbol{q}, w) + (\phi'(y)z, w) = (y - y_d, w), \qquad \forall\, w \in W, \tag{2.13}$$
$$(\nu u + z, \tilde{u} - u) \geq 0, \qquad \forall\, \tilde{u} \in U_{ad}, \tag{2.14}$$

where $(\cdot, \cdot)$ is the inner product of $L^2(\Omega)$.

In [9], the expression of the control variable is given. Here, we adopt the same method to derive the following equality:

$$u = (\max\{0, \bar{z}\} - z)/\nu, \tag{2.15}$$

where $\bar{z} = \int_\Omega z / \int_\Omega 1$ denotes the integral average on Ω of the function z.

Let $\mathcal{T}_h$ denote a regular triangulation of the polygonal domain Ω, h_T denotes the diameter of T and $h = \max h_T$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denotes the order $k = 1$ Raviart-Thomas mixed finite element space [11, 24], namely,

$$\forall\, T \in \mathcal{T}_h, \quad \boldsymbol{V}(T) = \boldsymbol{P}_1(T) \oplus \text{span}(xP_1(T)), \quad W(T) = P_1(T),$$

where $P_1(T)$ denote polynomials of total degree at most 1, $\boldsymbol{P}_1(T) = (P_1(T))^2$, $x = (x_1, x_2)$, which is treated as a vector, and

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \forall\, T \in \mathcal{T}_h, \boldsymbol{v}_h|_T \in \boldsymbol{V}(T)\}, \tag{2.16}$$
$$W_h := \{w_h \in W : \forall\, T \in \mathcal{T}_h, w_h|_T \in W(T)\}. \tag{2.17}$$

And the approximated space of control is given by

$$U_h := \{\tilde{u}_h \in U_{ad} : \forall\, T \in \mathcal{T}_h,\ \tilde{u}_h|_T = \text{constant}\}. \tag{2.18}$$

Before the mixed finite element scheme is given, we introduce two operators. Firstly, we define the standard $L^2(\Omega)$-projection [11] $P_h :\ W \to W_h$, which satisfies: for any $\phi \in W$

$$(P_h\phi - \phi, w_h) = 0, \qquad \forall\, w_h \in W_h, \tag{2.19}$$
$$\|\phi - P_h\phi\|_{0,\rho} \leq Ch^r\|\phi\|_{r,\rho},\ 1 \leq \rho \leq \infty, \qquad \forall\, \phi \in W^{r,\rho}(\Omega), \quad r = 1, 2, \tag{2.20}$$
$$\|\phi - P_h\phi\|_{-1} \leq Ch^3|\phi|_2, \qquad \forall\, \phi \in H^2(\Omega). \tag{2.21}$$

Next, recall the Fortin projection (see [2] and [11]) $\Pi_h :\ \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$(\text{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}), w_h) = 0, \qquad \forall\, w_h \in W_h, \tag{2.22}$$
$$\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\| \leq Ch^r\|\boldsymbol{q}\|_r, \qquad \forall\, \boldsymbol{q} \in (H^r(\Omega))^2, \quad r = 1, 2, \tag{2.23}$$
$$\|\text{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q})\| \leq Ch^r\|\text{div}\boldsymbol{q}\|_r, \qquad \forall\, \text{div}\boldsymbol{q} \in H^r(\Omega), \quad r = 1, 2. \tag{2.24}$$

We have the commuting diagram property

$$\text{div} \circ \Pi_h = P_h \circ \text{div} : \boldsymbol{V} \to W_h \quad \text{and} \quad \text{div}(I - \Pi_h)\boldsymbol{V} \perp W_h, \tag{2.25}$$

where and after, I denote the identity operator.

Furthermore, we also define the standard L^2-orthogonal projection $Q_h : U_{ad} \to U_h$, which satisfies: for any $u \in U_{ad}$

$$(u - Q_h u, u_h) = 0, \quad \forall\ u_h \in U_h. \tag{2.26}$$

We have the approximation property:

$$\|u - Q_h u\|_{-s,r} \le Ch^{1+s}|\phi|_{1,r},\ \ s = 0, 1,\ \forall\ u \in W^{1,r}(\Omega). \tag{2.27}$$

Then the mixed finite element discretization of (2.7)-(2.9) is as follows: find $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\min_{u_h \in U_h} \left\{ \frac{1}{2}\|y_h - y_d\|^2 + \frac{\nu}{2}\|u_h\|^2 \right\}, \tag{2.28}$$

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.29}$$

$$(\mathrm{div}\boldsymbol{p}_h, w_h) + (\phi(y_h), w_h) = (u_h, w_h), \qquad \forall\ w_h \in W_h. \tag{2.30}$$

The optimal control problem (2.28)-(2.30) again has a solution $(\boldsymbol{p}_h, y_h, u_h)$, and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (2.28)-(2.30) if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.31}$$

$$(\mathrm{div}\boldsymbol{p}_h, w_h) + (\phi(y_h), w_h) = (u_h, w_h), \qquad \forall\ w_h \in W_h, \tag{2.32}$$

$$(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.33}$$

$$(\mathrm{div}\boldsymbol{q}_h, w_h) + (\phi'(y_h)z_h, w_h) = (y_h - y_d, w_h), \qquad \forall\ w_h \in W_h, \tag{2.34}$$

$$(\nu u_h + z_h, \tilde{u}_h - u_h) \ge 0, \qquad \forall\ \tilde{u}_h \in U_h. \tag{2.35}$$

For the variational inequality (2.35) we have

$$u_h = Q_h \left(-\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}\right), \quad \overline{z_h} = \frac{\int_\Omega z_h}{\int_\Omega 1}. \tag{2.36}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in U_{ad}$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u})) \in (\boldsymbol{V} \times W)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) - (y(\tilde{u}), \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{2.37}$$

$$(\mathrm{div}\boldsymbol{p}(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (\tilde{u}, w), \qquad \forall\ w \in W, \tag{2.38}$$

$$(A^{-1}\boldsymbol{q}(\tilde{u}), \boldsymbol{v}) - (z(\tilde{u}), \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{2.39}$$

$$(\mathrm{div}\boldsymbol{q}(\tilde{u}), w) + (\phi'(y(\tilde{u}))z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), \qquad \forall\ w \in W. \tag{2.40}$$

Then, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})) \in (\boldsymbol{V}_h \times W_h)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) - (y_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.41}$$

$$(\mathrm{div}\mathbf{p}_h(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})), w_h) = (\tilde{u}, w_h), \qquad \forall\ w_h \in W_h, \tag{2.42}$$

$$(A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) - (z_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.43}$$

$$(\mathrm{div}\boldsymbol{q}_h(\tilde{u}), w_h) + (\phi'(y_h(\tilde{u}))z_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h), \qquad \forall w_h \in W_h. \tag{2.44}$$

Thus, as we defined before, the exact solution and its approximation can be written in the following way:

$$(\boldsymbol{p}, y, \boldsymbol{q}, z) = (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),$$
$$(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) = (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).$$

3. Superconvergence and postprocessing

In this section, we will give a detailed superconvergence analysis. Now, we are in the position of deriving the estimates for $\|P_h y(u_h) - y_h\|_{-1}$ and $\|P_h z(u_h) - z_h\|$.

Let $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h)) \in (\boldsymbol{V} \times W)^2$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) \in (\boldsymbol{V}_h \times W_h)^2$ be the solutions of (2.37)-(2.40) and (2.41)-(2.44) with $\tilde{u} = u_h$ respectively. We can easily obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \boldsymbol{v}_h) - (y(u_h) - y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.1}$$
$$(\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), w_h) + (\phi(y(u_h)) - \phi(y_h), w_h) = 0, \qquad \forall\, w_h \in W_h, \tag{3.2}$$
$$(A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{v}_h) - (z(u_h) - z_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.3}$$
$$(\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), w_h) + (\phi'(y(u_h))z(u_h) - \phi'(y_h)z_h, w_h) = (y(u_h) - y_h, w_h), \qquad \forall\, w_h \in W_h. \tag{3.4}$$

As a result of (2.19), we can rewrite (3.1)-(3.4) as

$$(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \boldsymbol{v}_h) - (P_h y(u_h) - y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.5}$$
$$(\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), w_h) + (\phi(y(u_h)) - \phi(y_h), w_h) = 0, \qquad \forall\, w_h \in W_h, \tag{3.6}$$
$$(A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{v}_h) - (P_h z(u_h) - z_h, \mathrm{div}\boldsymbol{v}_h) = 0, \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.7}$$
$$(\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), w_h) + (\phi'(y(u_h))z(u_h) - \phi'(y_h)z_h, w_h) = (P_h y(u_h) - y_h, w_h), \qquad \forall\, w_h \in W_h. \tag{3.8}$$

For sake of simplicity, we now denote

$$\tau = P_h y(u_h) - y_h, \qquad e = P_h z(u_h) - z_h. \tag{3.9}$$

Lemma 3.1. *Let $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h)) \in (\boldsymbol{V} \times W)^2$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) \in (\boldsymbol{V}_h \times W_h)^2$ be the solutions of (2.37)-(2.40) and (2.41)-(2.44) with $\tilde{u} = u_h$ respectively. Assume that the domain Ω is H^{s+2}-regular $(0 \le s \le 1)$, then we have*

$$\|P_h y(u_h) - y_h\|_{-1} + h\|P_h y(u_h) - y_h\| \le Ch^3(\|u\| + \|Q_h u - u_h\|). \tag{3.10}$$

Proof. As we can see,

$$\|\tau\|_{-1} = \sup_{\psi \in H^1(\Omega), \psi \ne 0} \frac{(\tau, \psi)}{\|\psi\|_1}, \tag{3.11}$$

we then need to bound (τ, ψ) for $\psi \in H^1(\Omega)$. Let $\xi \in H^3(\Omega) \cap H_0^1(\Omega)$ be the solution of (2.4) with $a_0 = \Phi$, where

$$\Phi = \begin{cases} \dfrac{\phi(y(u_h)) - \phi(y_h)}{y(u_h) - y_h}, & y(u_h) \ne y_h, \\ \phi'(y_h), & y(u_h) = y_h. \end{cases}$$

We can see from (2.4), (2.22) and (3.5)

$$\begin{aligned}(\tau, F) &= (\tau, -\mathrm{div}(A\mathbf{grad}\xi)) + (\tau, \Phi\xi)\\ &= -(\tau, \mathrm{div}(\Pi_h(A\mathbf{grad}\xi))) + (\tau, \Phi\xi)\\ &= -(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h(A\mathbf{grad}\xi)) + (\tau, \Phi\xi). \end{aligned} \tag{3.12}$$

Note that

$$(\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \xi) + (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), A\mathbf{grad}\xi) = 0. \tag{3.13}$$

Thus, from (3.6), (3.12) and (3.13), we derive

$$\begin{aligned}(\tau, F) =&(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), A\mathbf{grad}\xi - \Pi_h(A\mathbf{grad}\xi))\\ &+ (\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \xi - P_h\xi) - (\Phi(y(u_h) - P_h y(u_h)), \xi)\\ &+ (\phi(y(u_h)) - \phi(y_h), \xi - P_h\xi). \end{aligned} \tag{3.14}$$

From (2.23), we have

$$(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), A\mathbf{grad}\xi - \Pi_h(A\mathbf{grad}\xi)) \leq Ch^2\|\boldsymbol{p}(u_h) - \boldsymbol{p}_h\| \cdot \|\xi\|_3. \tag{3.15}$$

Let $\tilde{u} = u_h$ and $w = \mathrm{div}\boldsymbol{p}(u_h) + \phi(y(u_h)) - u_h$ in (2.38), we can find that

$$\mathrm{div}\boldsymbol{p}(u_h) + \phi(y(u_h)) - u_h = 0. \tag{3.16}$$

Similarly, by (2.19) and (2.32), it is easy to see that

$$\mathrm{div}\boldsymbol{p}_h = u_h - P_h\phi(y_h). \tag{3.17}$$

By (3.16), (3.17) and (2.20), we have

$$\begin{aligned}&(\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \xi - P_h\xi) + (\phi(y(u_h)) - \phi(y_h), \xi - P_h\xi)\\ =&(P_h\phi(y_h) - \phi(y_h), \xi - P_h\xi) \leq Ch^3\|\phi\|_1\|\xi\|_2. \end{aligned} \tag{3.18}$$

For the third term on the right side of (3.14), using (2.19), (2.20) and the assumption on ϕ, we get

$$\begin{aligned}&(\Phi(y(u_h) - P_h y(u_h)), \xi)\\ =&(\Phi(y(u_h) - P_h y(u_h)), \xi - P_h\xi) + (y(u_h) - P_h y(u_h), (\Phi - P_h\Phi)P_h\xi)\\ \leq& Ch\|\phi\|_{1,\infty}\|y(u_h) - P_h y(u_h)\| \cdot \|\xi\|_1 + Ch\|\phi\|_{2,\infty}\|y(u_h) - P_h y(u_h)\| \cdot \|\xi\|\\ \leq& Ch^3\|\xi\|_1\|y(u_h)\|_2. \end{aligned} \tag{3.19}$$

By (2.5), (3.11), (3.14), (3.15) and (3.18)-(3.19), we derive

$$\|P_h y(u_h) - y_h\|_{-1} \leq Ch^2\|\boldsymbol{p}(u_h) - \boldsymbol{p}_h\| + Ch^3\|y(u_h)\|_2. \tag{3.20}$$

Similarly, we arrive at

$$\|P_h y(u_h) - y_h\| \leq Ch\|\boldsymbol{p}(u_h) - \boldsymbol{p}_h\| + Ch^2\|y(u_h)\|_1. \tag{3.21}$$

Choosing $\boldsymbol{v}_h = \Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h$ in (3.5) and $w_h = P_h y(u_h) - y_h$ in (3.6), respectively. Then adding the two equations to get

$$\begin{aligned}&(A^{-1}(\Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h) + (\phi(P_h y(u_h)) - \phi(y_h), P_h y(u_h) - y_h)\\ =& - (A^{-1}(\boldsymbol{p}(u_h) - \Pi_h\boldsymbol{p}(u_h)), \Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h)\\ &- (\phi(y(u_h)) - \phi(P_h y(u_h)), P_h y(u_h) - y_h). \end{aligned} \tag{3.22}$$

Note that

$$(\phi(y(u_h)) - \phi(P_h y(u_h)), P_h y(u_h) - y_h) \leq Ch\|\phi\|_{1,\infty}\|y(u_h)\|_1\|P_h y(u_h) - y_h\| \tag{3.23}$$

Using (3.22), (3.23), (2.23) and the assumptions on A and ϕ, we find that

$$\|\Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}_h\| \leq Ch(\|\boldsymbol{p}(u_h)\|_1 + \|y(u_h)\|_1) + \|P_h y(u_h) - y_h\|. \tag{3.24}$$

Substituting (3.24) into (3.21), using (2.23), for sufficiently small h, we have

$$\|P_h y(u_h) - y_h\| \leq Ch^2(\|\boldsymbol{p}(u_h)\|_1 + \|y(u_h)\|_1). \tag{3.25}$$

Then, substituting (3.24) and (3.25) into (3.20), using (2.23), we find that

$$\|P_h y(u_h) - y_h\|_{-1} \leq Ch^3(\|\boldsymbol{p}(u_h)\|_1 + \|y(u_h)\|_2). \tag{3.26}$$

From Lemma 1.1, we have

$$\|\boldsymbol{p}(u_h)\|_1 + \|y(u_h)\|_2 \leq C\|y(u_h)\|_2 \leq C\|u_h\| \leq C(\|u\| + \|Q_h u - u_h\|). \tag{3.27}$$

By (3.25)-(3.27), we complete the proof. □

Lemma 3.2. *Let $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h)) \in (\boldsymbol{V} \times W)^2$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) \in (\boldsymbol{V}_h \times W_h)^2$ be the solutions of (2.37)-(2.40) and (2.41)-(2.44) with $\tilde{u} = u_h$ respectively. Assume that the domain Ω is H^{s+2}-regular $(0 \leq s \leq 1)$, then we have*

$$\|P_h z(u_h) - z_h\| \leq Ch^3(\|y_d\|_1 + \|u\| + \|Q_h u - u_h\|). \tag{3.28}$$

Proof. Since

$$\|e\| = \sup_{\psi \in L^2(\Omega), \psi \neq 0} \frac{(e, \psi)}{\|\psi\|}, \tag{3.29}$$

we then need to bound (e, ψ) for $\psi \in L^2(\Omega)$. Let ξ be the solution of (2.4) with $a_0 = \phi'(y(u_h))$. From (2.4), (2.22) and (3.7), we can see that

$$\begin{aligned}
(e, F) =& (e, -\mathrm{div}(A\mathbf{grad}\xi)) + (e, \phi'(y(u_h))\xi) \\
=& -(e, \mathrm{div}(\Pi_h(A\mathbf{grad}\xi))) + (e, \phi'(y(u_h))\xi) \\
=& -(A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \Pi_h(A\mathbf{grad}\xi)) + (e, \phi'(y(u_h))\xi).
\end{aligned} \tag{3.30}$$

Note that

$$(\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \xi) + (A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), A\mathbf{grad}\xi) = 0. \tag{3.31}$$

Thus, it follows from (2.19), (3.30) and (3.31), we derive

$$\begin{aligned}
(e, F) =& (A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), A\mathbf{grad}\xi - \Pi_h(A\mathbf{grad}\xi)) + (\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \xi - P_h\xi) \\
& - (P_h y(u_h) - y_h, P_h\xi) + (\phi'(y(u_h))z(u_h) - \phi'(y_h)z_h, \xi - P_h\xi) \\
& + (\phi'(y(u_h))(P_h z(u_h) - z(u_h)), \xi) + (z_h(\phi'(y_h) - \phi'(y(u_h))), \xi) \\
=:& \sum_{i=1}^{6} I_i.
\end{aligned} \tag{3.32}$$

For I_1, by (2.23), we have

$$I_1 \leq C\|\boldsymbol{q}(u_h) - \boldsymbol{q}_h\| \cdot \|A\mathbf{grad}\xi - \Pi_h(A\mathbf{grad}\xi)\| \leq Ch\|\boldsymbol{q}(u_h) - \boldsymbol{q}_h\| \cdot \|\xi\|_2. \quad (3.33)$$

Let $\tilde{u} = u_h$ and $w = \mathrm{div}\boldsymbol{q}(u_h) + \phi'(y(u_h))z(u_h) - y(u_h) + y_d$ in (2.40), we can find that

$$\mathrm{div}\boldsymbol{q}(u_h) + \phi'(y(u_h))z(u_h) = y(u_h) - y_d. \quad (3.34)$$

Similarly, by (2.19) and (2.34), it is easy to see that

$$\mathrm{div}\boldsymbol{q}_h = y_h - P_h y_d - P_h\phi'(y_h)z_h. \quad (3.35)$$

By (2.20) and (3.34)-(3.35), we have

$$\begin{aligned} I_2 =& (P_h\phi'(y_h)z_h - \phi'(y(u_h))z(u_h), \xi - P_h\xi) + (P_h y_d - y_d, \xi - P_h\xi) \\ &+ (y(u_h) - P_h y(u_h), \xi - P_h\xi) + (P_h y(u_h) - y_h, \xi - P_h\xi) \\ =& (P_h(\phi'(y(u_h))z(u_h)) - \phi'(y(u_h))z(u_h), \xi - P_h\xi) \\ &+ (P_h y_d - y_d, \xi - P_h\xi) + (y(u_h) - P_h y(u_h), \xi - P_h\xi) \\ \leq& Ch^3(\|\phi\|_2\|z(u_h)\|_{1,\infty} + \|y_d\|_1 + \|y(u_h)\|_1)\|\xi\|_2. \end{aligned} \quad (3.36)$$

From (2.19), we arrive at

$$I_3 = (\tau, \xi) \leq C\|\tau\|_{-1}\|\xi\|_1. \quad (3.37)$$

Note that

$$\phi'(y(u_h))z(u_h) - \phi'(y_h)z_h = z(u_h)(\phi'(y(u_h)) - \phi'(y_h)) + \phi'(y_h)(z(u_h) - z_h). \quad (3.38)$$

Then, by (2.20), (3.25) and the assumption on ϕ, we find that

$$\begin{aligned} I_4 \leq& C\|z(u_h)\|_{0,\infty}\|\phi\|_{2,\infty}\|y(u_h) - y_h\| \cdot \|\xi - P_h\xi\| \\ &+ C\|\phi\|_{1,\infty}\|z(u_h) - z_h\| \cdot \|\xi - P_h\xi\| \\ \leq& Ch^3\|z(u_h)\|_{1,\infty}\|\phi\|_{2,\infty}\|\xi\|_1 + Ch\|\phi\|_{1,\infty}\|\xi\|_2\|P_h z(u_h) - z_h\|. \end{aligned} \quad (3.39)$$

As for I_5, by the assumption on ϕ, (2.19) and (2.20), we derive

$$\begin{aligned} I_5 =& (\phi'(y(u_h))(P_h z(u_h) - z(u_h)), \xi - P_h\xi) \\ &+ (P_h z(u_h) - z(u_h), (\phi'(y(u_h)) - P_h(\phi'(y(u_h))))P_h\xi) \\ \leq& C\|\phi\|_{1,\infty}\|z(u_h) - P_h z(u_h)\| \cdot \|\xi - P_h\xi\| \\ &+ Ch\|\phi\|_{2,\infty}\|z(u_h) - P_h z(u_h)\| \cdot \|P_h\xi\| \\ \leq& Ch^3\|\phi\|_{2,\infty}\|z(u_h)\|_2\|\xi\|_2. \end{aligned} \quad (3.40)$$

For I_6, by (2.19), (2.20), the embedding $\|v\|_{0,\infty} \leq c\|v\|_2$ and the assumption on

ϕ, we obtain

$$\begin{aligned}
I_6 =&(\phi'(y_h)-\phi'(y(u_h)),(z_h-z(u_h))\xi)+(\phi'(y_h)-\phi'(P_hy(u_h)),z(u_h)\xi)\\
&+(\phi''(y(u_h))(P_hy(u_h)-y(u_h)),z(u_h)\xi)\\
&+\left(\frac{1}{2}\phi'''(y(u_h)+\theta(P_hy(u_h)-y(u_h)))(P_hy(u_h)-y(u_h))^2,z(u_h)\xi\right)\\
=&(\phi'(y_h)-\phi'(y(u_h)),(z_h-z(u_h))\xi)+(\phi'(y_h)-\phi'(P_hy(u_h)),z(u_h)\xi)\\
&+(\phi''(y(u_h))(P_hy(u_h)-y(u_h)),z(u_h)\xi-P_h(z(u_h)\xi))\\
&+(P_hy(u_h)-y(u_h),(\phi''(y(u_h))-P_h(\phi''(y(u_h))))P_h(z(u_h)\xi))\\
&+\frac{1}{2}(\phi'''(y(u_h)+\theta(P_hy(u_h)-y(u_h)))(P_hy(u_h)-y(u_h))^2,z(u_h)\xi)\\
\le&C\|\phi\|_{2,\infty}\|y(u_h)-y_h\|\cdot\|\xi\|_{0,\infty}\|z(u_h)-z_h\|\\
&+C\|\phi\|_{2,\infty}\|P_hy(u_h)-y_h\|_{-1}\|z(u_h)\|_{1,\infty}\|\xi\|_1\\
&+Ch\|z(u_h)\|_{1,\infty}\|y(u_h)-P_hy(u_h)\|(\|\phi\|_{2,\infty}\|\xi\|_1+\|\phi\|_3\|\xi\|_{0,\infty})\\
&+C\|\phi\|_3\|y(u_h)-P_hy(u_h)\|_{0,\infty}^2\|z(u_h)\|_{0,\infty}\|\xi\|\\
\le&Ch(h^2\|y(u_h)\|_2+h^2\|z(u_h)\|_2+\|e\|)\|\xi\|_2\\
&+C\|z(u_h)\|_{1,\infty}\|\tau\|_{-1}\|\xi\|_2,\ 0\le\theta\le 1.
\end{aligned}\tag{3.41}$$

Substituting the estimates I_1-I_6 in (3.32), for sufficiently small h, by (3.29), we derive

$$\|P_hz(u_h)-z_h\|\le Ch\|\boldsymbol{q}(u_h)-\boldsymbol{q}_h\|+C\|\tau\|_{-1}+Ch^3(\|y(u_h)\|_2+\|z(u_h)\|_2).\tag{3.42}$$

Next, using (2.22), we rewrite (3.7)-(3.8) as

$$\begin{aligned}
&(A^{-1}(\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\boldsymbol{v}_h)-(P_hz(u_h)-z_h,\mathrm{div}\boldsymbol{v}_h)\\
=&-(A^{-1}(\boldsymbol{q}(u_h)-\Pi_h\boldsymbol{q}(u_h)),\boldsymbol{v}_h), && \forall\ \boldsymbol{v}_h\in\boldsymbol{V}_h, \quad (3.43)\\
&(\mathrm{div}(\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h),w_h)+(\phi'(y(u_h))(P_hz(u_h)-z_h),w_h)\\
=&-(\phi'(y(u_h))(z(u_h)-P_hz(u_h)),w_h)+(P_hy(u_h)-y_h,w_h)\\
&+((\phi'(y(u_h))-\phi'(y_h))z_h,w_h), && \forall\ w_h\in W_h. \quad (3.44)
\end{aligned}$$

Choosing $\boldsymbol{v}_h=\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h$ in (3.43) and $w_h=P_hz(u_h)-z_h$ in (3.44), respectively. Then adding the two equations to get

$$\begin{aligned}
&(A^{-1}(\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h)+(\phi'(y(u_h))(P_hz(u_h)-z_h),P_hz(u_h)-z_h)\\
=&-(A^{-1}(\boldsymbol{q}(u_h)-\Pi_h\boldsymbol{q}(u_h)),\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h)+(P_hy(u_h)-y_h,P_hz(u_h)-z_h)\\
&-(\phi'(y(u_h))(z(u_h)-P_hz(u_h)),P_hz(u_h)-z_h)\\
&+((\phi'(y(u_h))-\phi'(y_h))z_h,P_hz(u_h)-z_h).
\end{aligned}\tag{3.45}$$

Note that

$$\begin{aligned}
&(\phi'(y(u_h))(z(u_h)-P_hz(u_h)),P_hz(u_h)-z_h)\\
\le&Ch\|\phi\|_{1,\infty}\|z(u_h)\|_1\|P_hz(u_h)-z_h\|
\end{aligned}\tag{3.46}$$

and

$$
\begin{aligned}
&((\phi'(y(u_h)) - \phi'(y_h))z_h, P_h z(u_h) - z_h) \\
\leq &((\phi'(y(u_h)) - \phi'(P_h y(u_h)))z_h, P_h z(u_h) - z_h) \\
&+ ((\phi'(P_h y(u_h)) - \phi'(y_h))z_h, P_h z(u_h) - z_h) \\
\leq &C\|\phi\|_{1,\infty}\|z_h\| \cdot \|P_h z(u_h) - z_h\|(h\|y(u_h)\|_{1,\infty} + \|P_h y(u_h) - y_h\|_{0,\infty}), \qquad (3.47)
\end{aligned}
$$

where

$$
\begin{aligned}
\|z_h\| &\leq \|z(u_h) - P_h z(u_h)\| + \|P_h z(u_h) - z_h\| + \|z(u_h)\| \\
&\leq C\|z(u_h)\|_1 + \|P_h z(u_h) - z_h\|. \qquad (3.48)
\end{aligned}
$$

Using (3.45)-(3.48), (2.23), the assumptions on A and ϕ, we find that

$$
\|\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h\| \leq Ch^2(\|\boldsymbol{q}(u_h)\|_2 + \|z(u_h)\|_2 + \|z(u_h)\|_{1,\infty}) + c\|\tau\| + \|e\|. \qquad (3.49)
$$

Substituting (3.49) into (3.42), for sufficiently small h, by (2.23), we derive

$$
\begin{aligned}
\|P_h z(u_h) - z_h\| \leq &Ch^3(\|z(u_h)\|_2 + \|y(u_h)\|_1 + \|y_d\|_1 + \|u\| \\
&+ \|Q_h u - u_h\| + \|z(u_h)\|_{1,\infty} + \|\boldsymbol{q}(u_h)\|_2). \qquad (3.50)
\end{aligned}
$$

Since the domain Ω is H^3-regular, we have

$$
\|z(u_h)\|_{1,\infty} + \|\boldsymbol{q}(u_h)\|_2 + \|z(u_h)\|_2 \leq C\|z(u_h)\|_3 \leq C(\|y(u_h)\|_1 + \|y_d\|_1). \qquad (3.51)
$$

Thus, using (3.27), (3.50) and (3.51), we complete the proof. □

Lemma 3.3. *Let $(\boldsymbol{p}(Q_h u), y(Q_h u), \boldsymbol{q}(Q_h u), z(Q_h u))$ and $(\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u))$ be the solutions of (2.37)-(2.40) with $\tilde{u} = Q_h u$ and $\tilde{u} = u$, respectively. Assume that $u \in H^1(\Omega)$. Assume that the domain Ω is H^2-regular, then we have*

$$
\|z(u) - z(Q_h u)\|_{0,\infty} \leq Ch^2. \qquad (3.52)
$$

Proof. First, in [7], we know that

$$
\|y(Q_h u) - y(u)\| \leq Ch^2. \qquad (3.53)
$$

Choosing $\tilde{u} = u$ and $\tilde{u} = Q_h u$ in (2.39) and (2.40), we have

$$
\begin{aligned}
&-\operatorname{div}(A\nabla(z - z(Q_h u))) + \phi'(y(Q_h u))(z - z(Q_h u)) \\
=&y - y(Q_h u) - z(\phi'(y) - \phi'(y(Q_h u))). \qquad (3.54)
\end{aligned}
$$

Using Lemma 1.1 and the classical imbedding theorem, we can see that

$$
\begin{aligned}
\|z - z(Q_h u)\|_{0,\infty} \leq &C\|z - z(Q_h u)\|_2 \leq C\|y - y(Q_h u) - z(\phi'(y) - \phi'(y(Q_h u)))\| \\
\leq &C\|y - y(Q_h u)\| + C\|z(\phi'(y) - \phi'(y(Q_h u)))\| \\
\leq &C\|y - y(Q_h u)\| + C\|z\|_{0,\infty}\|\phi\|_{2,\infty}\|y - y(Q_h u)\|. \qquad (3.55)
\end{aligned}
$$

Thus, using (3.53) and (3.55), we complete the proof. □

Let $y(u)$ be the solution of (2.7)-(2.9) and $J(\cdot) : L^2(\Omega) \to \mathbb{R}$ be a G-differential convex functional near the solution u which satisfies the following form:

$$
J(u) = \frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}\|u\|^2. \qquad (3.56)
$$

Then we can find that

$$(J'(u_h), v) = (\nu u_h + z(u_h), v), \tag{3.57}$$

$$(J'(Q_h u), v) = (\nu Q_h u + z(Q_h u), v). \tag{3.58}$$

In many applications, $J(\cdot)$ is local convex near the solution u. The convexity of $J(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem. Then, there exists a constant $c > 0$, independent of h, such that

$$(J'(Q_h u) - J'(u_h), Q_h u - u_h) \geq c\|Q_h u - u_h\|^2, \tag{3.59}$$

where u and u_h are solutions of (2.10)-(2.14) and (2.31)-(2.35) respectively, $Q_h u$ is the orthogonal projection of u which is defined in (2.26). We shall assume that the above inequality throughout this paper.

Now, we will discuss the following superconvergence property for the control variable.

Lemma 3.4. *Let u be the solution of (2.10)-(2.14) and u_h be the solution of (2.31)-(2.35), respectively. Assume that $u \in H^1(\Omega)$ and all the conditions in previous Lemmas are valid. Then, we have*

$$\|Q_h u - u_h\| \leq Ch^2. \tag{3.60}$$

Proof. We choose $\tilde{u} = u_h$ in (2.14) and $\tilde{u}_h = Q_h u$ in (2.35) to get the following two inequalities:

$$(\nu u + z, u_h - u) \geq 0, \tag{3.61}$$

$$(\nu u_h + z_h, Q_h u - u_h) \geq 0. \tag{3.62}$$

Note that $u_h - u = u_h - Q_h u + Q_h u - u$. Adding the above two inequalities to get

$$(\nu u_h + z_h - \nu u - z, Q_h u - u_h) + (\nu u + z, Q_h u - u) \geq 0. \tag{3.63}$$

Thus, by (3.63), (3.59) and (2.19), we find that

$$\begin{aligned}
&c\|Q_h u - u_h\|^2 \leq (J'(Q_h u) - J'(u_h), Q_h u - u_h)\\
=&\nu(Q_h u - u_h, Q_h u - u_h) + (z(Q_h u) - z(u_h), Q_h u - u_h)\\
=&\nu(Q_h u - u, Q_h u - u_h) + \nu(u - u_h, Q_h u - u_h) + (z(Q_h u) - z(u_h), Q_h u - u_h)\\
\leq&(z_h - z, Q_h u - u_h) + (\nu u + z, Q_h u - u) + (z(Q_h u) - z(u_h), Q_h u - u_h)\\
=&(z_h - P_h z(u_h), Q_h u - u_h) + (\nu u + z, Q_h u - u)\\
&+ (z(Q_h u) - z(u), Q_h u - u_h).
\end{aligned} \tag{3.64}$$

By Lemma 3.2 and Lemma 3.3, we arrive at

$$(z_h - P_h z(u_h), Q_h u - u_h) \leq Ch^6 + \frac{\nu}{4}\|Q_h u - u_h\|^2 + Ch^3\|Q_h u - u_h\|^2 \tag{3.65}$$

and

$$(z(Q_h u) - z(u), Q_h u - u_h) \leq Ch^4 + \frac{\nu}{4}\|Q_h u - u_h\|^2. \tag{3.66}$$

From (2.15), we know that

$$\nu u + z = \max\{0, \bar{z}\} = \text{constant}. \tag{3.67}$$

Thus, we have

$$(\nu u + z, Q_h u - u) = (\nu u + z) \int_\Omega (Q_h u - u) = 0. \tag{3.68}$$

Combining (3.64)-(3.66) with (3.68), for sufficiently small h, we derive (3.60). □

Let $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h))$ and $(\boldsymbol{p}(Q_h u), y(Q_h u), \boldsymbol{q}(Q_h u), z(Q_h u))$ be the solutions of (2.37)-(2.40) with $\tilde{u} = u_h$ and $\tilde{u} = Q_h u$, respectively. Then we have the following error equations

$$(A^{-1}(\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u_h)), \boldsymbol{v}) - (y(Q_h u) - y(u_h), \operatorname{div}\boldsymbol{v}) = 0, \tag{3.69}$$

$$(\operatorname{div}(\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u_h)), w) + (\phi(y(Q_h u) - \phi(y(u_h)), w) = (Q_h u - u_h, w), \tag{3.70}$$

$$(A^{-1}(\boldsymbol{q}(Q_h u) - \boldsymbol{q}(u_h)), \boldsymbol{v}) - (z(Q_h u) - z(u_h), \operatorname{div}\boldsymbol{v}) = 0, \tag{3.71}$$

$$\begin{aligned}&(\operatorname{div}(\boldsymbol{q}(Q_h u) - \boldsymbol{q}(u_h)), w) + (\phi'(y(Q_h u) z(Q_h u) - \phi'(y(u_h)) z(u_h), w)\\ &= (y(Q_h u) - y(u_h), w),\end{aligned} \tag{3.72}$$

for any $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$.

Similar to Lemma 3.3, using Lemma 3.4, we can prove the following estimate.

Lemma 3.5. *Assume that all the conditions in Lemma 3.4 are valid. Then we have*

$$\|z(Q_h u) - z(u_h)\|_{0,\infty} \le C h^2. \tag{3.73}$$

Lemma 3.6. *Assume that all the conditions in Lemma 3.4 are valid and $u \in W^{1,\infty}(\Omega)$. Let u and u_h be the solutions of (2.10)-(2.14) and (2.31)-(2.35), respectively. Then we have*

$$\|u - u_h\|_{0,\infty} \le C h. \tag{3.74}$$

Proof. By (2.27) and the inverse inequality, we arrive at

$$\begin{aligned}\|u - u_h\|_{0,\infty} \le& C(\|u - Q_h u\|_{0,\infty} + \|Q_h u - u_h\|_{0,\infty})\\ \le& C(h\|u\|_{1,\infty} + h^{-1}\|Q_h u - u_h\|).\end{aligned} \tag{3.75}$$

Gathering (3.75) and Lemma 3.4, we derive (3.74). □

Moreover, in order to improve the accuracy of the control approximation on a global scale, similar to the case in [21], we construct the following a postprocessing projection operator of the discrete co-state to the admissible set

$$\hat{u} = (\max\{0, \overline{z_h}\} - z_h)/\nu. \tag{3.76}$$

Now, we can prove the following global superconvergence result.

Theorem 3.1. *Assume that all the conditions in previous Lemmas are valid. Let u be the solution of (2.10)-(2.14) and $\hat{u}$ be the function constructed in (3.76). Then we have*

$$\|u - \hat{u}\|_{0,\infty} \le C h^2. \tag{3.77}$$

Proof. By use of (2.20), Lemma 3.2, Lemma 3.3, Lemma 3.5 and the inverse estimate, we find that

$$\begin{aligned} \|z-z_h\|_{0,\infty} \leq & \|z-z(Q_h u)\|_{0,\infty} + \|z(Q_h u) - z(u_h)\|_{0,\infty} + \|z(u_h) - P_h z(u_h)\|_{0,\infty} \\ & + \|P_h z(u_h) - z_h\|_{0,\infty} \leq Ch^2. \end{aligned} \tag{3.78}$$

From (2.15) and (3.76), we arrive at

$$|u-\hat{u}| \leq C|z-z_h| + C|\bar{z}-\overline{z_h}| \leq C\|z-z_h\|_{0,\infty}. \tag{3.79}$$

By (3.78) and (3.79), we complete the proof. □

4. Numerical experiments

In this section, we present below an example to illustrate the theoretical results. The optimization problems were solved numerically by projected gradient methods, with codes developed based on AFEPack [19]. The discretization was already described in previous sections: the control function u was discretized by piecewise constant functions, whereas the state $(y, \boldsymbol{p})$ and the co-state $(z, \boldsymbol{q})$ were approximated by the order $k = 1$ Raviart-Thomas mixed finite element functions. In our examples, we choose the domain $\Omega = [0,1] \times [0,1]$, $\phi(y) = y^3$, $\nu = 1$ and $A = E$, where E denotes the unit matrix.

Example 4.1. We consider the following two-dimensional elliptic optimal control problem

$$\min_{u \in U_{ad}} \left\{ \frac{1}{2}\|y-y_d\|^2 + \frac{1}{2}\|u\|^2 \right\} \tag{4.1}$$

subject to the state equation

$$\mathrm{div}\boldsymbol{p} + y^3 = f + u, \quad \boldsymbol{p} = -\mathbf{grad} y, \tag{4.2}$$

where

$$\begin{aligned} y &= \sin(\pi x_1)\sin(\pi x_2), \\ z &= \sin(2\pi x_1)\sin(2\pi x_2), \\ u &= \max(0, \bar{z}) - z, \\ f &= 2\pi^2 y + y^3 - u, \\ y_d &= y - 3y^2 z - 8\pi^2 z. \end{aligned} \tag{4.3}$$

In the numerical implementation, we choose the exact solution u which satisfies $\int_\Omega u dx = 0$. In Table 1, the errors $\|u-u_h\|_{0,\infty}$, $\|Q_h u - u_h\|$ and $\|u-\hat{u}\|_{0,\infty}$ obtained on a sequence of uniformly refined meshes are shown. In Figure 1, we show the convergence orders by slopes, and we denote $\hat{u}$ by u_{proj}. The theoretical results can be observed clearly from the data.

h	$\|u-u_h\|_{0,\infty}$	$\|Q_h u-u_h\|$	$\|u-\hat{u}\|_{0,\infty}$
1/16	9.4053e-02	1.2416e-04	3.5683e-02
1/32	4.6952e-02	2.5624e-05	8.9063e-03
1/64	2.3667e-02	6.3276e-06	2.2173e-03
1/128	1.1783e-02	1.4925e-06	5.5672e-04

Table 1. The errors of Example on a sequential uniform refined meshes.

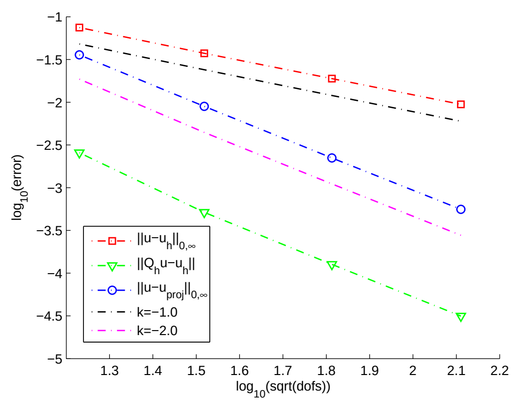

Figure 1. Convergence orders of $u-u_h$, $Q_h u-u_h$ and $u-u_{proj}$ in different norms.

5. Conclusions

In this paper, we discussed the order $k=1$ Raviart-Thomas mixed finite element methods for semilinear elliptic optimal control problem (1.1)-(1.4). We have derived a second order superconvergence result of mixed finite element methods for the control problem when the control was approximated by piecewise constant functions. In our future work, we will investigate the superconvergence of mixed finite element methods for optimal control problems governed by bilinear and quasilinear elliptic equations.

References

[1] N. Arada, E. Casas and F. Tröltzsch, *Error estimates for the numerical approximation of a semilinear elliptic control problem,* Comput. Optim. Appl., 23(2002), 201-229.

[2] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York., 1991.

[3] Y. Chen, *Superconvergence of mixed finite element methods for optimal control problems,* Math. Comp., 77(2008), 1269-1291.

[4] Y. Chen, *Superconvergence of quadratic optimal control problems by triangular mixed finite elements,* Inter. J. Numer. Meths. Eng., 75(2008), 881-898.

[5] Y. Chen, Y. Huang, W. B. Liu and N. Yan, *Error estimates and superconvergence of mixed finite element methods for convex optimal control problems,* J. Sci. Comput., 42(2009), 382-403.

[6] Y. Chen and Y. Dai, *Superconvergence for optimal control problems governed by semi-linear elliptic equations,* J. Sci. Comput., 39(2009), 206-221.

[7] Y. Chen and T. Hou, *Superconvergence and L^∞-error estimates of RT1 mixed methods for semilinear elliptic control problems with an integral constraint,* Numer. Math. Theor. Meth. Appl., 5(2012), 423-446.

[8] Y. Chen and T. Hou, *Error estimates and superconvergence of RT0 mixed methods for a class of semilinear elliptic optimal control problems,* Numer. Math. Theor. Meth. Appl., 6(2013), 637-656.

[9] Y. Chen, N. Yi and W. B. Liu, *A Legendre Galerkin spectral method for optimal control problems governed by elliptic equations,* SIAM J. Numer. Anal., 46(2008), 2254-2275.

[10] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam., 1978.

[11] J. Douglas and J. E. Roberts, *Global estimates for mixed finite element methods for second order elliptic equations,* Math. Comp., 44(1985), 39-52.

[12] F. S. Falk, *Approximation of a class of optimal control problems with order of convergence estimates,* J. Math. Anal. Appl., 44(1973), 28-47.

[13] M. D. Gunzburger and S. L. Hou, *Finite dimensional approximation of a class of constrained nonlinear control problems,* SIAM J. Control Optim., 34(1996), 1001-1043.

[14] T. Geveci, *On the approximation of the solution of an optimal control problem governed by an elliptic equation,* RAIRO. Anal. Numer., 13(1979), 313-328.

[15] T. Hou, *Superconvergence and L^∞-error estimates of the lowest order mixed methods for distributed optimal control problems governed by semilinear elliptic equations,* Numer. Math. Theor. Meth. Appl., 6(2013), 479-498.

[16] T. Hou and Y. Chen, *Superconvergence of RT1 mixed finite element approximations for elliptic control problems,* Sci China Math., 56(2013), 267-281.

[17] L. Hou and J. C. Turner, *Analysis and finite element approximation of an optimal control problem in electrochemistry with current density controls,* Numer. Math., 71(1995), 289-315.

[18] G. Knowles, *Finite element approximation of parabolic time optimal control problems,* SIAM J. Control Optim., 20(1982), 414-427.

[19] R. Li and W. B. Liu, *http://circus.math.pku.edu.cn/AFEPack*.

[20] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin., 1971.

[21] C. Meyer and A. Rösch, *Superconvergence properties of optimal control problems,* SIAM J. Control Optim., 43(2004), 970-985.

[22] C. Meyer and A. Rösch, *L^∞-error estimates for approximated optimal control problems,* SIAM J. Control Optim., 44(2005), 1636-1649.

[23] R. S. McKinght and J. Borsarge, *The Ritz-Galerkin procedure for parabolic control problems,* SIAM J. Control Optim., 11(1973), 510-542.

[24] P. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems, Aspecs of the Finite Element Method,* Lecture Notes in Math, Springer, Berlin., 606(1977), 292-315.