

PEAKON, PSEUDO-PEAKON, LOOP, AND PERIODIC CUSP WAVE SOLUTIONS OF A THREE-DIMENSIONAL 3DKP(2, 2) EQUATION WITH NONLINEAR DISPERSION*

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Abstract In this paper, we study the three-dimensional Kadomtsev-Petviashvili equation (3DKP(m, n)) with nonlinear dispersion for $m = n = 2$. By using the bifurcation theory of dynamical systems, we study the dynamical behavior and obtain peakon, pseudo-peakon, loop and periodic cusp wave solutions of the three-dimensional 3DKP(2, 2) equation. The parameter expressions of peakon, pseudo-peakon, loop and periodic cusp wave solutions are obtained and numerical graph are provided for those peakon, pseudo-peakon, loop and periodic cusp wave solutions.

Keywords Bifurcation theory of dynamical systems, peakon, pseudo-peakon, loop solution, periodic cusp wave, three-dimensional Kadomtsev-Petviashvili equation.

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1. Introduction

In this paper, we study the following three-dimensional Kadomtsev-Petviashvili equation with nonlinear dispersion for $m = n = 2$ (the 3DKP(2, 2) equation in short):

$$3DKP(2, 2): [u_t + a(u^2)_x + (u^2)_{xxx}]_x + (u)_{yy} + (u)_{zz} = 0, \quad (1.1)$$

where a is a non-zero real number. Recently, Xie and Yan [14] obtained solitary patterns, compact and noncompact solutions of 3DKP(m, n) by using some transformations. By using the ansatz method and the Exp-function method, M. Inc [6] considered some compact and noncompact solutions for Eq.(1.1) and obtain a new traveling wave solution for the 3DKP(2, 1) equation. The authors did not study the bifurcation behavior of the traveling wave solutions of the corresponding traveling wave equations in its parameter space. It is important to understand the dynamical behavior for the traveling wave solutions governed by a singular traveling wave

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equation. We emphasize that peakon, pseudo-peakon, loop and periodic cusp wave solutions have not been available as yet. To answer this question, we shall consider the bifurcations of traveling wave solutions of Eq.(1.1) in the three-parameter space (g, α, β) . We shall show that there exists peakon, pseudo-peakon, cuspon, and loop soliton solutions for 3DKP(2, 2) equation. Peakon is the so-called peaked soliton, i.e., soliton with discontinuous first order derivative at the peak point. Usually, the profile of a wave function is called peakon if at a continuous point its left and right derivatives are finite and have different sign. The pseudo-peakon solution is for the first time proposed in [7]. The so called ‘‘pseudo-peakon’’ means that the wave profile looks like peakon, but the solution still has continuous first order derivative.

Making the transformations $u(x, y, z, t) = \phi(k(x + ly + sz - \lambda t)) = \phi(\xi)$, where $k, l, s, \lambda \neq 0$. Thus, Eq.(1) becomes

$$[-\lambda\phi' + a(\phi^2)' + k^2(\phi^2)''']' + (l^2 + s^2)\phi'' = 0, \quad (1.2)$$

where ‘’’’ is the derivative with respect to ξ . Integrating (2) twice, we have

$$\alpha(g + \beta\phi + \phi^2) + 2(\phi')^2 + 2\phi\phi'' = 0, \quad (1.3)$$

where g is an integration constant, $\alpha = \frac{a}{k^2}, \beta = \frac{1}{a}(l^2 + s^2 - \lambda)$. Eq.(1.3) is equivalent to the following two-dimensional systems

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{1}{2\phi}[2y^2 + \alpha(\phi^2 + \beta\phi + g)]. \quad (1.4)$$

Systems (1.4) have the first integrals

$$y^2 = -\frac{\alpha}{\phi^2} \left(\frac{1}{4}\phi^4 + \frac{\beta}{3}\phi^3 + \frac{g}{2}\phi^2 + h \right) \quad (1.5)$$

and

$$H(\phi, y) = -\frac{1}{\alpha}\phi^2 y^2 - \phi^2 \left(\frac{1}{4}\phi^2 + \frac{\beta}{3}\phi + \frac{g}{2} \right) = h. \quad (1.6)$$

We emphasize that when $\phi = 0$, the right hand of the second equation of systems (1.4) is discontinuous. We call such systems the singular traveling wave systems. The straight line $\phi = 0$ in the $\phi - y$ -phase plane is called a singular straight line. There are some general theory and methods for investigating this type of singular traveling wave system [4,5,8,9,13]. It is now well known that the existence of the singular straight lines implies the occurrence of some nonsmooth dynamical behaviors and curve breaking phenomena of the traveling wave solutions of such system, more precisely the so-called peakon, pseudo-peakon, loop, and periodic cusp wave, etc.

The rest of this paper is organized as follows. In Section 2, we discuss the bifurcations of phase portraits of Eq.(1.4), where parametric conditions will be derived. In Section 3, we give explicit parametric representations for peakon, pseudo-peakon, loop and periodic cusp wave solutions of Eq.(1.1). Section 4 contains the concluding remarks.

2. Bifurcation set and all phase portraits of system (1.4)

We first make the transformation $d\xi = -2\phi d\tau$ for $\phi \neq 0$, such that system (1.4) becomes

$$\frac{d\phi}{d\tau} = -2\phi y, \quad \frac{dy}{d\tau} = 2y^2 + \alpha(\phi^2 + \beta\phi + g). \tag{2.1}$$

Clearly, system (2.1) has the same phase portraits as system (1.4). But, the phase orbits of two systems have different parametric representations.

System (2.1) has two equilibrium points $A_{\pm}(\phi_{\pm}, 0)$ on the ϕ -axis, where $\phi_{\pm} = \frac{1}{2}(-\beta \pm \sqrt{\Delta})$, $\Delta = \beta^2 - 4g$. System (2.1) has two equilibrium points $S_{\pm}(0, \pm\sqrt{Y})$ on the singular straight line $\phi = 0$, where $Y = -\frac{\alpha g}{2}$.

Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point (ϕ_e, y_e) and $J(\phi_e, y_e)$ be its Jacobin determinant. Then, we have

$$J(\phi_e, y_e) = -8y_e^2 + 2\alpha\phi_e(2\phi_e + g). \tag{2.2}$$

By the theory of planar dynamical systems, we know that for an equilibrium point (ϕ_e, y_e) of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $Trace(M(\phi_e, y_e)) = 0$ then it is a center point; if $J > 0$ and $(Trace(M(\phi_e, y_e)))^2 - 4J(\phi_e, y_e) > 0$ then it is a node; if $J = 0$ and the Poincare index of the equilibrium point is zero then it is a cusp.

For the Hamiltonian defined by (1.6), we write that

$$h_{\pm} = H(\phi_{\pm}, 0) = -\phi_{\pm}^2 \left(\frac{1}{4}\phi_{\pm}^2 + \frac{1}{3}\beta\phi_{\pm} + \frac{1}{2}g \right), \quad h_s = H(0, \pm\sqrt{Y}) = 0. \tag{2.3}$$

The relations $\Delta = 0$ and $H(0, \pm\sqrt{Y}) = H(\phi_{\pm}, 0)$ give the following two bifurcation curves:

$$(L_1) : g = \frac{1}{4}\beta^2; \quad (L_2) : g = \frac{2}{9}\beta^2. \tag{2.4}$$

Thus, in the (β, g) -plane, we have 8 or 6 different parameter regions partitioned by the curves $(L_i), i = 1, 2, \beta = 0$ and $g = 0$, which are shown in Figure 1.

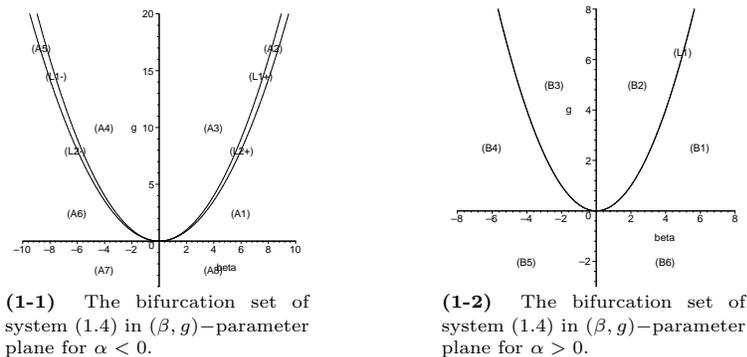


Figure 1. The bifurcation set of system (1.4) in (β, g) -parameter plane.

By using the above fact to do a qualitative analysis, we obtain all phase portraits of system (1.4) shown in Figure 2 and Figure 3 for the cases $\alpha < 0$ and $\alpha > 0$, respectively.

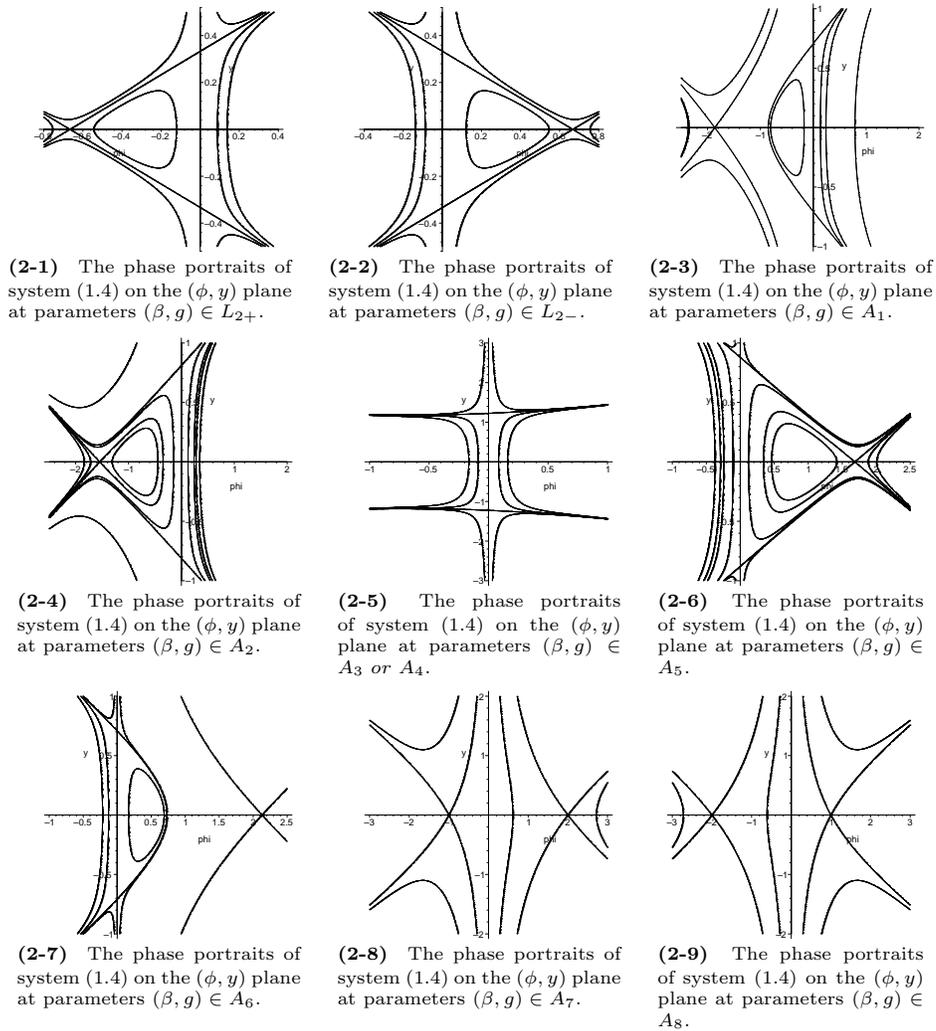


Figure 2. The phase portraits of system (1.4) for $\alpha < 0$.

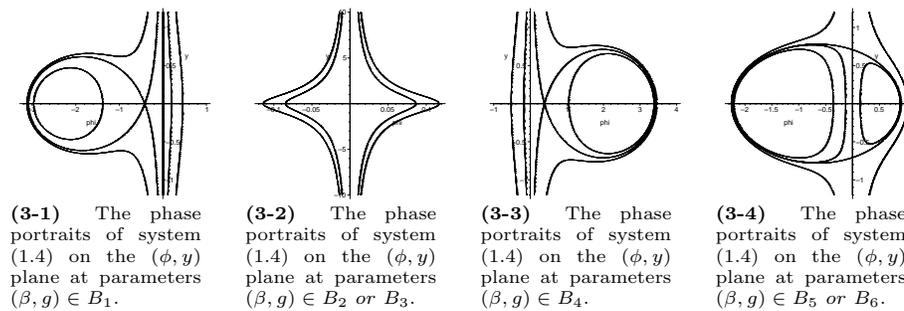


Figure 3. The phase portraits of system (1.4) for $\alpha > 0$.

Because the phase portraits Figs. 2(2-1), (2-3), (2-4), (2-8) and Fig. 3 (3-1) are the reflections of the phase portraits Figs. 2 (2-2),(2-7), (2-6), (2-9) and Fig. 3 (3-3) with respect to the y-axis, we consider the above phase portraits, the only discussion the phase portraits Figs. 2 (2-1), (2-3), (2-4), (2-8) and Fig. 3 (3-1).

3. Exact explicit peakon, pseudo-peakon, loop and periodic cusp wave solutions of Eq.(1.4)

In this section, we give all parametric representations of peakon, pseudo-peakon, loop and periodic cusp wave solutions.

Definition 3.1. A wave function $\phi(\xi)$ is called peakon if $\phi(\xi)$ is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} \phi'(\xi) = -\lim_{\xi \downarrow \xi_0} \phi'(\xi) = a, a \neq 0, a \neq \pm\infty$.

To discuss the existence of peakon, loop and periodic cusp wave solution, we need to use the following two lemmas relating to the singular straight line (see [5,8,9,13]).

Lemma 3.1. *When $h \rightarrow h_-$, the periodic orbits of the periodic annulus surrounding $(\phi_+, 0)$ approach to the boundary curves. Let $(\phi, y = \phi')$ be a point in a periodic orbit γ of system (1.4). Then, along the segment $B_{\gamma\epsilon}A_{\gamma\epsilon}$ near the straight line $\phi = 0$, in a very short time interval of $\xi, y = \phi'_\xi$ jumps up rapidly.*

Lemma 3.2 (Existence of finite time interval(s) of solutions with respect to ξ in the positive or (and) negative direction(s)). *Let $(y = \phi'(\xi))$ be the parametric representation of an orbit γ of system (1.4) and S_\pm be two points on the singular straight line $\phi = 0$. Suppose that the following conditions holds: $\alpha g < 0$ and, along the orbit γ , as ξ increases or (and) decreases, the phase point $(\phi(\xi), y(\xi))$ tends to the points $(0, \pm\sqrt{Y})$, respectively.*

Then, there is a finite value $\xi = \tilde{\xi}$ such that $\lim_{\xi \rightarrow \tilde{\xi}} \phi(\xi) = 0$.

We next consider the curve triangle $S_+A_-S_-$ in Fig. 2 (2-1). There exists an equilibrium point A_- of system (1.4) at the vertex of the triangle $S_+A_-S_-$, which is far from the singular straight line $\phi = 0$. If and only if $\xi \rightarrow \pm\infty$, along two curves S_+A_- and A_-S_- as ξ varies the phase point $(\phi(\xi), y(\xi))$ of system (1.4) tends to the equilibrium point A_- . Because the curve triangle $S_+A_-S_-$ in Fig. 2 (2-1) defined by $H(\phi, y) = 0$ is the limit curve of the $\{\Gamma^h\}$ of periodic orbits of system (1.4) given by $H(\phi, y) = h, h \in (h_s, h_+)$, as h is varied from h_+ to h_s the period of the periodic orbit $\{\Gamma^h\}$ tends to ∞ . Thus, by using Lemma 3.1 and Lemma 3.2, we know that, as the limit curve of the family of periodic cusp wave, this curve triangle $S_+A_-S_-$ gives rise to a solitary cusp wave of $\phi(\xi)$ (so called ‘‘peakon’’ [2]).

We see from the results of Section 2 and the theory of singular nonlinear traveling wave solutions developed by [4,5,8] that the following conclusions hold.

Theorem 3.1. *As limit boundary of a family of periodic orbits of system (1.4), the arch curve connecting S_\pm in the left or in the right side of the straight line $\phi = 0$ gives rise to a periodic cusp wave solution for Eq. (1.1).*

Theorem 3.2. *As limit boundary of a family of periodic orbits of system (1.4), the curved triangle gives rise to a peakon soliton solution for Eq. (1.1).*

In the following, we give parametric representations of peakon, pseudo-peakon, loop and periodic cusp wave solutions.

1. Peakon soliton solution.

Suppose that $\alpha < 0, \beta > 0, g = \frac{2}{9}\beta^2$. In this case, we have the phase portrait of Eq. (1.4) shown in Fig. 2 (2-1). Notice that $H(0, \pm Y) = H(\phi_-, 0) = 0$. We see from Eq. (1.6) that two arch curves connecting S_{\pm} in the left side of the straight line $\phi = 0$ has the algebraic equation

$$y^2 = -\frac{\alpha}{4} \left(\phi^2 + \frac{4}{3}\beta\phi + 2g \right) = -\frac{\alpha}{4} \left(\phi + \frac{2}{3}\beta \right)^2. \tag{3.1}$$

Thus, by using the first equation of (1.4) and Eq. (3.1), we obtain the parametric representation of two arch curves as follows:

$$\phi(\xi) = -\frac{2}{3}\beta + \left(\phi_+ + \frac{2}{3}\beta \right) \exp \left(-\frac{1}{2}\sqrt{-\alpha} |\xi| \right), \tag{3.2}$$

which gives rise to a peakon soliton solution of Eq. (1.1). The profile of peakon soliton solution is shown in Fig. 4(4-1). Therefore, we have

Theorem 3.3. (i) *When the parameter group (α, β, g) of system (1.4) satisfy the condition $g = \frac{2}{9}\beta^2$ with $\alpha < 0, \beta > 0$, there exists a heteroclinic loop of system (1.7) given by three branches of the curves $H(\phi, y) = 0$.*

(ii) *As the limit curves of a family of periodic orbits of system (1.4), the curve triangle (i.e., heteroclinic loop) in Fig. 2(2-1) gives rise to a solitary peaked wave solution (a peakon) of equation (1.1), which has the exact parametric representation given by (3.2).*

2. Pseudo peakon solution.

The phase portraits of system (1.4) on the (ϕ, y) plane at parameters $(\beta, g) \in A_2$.

Suppose that $\alpha < 0, (\beta, g) \in A_2$. In this case, we have the phase portrait of Eq. (1.4) shown in Fig. 2 (2-4). For the homoclinic orbit defined by $H(\phi, y) = h_-$, let $P_m(\phi_m, 0)$ be the intersection point of the homoclinic orbit with the ϕ -axis.

We know that $\phi_m = \frac{1}{2}[-(\frac{4}{3}\beta + 2\phi_-) - \sqrt{\frac{16}{9}\beta^2 + \frac{8}{3}\beta\phi_-}]$, Eq. (1.4) becomes $y = \pm \frac{\phi - \phi_-}{2\phi} \sqrt{-\alpha(\phi - \phi_m)(\phi - \phi_l)}$. Using the first equation of system (1.4) and taking initial value $\phi(0) = \phi_m$, we have

$$\xi = \sqrt{\frac{4}{-\alpha}} \int_{\phi}^{\phi_m} \frac{\phi d\phi}{(\phi - \phi_-)\sqrt{(\phi - \phi_m)(\phi - \phi_l)}}, \tag{3.3}$$

which leads to the parametric representation

$$\begin{aligned} \phi(\chi) &= \frac{1}{2}(\phi_m + \phi_l) + \sqrt{\phi_m^2 + \phi_l\phi_m + \phi_l^2} \cosh \chi, \\ \xi(\chi) &= \frac{2}{\sqrt{-\alpha}} [\cosh^{-1} M - \chi - N(\cosh^{-1} F(\phi_m) - \cosh^{-1} F(\phi(\chi)))] \end{aligned}, \tag{3.4}$$

for $\xi \in (-\infty, 0]$ and for $\xi \in [0, \infty)$, respectively, where $M = \frac{\phi_m - \phi_l}{2\sqrt{\phi_m^2 + \phi_l\phi_m + \phi_l^2}}, N =$

$$\frac{\phi_-}{\sqrt{(\phi_m - \phi_-)(\phi_l - \phi_-)}}, F(\phi) = \frac{2(\phi_m - \phi_-)(\phi_l - \phi_-) + 2\phi_- - \phi_m - \phi_l}{\phi_m + \phi_l - 2\phi_-}.$$

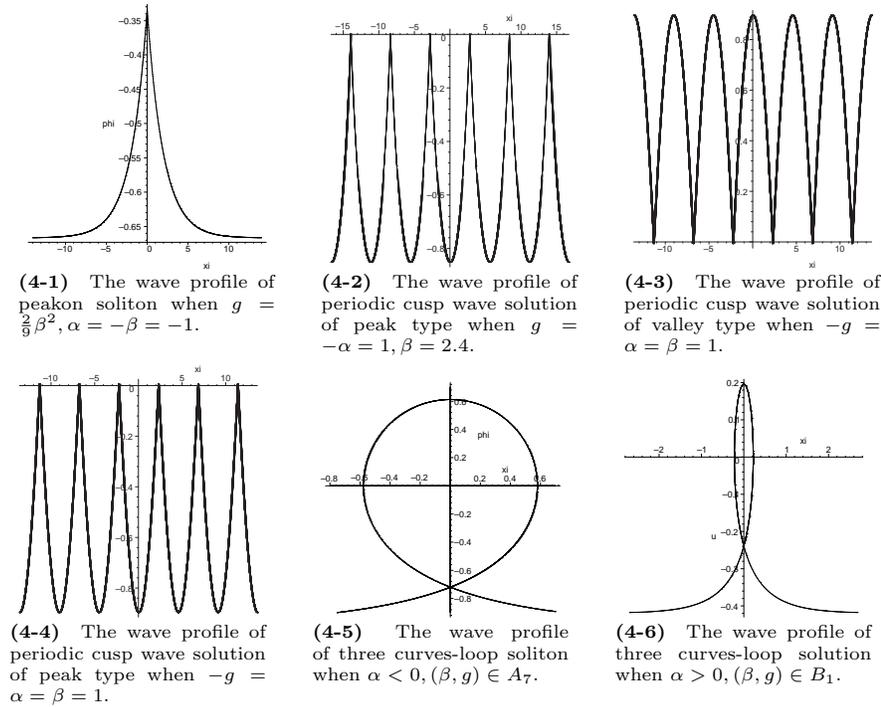


Figure 4. The profiles of wave solutions of Eq.(1.1).

For the family of periodic orbits of Eq.(1.4) defined by $H(\phi, y) = h, h \in (h_+, h_-)$ in (1.6), we see that

$$y^2 = -\frac{\alpha}{4\phi^2}(\phi^4 + \frac{4}{3}\beta\phi^3 + 2g\phi^2 + 4h) = -\frac{\alpha}{4\phi^2}(\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3)(\phi - \phi_4), \quad (3.5)$$

where $\phi_1 < \phi_2 < \phi_3 < 0 < \phi_4$. Thus, from the first equation of system (1.4) we have

$$\xi = \sqrt{\frac{4}{-\alpha}} \int_{\phi}^{\phi_3} \frac{\phi d\phi}{\sqrt{(\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3)(\phi - \phi_4)}}, \quad (3.6)$$

which leads to the parametric representation

$$\begin{aligned} \phi(\chi) &= \frac{\phi_2(\phi_3 - \phi_1) - \phi_1(\phi_3 - \phi_2)sn^2(\chi, k)}{(\phi_3 - \phi_1) - (\phi_3 - \phi_2)sn^2(\chi, k)}, \\ \xi(\chi) &= \frac{4}{\sqrt{-\alpha(\phi_4 - \phi_2)(\phi_3 - \phi_1)}} [(\phi_2 - \phi_1)\Pi(\arcsin(sn(\chi, k)), \alpha_0^2, k) + \phi_1\chi], \end{aligned} \quad (3.7)$$

where $k^2 = \frac{(\phi_3 - \phi_2)(\phi_4 - \phi_1)}{(\phi_4 - \phi_2)(\phi_3 - \phi_1)}, \alpha_0^2 = \frac{\phi_3 - \phi_2}{\phi_3 - \phi_1}, \Pi(\cdot, \alpha^2, k)$ is the elliptic integral of the third kind, $sn(u, k), cn(u, k), dn(u, k)$ are the Jacobian elliptic functions [1].

We notice from Fig. 2(2-4) that when parameter g is very close to $\frac{2}{9}\beta^2$, we have $0 < |h_-|$ and h_- is arbitrarily small. It implies that a segment of the homoclinic orbit defined by a branch of the level curve $H(\phi, y) = h_-$ completely lies in a left neighborhood of the singular straight line $\phi = 0$. By using Lemma 3.1, the state

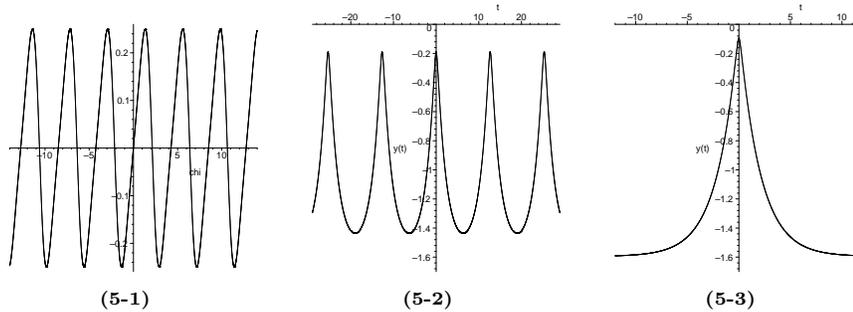


Figure 5. The change of wave profiles of $\phi(\xi)$.

coordinate y of the points in this segment rapidly jumps (following ξ varies) from a positive number to negative number, such that this homoclinic orbit gives rise to a solitary cusp wave.

In addition, from (3.7), one can easily see that

$$\frac{d\phi}{d\xi} = -\frac{\sqrt{-\alpha(\phi_3 - \phi_1)(\phi_4 - \phi_2)(\phi_2 - \phi_1)(\phi_3 - \phi_2)}sn(\chi, k)cn(\chi, k)dn(\chi, k)}{2\phi_2(\phi_3 - \phi_1)\left[1 - \frac{\phi_1(\phi_3 - \phi_2)}{(\phi_2(\phi_3 - \phi_1))}sn^2(\chi, k)\right]\left[1 - \frac{\phi_3 - \phi_2}{\phi_3 - \phi_1}sn^2(\chi, k)\right]} \tag{3.8}$$

When h very close to $h_s(= 0)$, i.e., k very close to 1, the graph of $\frac{d\phi}{d\xi}$ is shown in Fig. 4(4-1). Clearly, when $\chi = 4nK(k)$, $n = 0, \pm 1, \pm 2, \dots$, $\frac{d\phi}{d\xi} = 0$, where $K(k)$ is the complete elliptic integral of the first kind with the modulo k . By Lemma 3.1, when χ passes through $4nK(k)$, its sign changes from $+$ to $-$ and its value jumps rapidly from a positive maximum to a negative maximum. This fact implies that the wave profile of $\phi(\xi)$, determined by the periodic orbit closing to the homoclinic orbit, is a smooth periodic cusp wave (see Fig. 5(5-2)). Finally, when $h \rightarrow h_-, k \rightarrow 1$, the period of the periodic cusp wave solutions tends to ∞ , thus, the homoclinic orbit in Fig. 5(5-3) gives rise to a smooth solitary cusp wave solution of (1.4). Because for $\phi(\xi)$ given by (3.4), we have $\frac{d\phi}{d\xi}|_{\xi=0} = 0$, therefore, the solitary cusp wave solution defined by (3.4) is not a peakon, it is a pseudo-peakon.

In a short, we have the following conclusion.

Theorem 3.4. (i) When the parameter group (α, β, g) of system (1.4) satisfy the condition $(\beta, g) \in A_2$ with $\alpha < 0$, there exists a homoclinic orbit of system (1.7) given by a branch of the curves $H(\phi, y) = h_-$. The homoclinic orbit has the exact parametric representation given by (3.4).

(ii) When $0 < g - \frac{2}{9}\beta^2 \ll 0$ ($\ll 0$ means very close to zero), i.e., $0 < |h_-| \ll 0$ (namely, $|h_-|$ is strictly positive and arbitrarily small), as a limit curve of a family of periodic orbits of system (1.4) defined by the closed branch of the curves $H(\phi, y) = h$, $h \in (h_+, h_-)$ in Fig. 2(2-4), the homoclinic orbit gives rise to a smooth solitary cusp-like wave solution (a pseudo-peakon) of equation (1.1).

(iii) When h varies from h_- to $h_s(= 0)$, periodic wave solutions of equation (1.1) determined by periodic orbits of system (1.4) will gradually become peaked periodic wave, and evolve from nonpeaked periodic waves to the smooth periodic cusp-like waves and finally converge to a smooth solitary cusp-like wave (a pseudo-peakon).

3. Periodic cusp wave solutions.

3.1 Suppose that $\alpha < 0$, $(\beta, g) \in A_1$. In this case, we have the phase portrait of Eq. (1.4) shown in Fig. 2 (2-3). Notice that $H(0, \pm Y) = 0$. We see from Eq. (1.6) that the arch curve connecting S_{\pm} in the left side of the straight line $\phi = 0$ has the algebraic equation

$$y^2 = -\frac{\alpha}{4} \left(\phi^2 + \frac{4}{3}\beta\phi + 2g \right). \quad (3.9)$$

Thus, by using the first equation of (1.4) and Eq. (3.9), we obtain the parametric representation of this arch curve as follows:

$$\phi(\xi) = -\frac{2}{3}\beta \left(1 - \sqrt{1 - \frac{9g}{2\beta^2}} \cosh \frac{1}{2}\sqrt{-\alpha}\xi \right), \quad \xi \in \left(0, \frac{2}{\sqrt{-\alpha}} \cosh^{-1} \sqrt{\frac{2\beta^2}{2\beta^2 - 9g}} \right), \quad (3.10)$$

which gives rise to a periodic cusp wave solution of Eq. (1.1). The profile of periodic cusp wave solution is shown in Fig. 4(4-2).

3.2 Suppose that $\alpha > 0$, $(\beta, g) \in B_5$ or B_6 . In this case, we have the phase portrait of Eq. (1.4) shown in Fig. 3 (3-4). Notice that $H(0, \pm Y) = 0$. We see from Eq. (1.6) that two arch curves connecting S_{\pm} in the left and right sides of the straight line $\phi = 0$ has the algebraic equation

$$y^2 = -\frac{\alpha}{4} \left(\phi^2 + \frac{4}{3}\beta\phi + 2g \right). \quad (3.11)$$

Thus, by using the first equation of (1.4) and Eq. (3.11), we obtain the parametric representation of two arch curves as follows:

$$\phi(\xi) = -\frac{2}{3}\beta \left(1 - \operatorname{sgn}(\beta) \sqrt{1 - \frac{9g}{2\beta^2}} \cos \frac{1}{2}\sqrt{\alpha}\xi \right), \quad \xi \in \left(0, \frac{2}{\sqrt{\alpha}} \arccos \sqrt{\frac{2\beta^2}{2\beta^2 - 9g}} \right) \quad (3.12)$$

and

$$\phi(\xi) = -\frac{2}{3}\beta \left(1 + \operatorname{sgn}(\beta) \sqrt{1 - \frac{9g}{2\beta^2}} \cos \frac{1}{2}\sqrt{\alpha}\xi \right), \quad \xi \in \left(0, \frac{2}{\sqrt{\alpha}} \arccos \sqrt{\frac{2\beta^2}{2\beta^2 - 9g}} \right), \quad (3.13)$$

which gives rise to two periodic cusp wave solutions of peak type and valley type of Eq. (1.1). The profile of periodic cusp wave solutions are shown in Fig. 4(4-3) and 4(4-4).

4. Loop solutions.

From the viewpoint of dynamical systems, we shall show that for a given c value, corresponding to the level curves (stable and unstable manifolds of a saddle point $A_-(\phi_-, 0)$ or $A_+(\phi_+, 0)$ and an open curve) defined by $H(\phi, y) = h_-(h_+)$ of the traveling wave systems (ODEs) of some special nonlinear wave equations (PDEs), if we consider the parametric representations $u(x, t) = \phi(\chi)$ of these traveling wave solutions in the global interval $\chi \in (-\infty, \infty)$, we shall obtain a loop graph which has been called a loop solution by some references.

4.1 Suppose that $\alpha < 0$, $(\beta, g) \in A_7$. We notice that the curves defined by $H(\phi, y) = h_-$ correspond to 6 different orbits of Eq. (1.4) consisting of two stable

manifolds, two unstable manifolds of the saddle point $A_-(\phi_-, 0)$ and two open curves passing through the points $\left(-(\frac{2}{3}\beta + \phi_-) \pm \sqrt{\frac{4}{9}\beta^2 + \frac{2}{3}\beta\phi_-}, 0\right)$, respectively (see Figure 2 (2-8)). For these open curves, when $\xi \rightarrow \tilde{\xi}$, $\phi(\xi) \rightarrow 0$, respectively.

We next discuss the parametric representation of $\phi(\xi)$ for these curves. We see from Eq. (1.6) that the arch curve in the left side of the straight line $\phi = 0$ has the algebraic equation

$$y^2 = -\frac{\alpha}{4\phi^2}(\phi^4 + \frac{4}{3}\beta\phi^3 + 2g\phi^2 + 4h_-) = -\frac{\alpha}{4\phi^2}(\phi - \phi_-)^2(\phi - \phi_1)(\phi - \phi_2), \quad (3.14)$$

where $\phi_{1,2} = -(\frac{2}{3}\beta + \phi_-) \pm \sqrt{\frac{4}{9}\beta^2 + \frac{2}{3}\beta\phi_-}$.

We first consider the stable manifold of the saddle point $A_-(\phi_-, 0)$ in the third quadrant. Taking the initial value $\phi(0) = \frac{1}{2}\phi_-$, $y(0) = \sqrt{\alpha(\frac{11}{48}\beta\phi_- + \frac{9}{16}g)}$ (or $y(0) = -\sqrt{\alpha(\frac{11}{48}\beta\phi_- + \frac{9}{16}g)}$), we have from the Eq. (3.14) that

$$\xi = \int_{\phi}^{\frac{1}{2}\phi_-} f(\phi)d\phi = \frac{2}{\sqrt{-\alpha}} \left[\Phi_1(\phi) + \Phi_2(\phi) - \Phi_1\left(\frac{1}{2}\phi_-\right) - \Phi_2\left(\frac{1}{2}\phi_-\right) \right], \quad (3.15)$$

where $f(\phi) = \frac{2\phi}{\sqrt{-\alpha(\phi-\phi_-)}\sqrt{(\phi-\phi_1)(\phi-\phi_2)}}$, $\Phi_1(\phi) = \cosh^{-1} \frac{2(\phi - \frac{\phi_1+\phi_2}{2})}{\phi_1-\phi_2}$, $\Phi_2(\phi) = \frac{-\phi_-}{\sqrt{(\phi-\phi_1)(\phi-\phi_2)}} \cosh^{-1} \frac{\left(\frac{2(\phi-\phi_1)(\phi-\phi_2)}{(\phi_1-\phi_2)(\phi-\phi_-)} + \frac{2\phi_- - \phi_1 - \phi_2}{\phi_1 - \phi_2}\right)}{\phi_1 - \phi_2}$. By introducing a new variable χ , from (3.15) one obtains the parametric representation of the stable manifold

$$\begin{aligned} \phi(\chi) &= \frac{\phi_1 + \phi_2}{2} + \frac{\phi_1 - \phi_2}{2} \cosh \chi, \\ \xi(\chi) &= \frac{2}{\sqrt{-\alpha}} \left[\chi + \Phi_2(\chi) - \Phi_1\left(\frac{1}{2}\phi_-\right) - \Phi_2\left(\frac{1}{2}\phi_-\right) \right], \end{aligned} \quad (3.16)$$

for $\chi \in (\chi_0, \infty)$, where χ_0 satisfies $\frac{\phi_1+\phi_2}{2} + \frac{\phi_1-\phi_2}{2} \cosh \chi_0 = 0$.

Obviously, the parametric representation of the unstable manifold on the second quadrant has the same form as (3.16) with $\chi \in (-\infty, -\chi_0)$.

In addition, taking the initial condition $\phi(0) = \phi_2 = -(\frac{2}{3}\beta + \phi_-) - \sqrt{\frac{4}{9}\beta^2 + \frac{2}{3}\beta\phi_-}$, $y(0) = 0$, for the right open curve defined by a branch of $H(\phi, y) = h_-$, we have from the Eq. (3.14) that

$$\xi = \int_{\phi}^{\phi_2} f(\phi)d\phi = \frac{2}{\sqrt{-\alpha}} [\Phi_1(\phi) + \Phi_2(\phi) - \Phi_1(\phi_2) - \Phi_2(\phi_2)]. \quad (3.17)$$

By introducing a new variable χ , from (3.17) one obtains the parametric representation of the right open curve

$$\begin{aligned} \phi(\chi) &= \frac{\phi_1 + \phi_2}{2} + \frac{\phi_1 - \phi_2}{2} \cosh \chi, \\ \xi(\chi) &= \frac{2}{\sqrt{-\alpha}} [\chi + \Phi_2(\chi) - \Phi_1(\phi_2) - \Phi_2(\phi_2)], \end{aligned} \quad (3.18)$$

for $\chi \in (-\chi_0, \chi_0)$.

Employing the formulas (3.16) and (3.18), one obtains a loop solution. The profile of loop solution is shown in Fig. 4(4-5).

Remark 3.1. The loop solution, that is, the so-called loop soliton solution, is not one real soliton solution (see [1,2,9,11]).

4.2 Suppose that $\alpha > 0, (\beta, g) \in B_1$. We notice that the curves defined by $H(\phi, y) = h_+$ correspond to 6 different orbits of Eq. (1.4) consisting of two stable manifolds, two unstable manifolds of the saddle point $A_+(\phi_+, 0)$ and two open curves passing through the points $\left(-(\frac{2}{3}\beta + \phi_+) \pm \sqrt{\frac{4}{9}\beta^2 + \frac{2}{3}\beta\phi_+}, 0\right)$, respectively (see Figure 3 (3-1)). We see from Eq. (1.6) that the arch curve in the left side of the straight line $\phi = 0$ has the algebraic equation

$$y^2 = -\frac{\alpha}{4\phi^2}(\phi^4 + \frac{4}{3}\beta\phi^3 + 2g\phi^2 + 4h_+) = -\frac{\alpha}{4\phi^2}(\phi - \phi_+)^2(\phi - \phi_4)(\phi_3 - \phi), \quad (3.19)$$

where $\phi_{3,4} = -(\frac{2}{3}\beta + \phi_+) \pm \sqrt{\frac{4}{9}\beta^2 + \frac{2}{3}\beta\phi_+}$.

By the similar calculation in 4.1, we have the following parametric representation, respectively.

$$\begin{aligned} \phi(\chi) &= \frac{\phi_3 + \phi_4}{2} + \frac{\phi_3 - \phi_4}{2} \sin \chi, \\ \xi(\chi) &= \frac{2}{\sqrt{-\alpha}} \left[\chi + \Phi_4(\chi) - \Phi_3\left(\frac{1}{2}\phi_+\right) - \Phi_4\left(\frac{1}{2}\phi_+\right) \right], \end{aligned} \quad (3.20)$$

for $\chi \in (\chi_1, \infty)$ or $\chi \in (-\infty, -\chi_1)$, where χ_1 satisfies

$$\begin{aligned} \frac{\phi_3 + \phi_4}{2} + \frac{\phi_3 - \phi_4}{2} \sin \chi_1 &= 0, \quad \Phi_3(\chi) = \arcsin \frac{2\left(\frac{\phi_3 + \phi_4}{2} - \chi\right)}{\phi_3 - \phi_4}, \\ \Phi_4(\chi) &= \frac{-\phi_+}{\sqrt{(\phi_+ - \phi_3)(\phi_+ - \phi_4)}} \coth^{-1} \frac{\left(\frac{2(\phi_+ - \phi_3)(\phi_+ - \phi_4)}{(\phi_3 - \phi_4)(\chi - \phi_+)} + \frac{2\phi_+ - \phi_3 - \phi_4}{\phi_3 - \phi_4}\right)}{\phi_3 - \phi_4}, \end{aligned}$$

and

$$\begin{aligned} \phi(\chi) &= \frac{\phi_3 + \phi_4}{2} + \frac{\phi_3 - \phi_4}{2} \sin \chi, \\ \xi(\chi) &= \frac{2}{\sqrt{-\alpha}} [\chi + \Phi_4(\chi) - \Phi_3(\phi_3) - \Phi_4(\phi_3)], \end{aligned} \quad (3.21)$$

for $\chi \in (-\chi_1, \chi_1)$.

Employing the formulas (3.20) and (3.21), one obtains a loop solution. The profile of loop solution is shown in Fig. 4(4-6).

Remark 3.2. To the best of our knowledge, peakon, pseudo peakon, loop and periodic cusp wave solutions obtained for 3DKP(2, 2) equation have not been reported before in the literature.

4. Conclusions

In this paper, we used the qualitative analysis methods of a dynamical system to investigate the peakon, pseudo peakon, loop and periodic cusp wave solutions of 3DKP(2, 2) equation. The conditions of existence of peakon, pseudo peakon, loop and periodic cusp wave solutions are given by using phase portrait analytical

technique. Our procedure shows that the 3DKP(2, 2) equation either has peakon, pseudo peakon and loop or possesses periodic cusp wave solutions. We obtain all peakon, pseudo peakon, loop and periodic cusp wave solutions of 3DKP(2, 2) and analyze their analytic and dynamical behavior. The phase portrait bifurcation of the traveling wave system corresponding to the equation is given. The graph of the solutions are given with the numerical simulation. The mathematical results about singular traveling wave equation provide a deep insight into nonlinear wave model and is useful for physicists to comprehend dynamical behavior of nonlinear wave models.

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