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TWO GENERAL CENTRE PRODUCING SYSTEMS FOR THE POINCARÉ PROBLEM

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Abstract We consider the polynomial system $\frac{dx}{dt} = -y - ax^{s+3}y^{n-s-3} - bx^{s+1}y^{n-s-1}$, $\frac{dy}{dt} = x + cx^{s+2}y^{n-s-2} + dx^s y^{n-s}$ where $n \geq 3$ is an odd integer and $s = 0, \dots, n-3$ is an even integer. We calculate the first three nonzero Lyapunov coefficients for the system and obtain a Gröbner basis for the ideal generated by them. Potential centre conditions for the system are obtained by setting the basis elements equal to zero and solving the resulting system. This gives five basic solutions and within this set we find two well known classes of centres and three new centre producing systems. One of the three is a variant of one of the other new systems, so we obtain two general independent systems which produce multiple centre conditions for each $n \geq 5$.

Keywords Centre-focus problem, Lyapunov coefficients, Gröbner basis, hypergeometric function.

MSC(2000) 34A05, 34C25.

1. Introduction

In this work we consider centre conditions for the system

$$\frac{dx}{dt} = -y - p(x,y), \qquad \frac{dy}{dt} = x + q(x,y) \tag{1.1}$$

where p and q are homogeneous polynomials of degree $n \geq 2$. That is, conditions such that all trajectories of the system on a sufficiently small neighborhood of the origin are closed curves. Specifically, we consider general systems for which

$$\begin{aligned} p(x,y) &= ax^{s+3}y^{n-s-3} + bx^{s+1}y^{n-s-1} = x^s y^{n-s-3}(ax^3 + bxy^2), \\ q(x,y) &= cx^{s+2}y^{n-s-2} + dx^s y^{n-s} = x^s y^{n-s-3}(cx^2 y + dy^3) \end{aligned} \tag{1.2}$$

where $0 \leq s \leq n-3$ is an integer. If n is even or if n and s are both odd, the system always defines *symmetric* centres. That is, its phase portrait has a line of symmetry which passes through the origin. To avoid these well known centres we restrict our attention to those cases for which n is odd and s is even. We shall show that in addition to generating Hamiltonian and *constant invariant* (see Section 2) forms, the system also has two other independent centre producing forms. One of these produces $(n-1)/2$ centre conditions for each n and the other, because of the symmetrical form of the system, gives only $[(n+1)/4]$ conditions where $[\dots]$ is the greatest integer function. The only forms of these systems which are integrable in

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the sense that an integrating factor can be found are the $s = 0$ cases [12] along with the $s = (n-3)/2$ and $s = n-3$ cases of one of the systems. Also associated with (1.1) is the ordinary differential equation

$$\frac{dy}{dx} = -\frac{x + q(x,y)}{y + p(x,y)}. \tag{1.3}$$

Throughout the paper any integrating factor μ of (1.3) will be such that

$$\frac{\partial}{\partial y}\big(\mu(x,y)(x + q(x,y))\big) - \frac{\partial}{\partial x}\big(\mu(x,y)(y + p(x,y))\big) = 0.$$

One of the major problems encountered in showing that a particular system gives rise to a centre is to establish the sufficiency of the conditions which have been obtained. For the system (1.2) the problem is made more difficult because we must not only show it for general values of n but also for the subcases generated by allowing s to vary. A few general systems valid for arbitrary values of n are known. In each case the centre nature of the system is established either by showing the existence of an integrating factor for it or by demonstrating that certain parity conditions with regard to related differential equations are satisfied. The system $p(x,y) = ax^{n-1}y - 2bx^2y^{n-2} + by^n$, $q(x,y) = ax^n - 2ax^{n-2}y^2 + bxy^{n-1}$ having integrating factor

$$\mu(x,y) = (1 + 2(ax^{n-1} + by^{n-1}) + (ax^{n-1} - by^{n-1})^2)^{-(n+3)/(2n-2)}$$

is obtained in [2] and several other systems are given in [12, 13]. In particular, the $s = 0$ forms of the systems obtained herein are shown to be integrable in [12] and this led, at least in part, to the study of more general systems of the type defined by (1.2). Examples of the use of parity properties can be found in [12]. Other criteria can be used to establish whether a given system produces a centre.

In his original work [14], Poincaré developed a method for determining if the origin is a centre for (1.1) by seeking an analytic solution of (1.3). This takes the form

$$U(x,y) = \frac{1}{2}(x^2 + y^2) + \sum_{k=2}^{\infty} U_{\alpha_k}(x,y)$$

where $U_{\alpha_k}(x,y)$ is a homogeneous polynomial of degree $\alpha_k = k(n-1) + 2$. This solution is required to satisfy the condition

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} = \sum_{k=1}^{\infty} V_{\alpha_k}(x^2 + y^2)^{\alpha_k/2} \equiv \sum_{\ell=2}^{\infty} \widetilde{V}_{2\ell}(x^2 + y^2)^{\ell}.$$

Here, the V_{α_k}, $\widetilde{V}_{2\ell}$ are called *Lyapunov coefficients* and they are homogeneous polynomials in the coefficients of the system. We note that if α_k is odd, then $V_{\alpha_k} = 0$. The origin is said to be a *fine focus* of order N if $\widetilde{V}_{2\ell} = 0$ for $\ell \le N$ but $\widetilde{V}_{2N+2} \ne 0$. A necessary and sufficient condition for the existence of a centre is the vanishing of all the Lyapunov coefficients.

It has been conjectured [4] that all systems which produce centres are either integrable in the sense of *Darboux* (as are the $s = 0$ systems obtained herein) or *Liouville* or they are rationally (algebraically) reversible. Although it is now known that this is not true in general, it was reiterated in [15] for the case of elementary centres. Another possibility is that the given system can be transformed to one of Liénard type and the known centre conditions (see [5]) for these types of systems applied. The general system (1.1) can be transformed to such a system, but having rational rather than polynomial coefficients and all attempts to show rational reversibility of the systems obtained in this paper have produced no worthwhile results. Another possible route to showing the sufficiency of centre conditions is to transform the system to various types of Abel equations (see [6,11]) or to transform the system to a corresponding complex form.

To obtain necessary conditions for a centre for (1.2), we calculate as many of the initial Lyapunov coefficients as possible. Although the system depends upon four parameters, we are actually able to show that the centre conditions are defined by the vanishing of the first three nonzero Lyapunov coefficients. Due to parity conditions, two of the first five are identically zero which results in nonzero Lyapunov coefficients V_{α_1}, V_{α_3}, V_{α_5} of degrees 1, 3, and 5 respectively. We note that this is very similar to what happens for the reduced cubic system ($n = 3$, $s = 0$) derived from (1.2). The V_{α_k} are calculated in terms of trigonometric integrals, primarily because this seemed the only possible way to do it, and from these a Gröbner basis is computed. This has a much simpler form than the original set of coefficients and easily allows for the determination of possible centre conditions.

In the next Section we provide some of necessary background details and present the main result. The following two Sections are devoted to obtaining the necessary and sufficient conditions for the systems which we obtain to be centres. In particular, the proof of sufficiency is aided by the fact that the system (1.2) includes some easily identifiable centres which means that the general coefficient V_{α_k} can be expressed in a specific fashion with respect to the basis generators of the ideal $<V_{\alpha_1}, V_{\alpha_3}, V_{\alpha_5}>$. We also give a probable generalization of one of the systems obtained, although a direct proof of this is not yet available.

2. The Main Result

Setting $x = r(\theta)\cos\theta$, $y = r(\theta)\sin\theta$ in (1.3), we obtain

$$\frac{dr}{d\theta} = \frac{\xi(\theta)r^n}{1+\eta(\theta)r^{n-1}} \tag{2.1}$$

where

$$\begin{aligned}\xi(\theta) &= \sin\theta\, q(\cos\theta,\sin\theta) - \cos\theta\, p(\cos\theta,\sin\theta),\\ \eta(\theta) &= \sin\theta\, p(\cos\theta,\sin\theta) + \cos\theta\, q(\cos\theta,\sin\theta).\end{aligned} \tag{2.2}$$

For the system (1.2), (2.2) becomes

$$\begin{aligned}\xi(\theta) &= -(\cos\theta)^s(\sin\theta)^{n-s-3}(a\cos^4\theta + (b-c)\cos^2\theta\sin^2\theta - d\sin^4\theta),\\ \eta(\theta) &= (\cos\theta)^{s+1}(\sin\theta)^{n-s-2}((a+c)\cos^2\theta + (b+d)\sin^2\theta).\end{aligned} \tag{2.3}$$

From this we can note that ξ, η are even, odd functions of θ respectively. Substituting $u = r^{n-1}$ in (2.1) produces an Abel equation of the second kind which can be further transformed to an Abel equation of the first kind using the standard transformation [3]

$$\rho(\theta) = \frac{u(\theta)}{1+\eta(\theta)u(\theta)}.$$

This gives

$$\frac{d\rho}{d\theta} = -(n-1)\xi(\theta)\eta(\theta)\rho^3 + ((n-1)\xi(\theta) - \eta'(\theta))\rho^2. \tag{2.4}$$

If the coefficient functions in (2.4) satisfy a relation of the form $\xi(\theta) = K\eta'(\theta)$ where $K \neq 0$ is a constant, the equation will have a constant first invariant I_1. Systems which satisfy this condition are centres for (1.1) and in the following we shall refer to them as *constant invariant centres*. In particular, the Hamiltonian solution has $K = -1/(n+1)$.

At this time we present the main result of the paper.

Theorem 2.1. *Let $n \geq 3$ be an odd integer, $0 \leq s \leq n-3$ an even integer and C a nonzero constant. If either*

$$\begin{aligned} a &= (n-s-2)^2 C, & b &= -((n-3)s - s^2 + 2n - 3)C, \\ c &= ((n-3)s - s^2 + 2n - 3)C, & d &= -(s+1)^2 C, \end{aligned} \tag{2.5}$$

or

$$\begin{aligned} a &= (n-s)(n-s-2)C, & b &= (s^2 + s - ns - 3n + 2)C, \\ c &= (s-1)(n-s-2)C, & d &= -(s^2-1)C \end{aligned} \tag{2.6}$$

in (1.2) *then the origin is a centre.*

The $s = 0$ forms of these systems were obtained in [12] where it was shown that (2.5), (2.6) have integrating factors given by

$$\mu(x,y) = \left[1 + 2\,(n-1)Cxy^{n-2} + (n-1)^2C^2x^2y^{2n-4} - 2\,(n-1)C^2y^{2n-2}\right]^{\alpha}$$

where $\alpha = (n-3)/(2(n-1))$ and

$$\mu(x,y) = \left[1 - 2\,(n-1)Cxy^{n-2} + 2\,(n-1)C^2y^{2n-2}\right]^{-(2n-1)/(n-1)}$$

respectively. Moreover, the $s = (n-3)/2$ case of (2.5) is always Hamiltonian (even for s odd) and due to the symmetry of its coefficients, $s = n-3$ is also an integrable case. We have spent considerable time seeking other integrable forms, but without success. For $s \neq 0$, $n-3$ the systems do not seem to possess *invariant curves* of the form $f(x,y) = 0$ which are necessary for the construction of integrating factors of Darboux type. We further observe that since the coefficients of (1.2) defined in Theorem 2.1 are functions of n and s they have the elementary property that if we let $(n,s) \to (n+\nu, s+\sigma)$ where ν and σ are even integers which satisfy $n+\nu \geq 3, 0 \leq s+\sigma \leq n+\nu-3$, they generate the coefficients of the similarly transformed system. In the end it is this translation property along with the presence of the integrable $s = 0$ systems which allow for the proof of sufficiency of the centre nature of the systems defined by (2.5) and (2.6).

3. Calculation of the Lyapunov Coefficients

At this point we need some method for determining the Lyapunov coefficients for (1.2). Since n and s can take on arbitrary values, no method which assumes a particular value of n is applicable. The only suitable approach that we found was in terms of trigonometric integrals. For this purpose we adapted the conditions given in [1] to our problem. In this form they apply to the coefficients of the Abel equation (2.4), although equivalent conditions can easily be determined from the recurrence relation generated by assuming a series solution for (2.1). The vanishing of the first five Lyapunov coefficients is given by

$$\int_0^{2\pi} \xi(\theta)\, d\theta = 0, \tag{3.1}$$

$$\int_0^{2\pi} \xi(\theta)\eta(\theta)\, d\theta = 0, \tag{3.2}$$

$$\int_0^{2\pi} [(n-1)\xi_1(\theta) - \eta(\theta)]\xi(\theta)\eta(\theta)\, d\theta = 0, \tag{3.3}$$

$$\int_0^{2\pi} [(n-1)\xi_1(\theta) - \eta(\theta)]^2\xi(\theta)\eta(\theta)\, d\theta = 0, \tag{3.4}$$

$$\int_0^{2\pi} \xi(\theta)\eta(\theta)[(n-1)\xi_1(\theta) - \eta(\theta)][((n-1)\xi_1(\theta) - \eta(\theta))^2 - (n-1)\Psi(\theta)]\, d\theta = 0 \tag{3.5}$$

where

$$\xi_1(\theta) = \int \xi(\theta)\, d\theta, \qquad \Psi(\theta) = \int \xi(\theta)\eta(\theta)\, d\theta.$$

In view of the parity conditions noted earlier for ξ, η the integrals in (3.2) and (3.4) vanish identically, so those remaining provide the conditions for the first three nonzero Lyapunov coefficients.

To establish the conditions for the vanishing of the Lyapunov coefficients, each of the nonzero integrals in (3.1)–(3.5) is reduced to a single integral. Setting the coefficient of the integral to zero gives the necessary condition. In order to do this, we make repeated use of the result

$$\int (\cos x)^\alpha (\sin x)^\beta\, dx = \frac{(\cos x)^{\alpha-1}(\sin x)^{\beta+1}}{\beta+1} + \frac{\alpha-1}{\beta+1}\int (\cos x)^{\alpha-2}(\sin x)^{\beta+2}\, dx. \tag{3.6}$$

Integrating ξ from (2.3) and using (3.6), we obtain

$$\begin{aligned} \xi_1(\theta) = & \frac{-a}{n-s-2}(\cos\theta)^{s+3}(\sin\theta)^{n-s-2} \\ & + \frac{1}{n-s}\left(c - b - \frac{s+3}{n-s-2}a\right)(\cos\theta)^{s+1}(\sin\theta)^{n-s} \\ & + \left[\frac{s+1}{n-s}\left(c - b - \frac{s+3}{n-s-2}a\right) + d\right]\int (\cos\theta)^s(\sin\theta)^{n-s+1}\, d\theta. \end{aligned} \tag{3.7}$$

Evaluating this at the limits 0 and 2π and noting that the integral will be nonzero, we see that the condition for the vanishing of the first Lyapunov coefficient is coincident with the coefficient of the integral being zero. Ordinarily, the Lyapunov coefficient would include the value of the integral. However, this is of no interest in this case, so in this and the following we shall denote the coefficients of these integrals by $\overline{V}_k$ but continue to refer to them as Lyapunov coefficients. Thus

$$\overline{V}_1 = \frac{s+1}{n-s}\left(c - b - \frac{s+3}{n-s-2}a\right) + d \tag{3.8}$$

and if $\overline{V}_1 = 0$, we can express ξ_1 in the simpler form

$$\xi_1(\theta) = \frac{-a}{n-s-2}(\cos\theta)^{s+3}(\sin\theta)^{n-s-2} - \frac{d}{s+1}(\cos\theta)^{s+1}(\sin\theta)^{n-s}. \tag{3.9}$$

To compute the next nonzero coefficient, we consider the integral (3.3) having integrand $[(n-1)\xi_1(\theta) - \eta(\theta)]\xi(\theta)\eta(\theta)$. From (2.3)

$$\begin{aligned}\xi(\theta)\eta(\theta) =& (\cos\theta)^{2s+1}(\sin\theta)^{2n-2s-5}\\ &\times (A\cos^6\theta + B\cos^4\theta\sin^2\theta + C\cos^2\theta\sin^4\theta + D\sin^6\theta)\end{aligned} \tag{3.10}$$

where

$$\begin{aligned} A &= -a(a+c), & B &= ac - bc + c^2 - 2ab - ad,\\ C &= bc - b^2 + 2cd - bd + ad, & D &= (b+d)d.\end{aligned} \tag{3.11}$$

Also

$$(n-1)\xi_1(\theta) - \eta(\theta) = (\cos\theta)^{s+1}(\sin\theta)^{n-s-2}(A'\cos^2\theta + B'\sin^2\theta) \tag{3.12}$$

where

$$A' = \frac{as + cs - cn + 3a - 2na + 2c}{n-s-2}, \qquad B' = -\frac{bs + ds + b + nd}{s+1}. \tag{3.13}$$

Combining (3.10)–(3.13), we finally obtain

$$\begin{aligned}&[(n-1)\xi_1(\theta) - \eta(\theta)]\xi(\theta)\eta(\theta) = (\cos\theta)^{3s+2}(\sin\theta)^{3n-3s-7}\\ &\times (A_1\cos^8\theta + A_2\cos^6\theta\sin^2\theta + A_3\cos^4\theta\sin^4\theta + A_4\cos^2\theta\sin^6\theta + A_5\sin^8\theta)\end{aligned} \tag{3.14}$$

where

$$A_1 = AA',\ A_2 = A'B + AB',\ A_3 = A'C + BB',\ A_4 = A'D + B'C,\ A_5 = B'D. \tag{3.15}$$

Integrate (3.14) from 0 to 2π and convert all integrals to

$$I = \int_0^{2\pi} (\cos\theta)^{3s+2}(\sin\theta)^{3n-3s+1}\,d\theta$$

using (3.6). All explicitly evaluated terms vanish due to periodicity but I is nonzero, so its coefficient becomes the next Lyapunov coefficient. We have

$$\overline{V}_3 = \left(\left[\left(A_1\frac{q_1-1}{p_1+1} + A_2\right)\frac{q_1-3}{p_1+3} + A_3\right]\frac{q_1-5}{p_1+5} + A_4\right)\frac{q_1-7}{p_1+7} + A_5 \tag{3.16}$$

where $q_1 = 3s+10$ and $p_1 = 3n-3s-7$. Assuming $\overline{V}_3 = 0$, we also obtain

$$\begin{aligned}\psi(\theta) = \int \xi(\theta)\eta(\theta)[(n-1)\xi_1(\theta) - \eta(\theta)]\,d\theta = (\cos\theta)^{3s+3}(\sin\theta)^{3n-3s-6}\\ \times (B_1\cos^6\theta + B_2\cos^4\theta\sin^2\theta + B_3\cos^2\theta\sin^4\theta + B_4\sin^6\theta)\end{aligned} \tag{3.17}$$

where

$$\begin{aligned} B_1 &= \frac{A_1}{p_1+1}, & B_2 &= \frac{1}{p_1+3}\left(\frac{q_1-1}{p_1+1}A_1 + A_2\right),\\ B_3 &= -\frac{1}{q_1-5}\left(A_4 + \frac{p_1+7}{q_1-7}A_5\right), & B_4 &= -\frac{A_5}{q_1-7}.\end{aligned} \tag{3.18}$$

The calculation of the next Lyapunov coefficient is similar to that of the previous two, but it is first necessary to express the condition in a manner in which it can be fully evaluated. There appears to be no simple way to express the function Ψ, but an integration by parts removes the necessity to do so. We can replace the current integral with

$$\int_0^{2\pi} [2((n-1)\xi(\theta) - \eta'(\theta))[(n-1)\xi_1(\theta) - \eta(\theta)] - (n-1)\xi(\theta)\eta(\theta)]\psi(\theta)\, d\theta$$

where ψ is given by (3.17). All component functions have already been evaluated and it is a straightforward matter to express the nonvanishing portion of the integral in terms of a single integral. We obtain

$$\overline{V}_5 = \left(T\frac{q_2-9}{p_2+9} + G_6\right)\frac{q_2-11}{p_2+11} + G_7 \tag{3.19}$$

where

$$T = \left(\left[\left(G_1\frac{q_2-1}{p_2+1} + G_2\right)\frac{q_2-3}{p_2+3} + G_3\right]\frac{q_2-5}{p_2+5} + G_4\right)\frac{q_2-7}{p_2+7} + G_5.$$

In these $q_2 = 5s+16$ and $p_2 = 5n-5s-11$. Also

$$\begin{aligned}
&G_1 = B_1F_1, \quad G_2 = B_2F_1 + B_1F_2, && G_3 = B_3F_1 + B_2F_2 + B_1F_3,\\
&G_4 = B_4F_1 + B_3F_2 + B_2F_3 + B_1F_4, && G_5 = B_4F_2 + B_3F_3 + B_2F_4,\\
&G_6 = B_4F_3 + B_3F_4, \quad G_7 = B_4F_4
\end{aligned}$$

where the B_k's are given by (3.18) and

$$\begin{aligned}
F_1 &= 2(n-s-2)(A')^2 - (n-1)A,\\
F_2 &= 2\left(2(n-s-1)A'B' - (s+3)(A')^2\right) - (n-1)B,\\
F_3 &= -2\left(2(s+2)A'B' - (n-s)(B')^2\right) - (n-1)C,\\
F_4 &= -2(s+1)(B')^2 - (n-1)D.
\end{aligned}$$

$A, \ldots, D$ are given by (3.11) and A', B' by (3.13).

In Section 5 we calculate a form of the next coefficient $\overline{V}_7$ which is specifically related to the systems given in Theorem 2.1 and show that it does vanish for these cases. In the next Section we consider the ideal generated by $\overline{V}_1$, $\overline{V}_3$, $\overline{V}_5$ and obtain a Gröbner basis for it.

4. Determination of Possible Centre Conditions

We consider the ideal $\mathcal{V} = <\overline{V}_1, \overline{V}_3, \overline{V}_5, \ldots>$ generated by the sequence of Lyapunov coefficients arising from the system (1.2) and the ideal $\overline{\mathcal{V}} = <\overline{V}_1, \overline{V}_3, \overline{V}_5>$ generated by (3.8), (3.16), (3.19). By Hilbert's basis theorem, it is known that $\mathcal{V}$ has a finite number of generators, but it is not possible to say *ab initio* how many are required. In the following we show that the generators of $\overline{\mathcal{V}}$ also generate $\mathcal{V}$. All computations in this Section and throughout the paper were carried out using Maple 13.

A necessary condition that (1.2) has a centre at the origin is that the first three nonzero Lyapunov coefficients should vanish. The forms $\overline{V}_3$, $\overline{V}_5$ are not particularly amenable to this purpose, so we used the Groebner package of Maple to obtain a basis for $\overline{\mathcal{V}}$. This results in a surprisingly simple form for the basis elements. Denoting the ideal of generators by $\mathfrak{B} = <\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3>$, we find

$$\begin{aligned}\mathfrak{B}_1 =&(s+1)(s+3)a + (s+1)(n-s-2)b\\ &- (s+1)(n-s-2)c - (n-s)(n-s-2)d,\\ \mathfrak{B}_2 =&[(s+1)b - (n-s)d]\\ &\times [(s^2-1)b - (s+1)(3s+5)c + 4(s^2+3s-ns-2n+3)d]\\ &\times [(s+1)b - (s+1)c - (n-2s-3)d],\\ \mathfrak{B}_3 =&[(s+1)(s+3)c + (ns+5n-s^2-5s-8)d]\\ &\times [(s+1)^2c + (ns+2n-s^2-3s-3)d][(s+1)c + (n-s-2)d]\\ &\times [(s+1)b - (n-s)d][(s+1)b - (s+1)c - (n-2s-3)d].\end{aligned} \tag{4.1}$$

Potential centre conditions are now found by solving the system $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}_3 = 0$ for a, b, c, d. We obtain

$$a = \frac{n-s-2}{s+3}c, \quad b = \frac{n-s}{s+1}d; \tag{4.2}$$

$$a = \frac{n-s-2}{s+1}d, \quad b = c + \frac{n-2s-3}{s+1}d; \tag{4.3}$$

$$a = \frac{(n-s-2)^2}{ns+2n-s^2-3s-3}c, \quad b = -c, \quad d = -\frac{(s+1)^2}{ns+2n-s^2-3s-3}c; \tag{4.4}$$

$$a = \frac{n-s}{s-1}c, \quad b = -\frac{ns+3n-s^2-s-2}{(s-1)(n-s-2)}c, \quad d = -\frac{s+1}{n-s-2}c; \tag{4.5}$$

$$\begin{aligned}a &= \frac{(n-s-2)(n-s-4)}{ns+5n-s^2-5s-8}c, \quad b = -\frac{(s+1)(n-s-4)}{ns+5n-s^2-5s-8}c,\\ d &= -\frac{(s+1)(s+3)}{ns+5n-s^2-5s-8}c.\end{aligned} \tag{4.6}$$

In these c is arbitrary and d is arbitrary as well in (4.2) and (4.3). Solution (4.2) leads to Hamiltonian systems for (1.2) and (4.3) gives *constant invariant* forms, so these define centre conditions. The remaining solutions are those of interest, and of these, (4.6) is a variant of (4.5) obtained by replacing s by $n-s-3$. This produces the same system as (4.5) but rotated through 90°, so we do not include it in the statement of Theorem 2.1. If $s = (n-3)/2$, (4.4) reduces to a particular case of (4.2).

The nature of the problem being considered requires that n and s be non–negative integers, however (4.2)–(4.6) are defined for almost all values of the variables and the more general choices will still cause the expressions for the Lyapunov coefficients to vanish. That is to say, the vanishing of these coefficients produces an identity in n and s which holds for all values of the variables except those for which a, b, d fail to be defined. This observation is key to establishing the sufficiency of the conditions (4.4)–(4.6). For future reference we also include the solution of the system $\mathfrak{B}_1 = \mathfrak{B}_2 = 0$. In addition to (4.2) and (4.3) this also gives

$$
\begin{aligned}
a &= -\frac{2(n-s-2)}{s-1}c - \frac{(3n-3s-4)(n-s-2)}{s^2-1}d,\\
b &= \frac{3s+5}{s-1}c + \frac{4(ns-3s-s^2+2n-3)}{s^2-1}d.
\end{aligned}
\tag{4.7}
$$

5. Sufficiency of Conditions

In this Section we consider the sufficiency of the conditions given by (4.4)–(4.6). That is, we want to show that $\overline{V}_k$ is in $\mathfrak{B}$ for all odd values $k \geq 1$. We have already indicated that this must be true for (4.2) and (4.3) as well as for certain cases of the other solutions. In general we have

$$\overline{V}_k = \mathfrak{b}_k^{(1)}\mathfrak{B}_1 + \mathfrak{b}_k^{(2)}\mathfrak{B}_2 + \mathfrak{b}_k^{(3)}\mathfrak{B}_3 + \mathfrak{R}_k \tag{5.1}$$

where the $\mathfrak{B}_k$'s are given by (4.1) and $\mathfrak{R}_k$ is a remainder term which could include one or more previously undetermined generators of the ideal $\mathcal{V}$.

We want to show that $\mathfrak{R}_k = 0$ for each of the proposed centre conditions. In order to do this it is convenient to consider the ideal $\overline{\mathcal{V}}^* = <\mathfrak{B}_1, \mathfrak{B}_2>$. The only centres defined in $\overline{\mathcal{V}}^*$ are given by (4.2) and (4.3), so in order to obtain any one of (4.4)–(4.6) it is necessary to impose an additional condition. This can always be expressed in the form $d = \alpha(n,s)c$ and results in the vanishing of $\mathfrak{B}_3$.

The ideal $\overline{\mathcal{V}}^*$ admits a prime decomposition in terms of $\mathcal{I}_{\mathcal{A}} = <\mathcal{A}_1, \mathcal{A}_2>$, $\mathcal{I}_{\mathcal{B}} = <\mathcal{B}_1, \mathcal{B}_2>$ and $\mathcal{I}_{\mathcal{C}} = <\mathcal{C}_1, \mathcal{C}_2>$ where

$$
\begin{aligned}
&\mathcal{A}_1 = -(s+1)b + (n-s)d,\\
&\mathcal{A}_2 = (s+3)a - (n-s-2)c,\\
&\mathcal{B}_1 = -(s+1)b + (s+1)c + (n-2s-3)d,\\
&\mathcal{B}_2 = (s+1)a - (n-s-2)d,\\
&\mathcal{C}_1 = (s+1)(3s+5)a + 2(s+1)(n-s-2)b + (n-s-2)(n-s-4)d,\\
&\mathcal{C}_2 = -(s^2-1)b + (s+1)(3s+5)c + (4(n-3)s - 4s^2 + 8n - 12)d.
\end{aligned}
\tag{5.2}
$$

For general values of the parameters a, b, c, d these have the property that $\mathcal{I}_{\mathcal{A}}$ vanishes only for (4.2), $\mathcal{I}_{\mathcal{B}}$ vanishes only for (4.3) and $\mathcal{I}_{\mathcal{C}}$ vanishes only for (4.4)–(4.6). However, if we allow certain relations to be defined between the parameters c and d, there are two centre conditions which are defined in a dual fashion. One of these is defined for $\mathcal{I}_{\mathcal{A}}$, $\mathcal{I}_{\mathcal{C}}$ and the other for $\mathcal{I}_{\mathcal{B}}$, $\mathcal{I}_{\mathcal{C}}$ and they provide some useful information regarding the structure of the Lyapunov coefficients.

Since the centres defined by (4.2) and (4.3) can be obtained by solving $\mathfrak{B}_1 = \mathfrak{B}_2 = 0$, it follows that we can write (5.1) as

$$
\begin{aligned}
\overline{V}_k &= \alpha_k^{(1)}\mathcal{A}_1 + \alpha_k^{(2)}\mathcal{A}_2 = \beta_k^{(1)}\mathcal{B}_1 + \beta_k^{(2)}\mathcal{B}_2\\
&= \gamma_k^{(1)}\mathcal{C}_1 + \gamma_k^{(2)}\mathcal{C}_2 + \mathfrak{R}_k^{(\mathcal{C})}
\end{aligned}
\tag{5.3}
$$

where none of the coefficient functions $\alpha_k^{(1)}, \ldots, \gamma_k^{(2)}$ vanish for any centre condition. The reduction of $\overline{V}_k$ modulo $\mathcal{I}_{\mathcal{C}}$ results in $\mathfrak{R}_k^{(\mathcal{C})}$ being expressed in terms of c and d.

One advantage of this is that whenever any one of (4.2), (4.3) or (4.7) is substituted into $\mathfrak{R}_k^{(\mathcal{C})}$ its value remains unchanged. Also, from (4.1) and (5.2) we have

$$\begin{aligned}\mathfrak{B}_1 &= -(n-s-2)\mathcal{A}_1 + (s+1)\mathcal{A}_2 = -(n-s-2)\mathcal{B}_1 + (s+3)\mathcal{B}_2 \\ &= \frac{s+3}{3s+5}\mathcal{C}_1 - \frac{n-s-2}{3s+5}\mathcal{C}_2.\end{aligned} \tag{5.4}$$

Substituting the non-centre conditions (4.7) into $\mathcal{C}_1$ and $\mathcal{C}_2$ causes them to vanish and (5.3) becomes

$$\widetilde{\overline{V}}_k = (\widetilde{\alpha}_k^{(1)} + A\widetilde{\alpha}_k^{(2)})\widetilde{\mathcal{A}}_1 = (\widetilde{\beta}_k^{(1)} + B\widetilde{\beta}_k^{(2)})\widetilde{\mathcal{B}}_1 = \mathfrak{R}_k^{(\mathcal{C})}$$

where the tildes indicate the transformed values due to the substitution. Since $\overline{V}_k \neq 0$ for $k \geq 5$ we see that $\mathfrak{R}_k^{(\mathcal{C})}$ is also nonzero. The substitution means that $\mathfrak{B}_1 = 0$ and from (5.4) we have $\widetilde{\mathcal{A}}_2/\widetilde{\mathcal{A}}_1 = A = (n-s-2)/(s+1)$, $\widetilde{\mathcal{B}}_2/\widetilde{\mathcal{B}}_1 = B = (n-s-2)/(s+3)$ where

$$\begin{aligned}\widetilde{\mathcal{A}}_1 &= -\frac{(s+1)(3s+5)}{s-1}c - \frac{(s+3)(3n-3s-4)}{s-1}d, \\ \widetilde{\mathcal{B}}_1 &= -2\frac{(s+1)(s+3)}{s-1}c - \frac{(s+3)(3n-2s-5)}{s-1}d.\end{aligned} \tag{5.5}$$

We can make either $\widetilde{\mathcal{A}}_1$ or $\widetilde{\mathcal{B}}_1$ vanish by imposing conditions of the form $d = \beta_1 c$, $d = \beta_2 c$ respectively where

$$\beta_1 = -\frac{(s+1)(3s+5)}{(s+3)(3n-3s-4)}, \qquad \beta_2 = -2\frac{s+1}{3n-2s-5}. \tag{5.6}$$

These conditions define the dually defined centres mentioned earlier. The conditions given by (5.6) are similar to those defined in (4.4)–(4.6) where we have respective values

$$\alpha_1 = -\frac{(s+1)^2}{ns+2n-s^2-3s-3}, \quad \alpha_2 = -\frac{s+1}{n-s-2}, \quad \alpha_3 = -\frac{(s+1)(s+3)}{ns+5n-s^2-5s-8} \tag{5.7}$$

for substitutions of the form $d = \alpha c$.

The remainder $\mathfrak{R}_k^{(\mathcal{C})}$ is a homogeneous polynomial of degree k in c and d. So if we make a substitution of the form $d = \alpha c$, we can write $\mathfrak{R}_k^{(\mathcal{C})}\big|_{d=\alpha c} = \widetilde{\mathfrak{R}}_k^{(\mathcal{C})}(\alpha) = Q_k(\alpha)c^k$ where Q_k is a polynomial of degree k in α which must vanish for any centre condition arising from (4.7). In particular this must be true for the conditions defined by (5.5), (5.6). Also, Q_{k+2} is a polynomial which has a degree that is 2 greater than the degree of Q_k, so there are always two more values of α for which $\widetilde{\mathfrak{R}}_{k+2}^{(\mathcal{C})}$ vanishes but $\widetilde{\mathfrak{R}}_k^{(\mathcal{C})}$ does not. Since they cannot represent centre conditions, these additional values do not interest us. In general we have

$$Q_{5+j}(\alpha) = q_j(\alpha)(\alpha-\beta_1)(\alpha-\beta_2)(\alpha-\alpha_1^{(j)})(\alpha-\alpha_2^{(j)})(\alpha-\alpha_3^{(j)}) \tag{5.8}$$

for $j = 0, 2, 4, \ldots$ where q_j is a polynomial of degree j. We have extended the results in [1] to show that a condition for the seventh Lyapunov coefficient to be zero can be expressed (after many rearrangements and integrations by parts) as

$$\int_0^{2\pi} \left[(n-1)\lambda(\theta)\Phi(\theta) + 2\,\xi(\theta)\eta(\theta)\Lambda^5(\theta)\right]\,d\theta = 0$$

where $\Phi(\theta) = 9\,\Lambda^2(\theta)\Psi^2(\theta) + \psi^2(\theta) - (n-1)\Psi^3(\theta)$, $\Lambda(\theta) = (n-1)\xi_1(\theta) - \eta(\theta)$ and $\lambda(\theta) = \Lambda'(\theta)$. The evaluation of the function Ψ defined in (3.5) requires a general sum and although it is possible to rewrite the integral in such a manner that Ψ appears only to the first power, we have been unable to eliminate the sum entirely. When the integral is evaluated it gives the expected form $Q_7(\alpha) = (a_2\alpha^2 + a_1\alpha + a_0)(\alpha-\beta_1)(\alpha-\beta_2)(\alpha-\alpha_1)(\alpha-\alpha_2)(\alpha-\alpha_3)$ where the coefficients a_2, a_1, a_0 depend upon seven ${}_3\mathcal{F}_2$ hypergeometric functions of unit argument. We had hoped to give the explicit form for our q_2, but the expression is too large to reproduce in a reasonable fashion. A secondary issue which we do not consider here is the possibility of summing the hypergeometrics which are not *well–poised*. Since q_2 has just three coefficients, there are certainly relations (quite complicated) amongst the hypergeometrics which would reduce the number which appear. These relations do not seem to be of the form defined by the usual *contiguous* relations but perhaps can be reduced to such. One of the hypergeometrics is

$$ {}_3\mathcal{F}_2\left(\left[-s, n-s+1, \tfrac{7n}{2} - \tfrac{7s}{2}\right], \left[n-s+2, \tfrac{7n}{2} - s + \tfrac{9}{2}\right]; 1\right) $$

with the others having similar forms.

For $j = 0$, 2 and $i = 1$, 2, 3 we can take $\{\alpha_i^{(j)}\} = \{\alpha_i\}$ and we need to establish that the same relation holds for all values of j. In addition to the two known cases, we have shown by direct calculation that this is true in every case $n = 5, \ldots, 25$, $s = 0, 2, \ldots, n-3$ for $j = 4, 6, \ldots, 20$ and also for $n = 27, \ldots, 61$, $s = 0, 2, \ldots, n-3$ for $j = 4$, 6. It is difficult to carry out the symbolic calculations much beyond this point because the expressions being calculated become very large and Maple ultimately returns the error message "object too large."

The system (1.2) with a and b given by (4.7) and $d = \alpha$, $c = 1$ has (5.8) for $j = 0, 2, 4, \ldots$ as its general Lyapunov coefficient. This reduced system always has two centres corresponding to the choices $\alpha = \beta_1$, β_2 and a third possible centre for a specific choice $\alpha = \alpha_i$, $i = 1$, 2, 3. Since each of these conditions depends upon n and s, we suppose that $Q_{5+j} = Q_{5+j}(n, s)$ (clearly the Lyapunov coefficients are functions of n and s) and note that each individual term can be expressed in a similar fashion. In the following we make the obvious association of $\alpha_i^{(j)}$ with α_i for each i wherever it occurs. We are now in a position to proceed with the proof of Theorem 2.1.

Proof of Theorem 2.1. The necessity of the conditions of Theorem 2.1 are provided by equations (4.4)–(4.6) and we can now also show that these are sufficient as well.

Set $\mathcal{Q}_{i,j}(n,s) = \alpha(n,s) - \alpha_i^{(j)}(n,s)$ for each $j = 0, 2, 4, \ldots$ and $i = 1$, 2, 3 and note that $\mathcal{Q}_{i,j}(n,s) = 0$ defines an identity in n and s which is satisfied for arbitrary values of the variables. Hence it follows that if $\mathcal{Q}_{i,j}(n,s) = 0$ for any fixed values of n and s then $\mathcal{Q}_{i,j}(n, s+\sigma) = 0$ for any even integer σ such that $0 \le s+\sigma \le n-3$. (We note that all of our general results clearly exhibit this behaviour.) For $i = 1$, 2; all n; and each j we have $\alpha_i^{(j)}(n,0) = \alpha_i(n,0)$ since these systems define centres and a similar result holds for α_3 with $\alpha_3^{(j)}(n,n-3) = \alpha_3(n,n-3)$. So for $i = 1$, 2 we have $\mathcal{Q}_{i,j}(n,0) = 0$ and this gives $\mathcal{Q}_{i,j}(n,\sigma) = 0 = \alpha(n,\sigma) - \alpha_i(n,\sigma)$. The general result then follows because the remaining system (4.6) depending on α_3 is essentially the same as (4.5). In addition to the preceding, we note that we can uniquely invert the equations (5.7) defining α_1, α_2 on the region $n \ge 3$, $0 \le s \le n-3$ and express Q_{5+j} as a function of α_1, α_2. This further shows that if either of these roots ever appears in (5.8) for some j, it must appear in full form and not some modified (e.g. $s = 0$) form. □

Remark 5.1. Denote the set of general systems (4.4)–(4.6) by $\mathcal{S}_{n,s}$. Then we can see that, ultimately, the reason the $\mathcal{S}_{n,s}$ systems are centres is because the corresponding $\mathcal{S}_{3,0}$ systems are centres. For if $\mathcal{Q}_{i,j}(n,s) = 0$ for any fixed values n and s, then $\mathcal{Q}_{i,j}(n+\nu, s+\sigma) = 0$ for all values $n+\nu$ and $s+\sigma$ for which the expressions (5.7) are defined. Since $\mathcal{Q}_{i,j}$ vanishes in the $n = 3$ case, it must vanish for all other cases as well. From this point of view the $\mathcal{S}_{n,s}$ systems can be considered as a consequence (at least with respect to the vanishing of the Lyapunov coefficients) of the $\mathcal{S}_{3,0}$ systems. This relationship is only evident when we consider the trigonometric form for the Lyapunov coefficients and we might well ask if there are other systems which behave in a similar manner. In the following we give a probable extension of (2.5) which has a similar format, although we have been unable thus far to compute the necessary Lyapunov coefficients for it. There do seem to be other systems similar in form to those discussed in this paper which are valid for odd integers $n \geq 5$. Specifically, we believe that there are systems satisfying $p(x,y) = p(x,-y)$, $q(x,-y) = -q(x,y)$ and having length $(n+1)/2$ (i.e. number of terms) which originate for each odd integer n. This is certainly true for $n = 3$, 5 and it may also be true for $n = 7$, but for this latter case we have thus far obtained only a single system. This particular system, for which a specific choice of parameters gives $p(x,y) = 27\,x^7 - 347\,x^5y^2 + 161\,x^3y^4 - 33\,xy^6$, $q(x,y) = 27\,x^6y - 739\,x^4y^3 + 553\,x^2y^5 - 33\,y^7$, has several properties in common with one of our known three term systems, however its general form is more complex than any of the two or three term systems known to us. This could mean that the system does not belong to such a family or (and we believe this more likely) it could simply be the result of extending this idea to larger values of n. The first one hundred fifty Lyapunov coefficients for the given system are zero, so it probably is a centre. However, for this along with the other general systems we have mentioned, no general proof of their centre nature is yet available. We do not know of any systems of this type for even values of $n \geq 4$, primarily because of the need to calculate Lyapunov coefficients having twice the degree of a system having similar form for n odd. Also, any systems of the form (1.2) are always symmetric if n is even, regardless of the parity of s, so we would require a different form than this as a generator.

Remark 5.2. In the Introduction we mentioned the conjecture [15] that all elementary centres are integrable in some form or are rationally (algebraically) reversible. For the $\mathcal{S}_{n,s}$ systems described in Theorem 2.1 we have not found any evidence of general integrability, except for the specific cases mentioned, and neither have we been able to establish that they are rationally reversible. It would be very interesting from the point of view of solving a related Abel differential equation [12] if these systems could be shown to be integrable, but we doubt that this is true and we also do not believe they are reversible, although an exhaustive study of either of these possibilities has not been undertaken. Their centre nature derives from the fact that using a trigonometric form, they generate Lyapunov coefficients having similar functional form to that of the defining (and integrable) $\mathcal{S}_{3,0}$ systems. Based on these considerations, we believe that the conjecture is probably not true.

There is extensive computational evidence that the system (2.5) can be generalized. Our investigation using specific odd values of n with three term systems led us to the conclusion that a more complete form of the system can be written as

$$\begin{aligned} p(x,y) =& A(n-s-2)^2 x^{s+3} y^{n-s-3} + B(n-s-2) x^{s+2} y^{n-s-2} \\ & - A[(n-3)s - s^2 + 2\,n - 3] x^{s+1} y^{n-s-1}, \\ q(x,y) =& A[(n-3)s - s^2 + 2\,n - 3] x^{s+2} y^{n-s-2} \\ & + B(s+1) x^{s+1} y^{n-s-1} - A(s+1)^2 x^s y^{n-s} \end{aligned} \tag{5.9}$$

where A, B are arbitrary parameters and $n \geq 2$. Since this system was developed by considering only odd values of n, it is somewhat surprising that it is also valid for n even. If n is odd, we again have $s = 0, 2, \ldots, n-3$, but if n is even any value $s = -1, 0, \ldots, n-2$ is allowed. The basic form of the defining system would have six parameters, one of which can be removed by scaling, so to establish this form we believe we would have to show that the first five nonzero Lyapunov coefficients would vanish. Since the parity conditions found in (1.2) do not hold for a general system of the form (5.9), these would be given by the trigonometric integrals in (3.1)–(3.5) for odd values of n. At this time we have shown that the first three of these are zero, however the inclusion of the additional terms ($B \neq 0$) makes their calculation much more difficult. For example, there is no simple counterpart for ξ_1 given by (3.9). The problem for n even seems completely out of reach as it would require calculation of Lyapunov coefficients up to degree 10 (i.e. $V_{\alpha_{10}}$). On the other hand, we believe this system is valid and that its validity suggests some type of regularity between the Lyapunov coefficients for systems of odd or even degree. We state the extended result as a conjecture because we have not actually shown that the required Lyapunov coefficients are zero, although the many calculations carried out with specific values strongly suggest that this is true.

Conjecture 5.1. *The system* (5.9) *is a centre of* (1.1) *for the indicated values of n and s.*

For fixed values of A and B it is obvious that the system's Lyapunov coefficients satisfy $V_{\alpha_k} = V_{\alpha_k}(n,s)$. Since the $n = 2$ and $n = 3$ cases are centres, the same argument as used above would show that all other cases must be centres as well. Also, the $s = 0$ case is integrable for each n. An integrating factor for it using a different parametrization is given in [12]. The system (2.6) can also be extended, but in a somewhat different fashion. We will discuss this and several other related systems in another work.

6. Another General System

The simple system (1.1) with $p(x,y) = ax^n$, $q(x,y) = by^n$ was considered in [7, Proposition 5.2]. There it was suggested that a necessary (and sufficient) condition that the origin be a centre for even values of n satisfying $2 \leq n \leq 100$ is that $ab(a^2 - b^2) = 0$. In the following we show that this is both a necessary and sufficient condition for a centre for all even values of n. We felt that some of the methodology used in this paper would lead naturally into a discussion of this problem, however we were somewhat surprised at the extensive calculations required to produce the final result. In the following we assume that n is even which means that all $V_{\alpha_k} = 0$ if k is odd.

From (2.2) we have

$$\xi(\theta) = b(\sin\theta)^{n+1} - a(\cos\theta)^{n+1}, \qquad \eta(\theta) = a\sin\theta(\cos\theta)^n + b\cos\theta(\sin\theta)^n$$

and

$$\xi_1(\theta) = -\sum_{k=0}^{n/2} \frac{(-1)^k}{2k+1}\binom{n/2}{k}\left(b(\cos\theta)^{2k+1} + a(\sin\theta)^{2k+1}\right).$$

It is interesting to note the relation $\xi'(\theta) = (n+1)\eta(\theta)$ which can be used to simplify some of the calculations. Thus

$$\begin{aligned}\Psi(\theta) &= \int \xi(\theta)\eta(\theta)\,d\theta = \frac{1}{n+1}\int \xi(\theta)\xi'(\theta)\,d\theta = \frac{1}{2(n+1)}\xi^2(\theta)\\ &= \frac{1}{2(n+1)}\left(a^2(\cos\theta)^{2n+2} + b^2(\sin\theta)^{2n+2}\right) - \frac{ab}{n+1}(\sin\theta)^{n+1}(\cos\theta)^{n+1}\end{aligned}$$

from which we can see from (3.2) that $V_{\alpha_2} = 0$ regardless of the parity of n. So the first nonzero Lyapunov coefficient is given by (3.4). In order to avoid the squaring of sums, we perform an integration by parts and consider the integral

$$-2\int_0^{2\pi} [(n-1)\xi_1(\theta) - \eta(\theta)][(n-1)\xi(\theta) - \eta'(\theta)]\Psi(\theta)\,d\theta. \tag{6.1}$$

To calculate the coefficient we determine the even part of this integral. This consists of terms of the form $(\cos\theta)^\alpha(\sin\theta)^\beta$ where α, β are nonnegative integers such that β is even. In this case we have

$$\begin{aligned}I_{\alpha,\beta} = I_{\beta,\alpha} &= \int_0^{2\pi}(\cos\theta)^\alpha(\sin\theta)^\beta\,d\theta = 4\int_0^{\frac{\pi}{2}}(\cos\theta)^\alpha(\sin\theta)^\beta\,d\theta\\ &= 2\mathrm{B}\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right) = 2\frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\frac{\alpha+\beta}{2}+1\right)}.\end{aligned}$$

(This is the same method that was used to evaluate Q_7 in the previous Section.) Evaluating (6.1), we obtain the form $V_{\alpha_4} = K_n\pi ab(a^2 - b^2)$ where

$$\begin{aligned}K_n\pi = &-\frac{n(n-1)}{n+1}\sum_{k=0}^{n/2}\frac{(-1)^k}{2k+1}\binom{n/2}{k}I_{3n+2k+4,0}\\ &+\frac{n(n-1)}{n+1}\sum_{k=0}^{n/2}\frac{(-1)^k}{2k+1}\binom{n/2}{k}I_{3n+2k+2,2}\\ &-\frac{n(n-1)}{n+1}\sum_{k=0}^{n/2}\frac{(-1)^k}{2k+1}\binom{n/2}{k}\left(2I_{n+2k+4,2n} + I_{n+2k,2n+4} - 3I_{n+2k+2,2n+2}\right)\\ &-\frac{2n}{n+1}\left(I_{3n+4,n} - 2I_{3n+2,n+2} + I_{3n,n+4}\right).\end{aligned} \tag{6.2}$$

The first term (involving $I_{3n+2k+4,0}$) is the only one which does not tend to zero as $n \to \infty$. Denoting this by S_1 (including sign) and the others in order by S_2, S_3 and

S_4 where S_4 does not involve a summation, we can write $V_{\alpha_4} = (S_1 + S_2 + S_3 + S_4)ab(a^2 - b^2)$.

Each of the sums in (6.2) can be evaluated in terms of hypergeometric functions. Since S_1 is the dominant sum, we demonstrate using it. Converting the terms being summed to Gamma functions, we find that the ratio of consecutive terms is

$$\frac{a_{k+1}}{a_k} = -\frac{2k+1}{2k+3}\frac{\left(\frac{n}{2}-k\right)\left(\frac{3n}{2}+k+\frac{5}{2}\right)}{\left(\frac{3n}{2}+k+3\right)(k+1)} = \frac{\left(k+\frac{1}{2}\right)\left(k-\frac{n}{2}\right)\left(k+\frac{3n}{2}+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)\left(k+\frac{3n}{2}+3\right)(k+1)}.$$

Since this is a rational function of k, it leads to a hypergeometric function ${}_3\mathcal{F}_2$ having upper parameters $1/2$, $-n/2$, $3n/2+5/2$, lower parameters $3/2$, $3n/2+3$ and which is evaluated at 1. Normalizing so that the leading term in the sum is 1, we obtain

$$S_1 = -\frac{2n(n-1)\sqrt{\pi}}{n+1}\frac{\Gamma\left(\frac{3n}{2}+\frac{5}{2}\right)}{\Gamma\left(\frac{3n}{2}+3\right)}{}_3\mathcal{F}_2\left(\left[\tfrac{1}{2},-\tfrac{n}{2},\tfrac{3n}{2}+\tfrac{5}{2}\right],\left[\tfrac{3}{2},\tfrac{3n}{2}+3\right];1\right)$$

and the remaining sums can be evaluated in a similar fashion.

The initial values of K_n are $K_2 = -1/4$, $K_4 = -125/256$, $K_6 = -84057/131072$ and we note that they agree exactly with those obtained by direct calculation of V_{α_4}. They also have the same numerator values as those given in [7]. Using (6.2) we have shown by direct evaluation that $K_n < 0$ for $n \leq 10000$ with the sign remaining constant due to the dominance of S_1. By this point asymptotic forms of the expression are valid. After some work, we can show that

$${}_3\mathcal{F}_2\left(\left[\tfrac{1}{2},-\tfrac{n}{2},\tfrac{3n}{2}+\tfrac{5}{2}\right],\left[\tfrac{3}{2},\tfrac{3n}{2}+3\right];1\right) \sim \sqrt{\frac{\pi}{2n}} + \mathrm{O}\left(\frac{1}{n^{3/2}}\right) \quad \text{as } n \to \infty.$$

Also, from the standard asymptotic for Γ functions, $\Gamma(x+a)/\Gamma(x+b) \sim x^{a-b} + \mathrm{O}(x^{a-b-1})$ as $x \to \infty$, we have

$$\frac{\Gamma(\frac{3n}{2}+\frac{5}{2})}{\Gamma(\frac{3n}{2}+3)} \sim \sqrt{\frac{2}{3n}} + \mathrm{O}\left(\frac{1}{n^{3/2}}\right) \quad \text{as } n \to \infty.$$

Combining these results, we find that

$$S_1 \sim -\frac{2\pi}{\sqrt{3}} + \mathrm{O}\left(\frac{1}{n}\right) \quad \text{as } n \to \infty,$$

a result which is well supported numerically. The remaining S_k tend to zero with S_3, S_4 being pretty well negligible by $n = 20$. Thus $K_n \sim -2/\sqrt{3} = -1.1547005\ldots$ as $n \to \infty$. The exact value of K_n for $n = 10000$ is $-1.1543157\ldots$ which is in good agreement with both the asymptotic value and the order estimate. Hence it follows that $K_n \neq 0$ for any n.

If $a = 0$, $b \neq 0$ then ξ is odd and η is even. These conditions give *symmetric centres* since the solution $r(\theta)$ of (2.1) is an even function of θ and the phase portrait has the x–axis as a line of symmetry. If $a \neq 0$, $b = 0$ the transformation $\theta \to \theta + \pi/2$ will produce a new system which again has symmetry about the x–axis. If $b = \pm a \neq 0$, then respective transformations of the form $\theta \to \theta \pm \pi/4$ will make ξ odd and η even and these are symmetric centres as well. This leads to the following.

Proposition 6.1. *Let $n \geq 2$ be an even integer and $p(x,y) = ax^n$, $q(x,y) = by^n$ for constants a, b. If $ab(a^2 - b^2) \neq 0$, then the origin for the system* (1.1) *is a fine focus of order $N = 4n - 3$. A necessary and sufficient condition for the origin to be a centre is $ab(a^2 - b^2) = 0$.*

The system has at most one independent parameter and as we have seen, the vanishing of one nonzero Lyapunov coefficient is sufficient to ensure that all Lyapunov coefficients will be zero.

7. Final Comments

One of the most difficult tasks in proving that a proposed set of conditions yields a centre for a general system of the type (1.1) is that of establishing the sufficiency of the conditions. If the system possesses an analytic solution or an analytic, nonzero integrating factor on a neighborhood of the origin, the origin will be a centre. Other possibilities include symmetry in the system itself or in a related equation such as (2.4) or systems which are rationally or algebraically reversible. But if we cannot place the system within one of these or several other known categories, the problem remains.

In the proof of the homogeneous $n = 2, 3$ cases, the centre conditions are classified according to several irreducible components. One of these is the Hamiltonian component and another is the Sibiriskii or symmetric component. It is generally accepted (see [10]) that these two components always form irreducible parts of the centre stratum for a given n. The systems discussed here do not contain any elements of the symmetric component, but they do contain a portion of the Hamiltonian component. We observe that the systems defined by (2.5) are Hamiltonian if $s = (n-3)/2$, otherwise they are not. That is, they span more than one irreducible component of the centre stratum. We believe this leads to obvious questions about how many of the individual irreducible components are actually involved. There has been some interesting probalistic work [9] done on various systems and this correctly predicts the classification of irreducible components for the homogeneous $n = 2, 3$ cases. It would be interesting to apply this technique to (1.1) for other values of n in conjunction with the systems discussed herein to see if it is possible to obtain a better understanding of the breakdown of irreducible components for these systems.

In [8] the author obtained a relation involving the Lyapunov coefficients of a system by expressing a specific set of Lyapunov coefficients in terms of the basis of generators formed by the parameters of the system. In terms of the results in this paper, this can be put in the form

$$(\det M)\,\overline{V}_k = \alpha_k \mathfrak{B}_1 + \beta_k \mathfrak{B}_2 + \gamma_k \mathfrak{B}_3$$

where $\det M$ is the determinant of a particular matrix and α_k, β_k, γ_k are homogeneous polynomials of degree $k + 6$, $k + 4$, $k + 2$ respectively. For the systems considered in this paper we have $\det M = \lambda \mathfrak{B}_3$ where λ is a homogeneous polynomial of degree 2 which does not vanish at any centre condition. An expression of this form is always possible and the author poses the question as to whether or not it can be used to obtain a desired result for the Lyapunov coefficients, namely expressing them in terms of a particular set of generators. The answer, with regards to the results in this paper, is that it cannot be used in such a manner. Its application leads, after some time, to a rather trivial identity. The main problem is that $\det M$ is in $\mathfrak{B}$ (a general condition of this type is always true) and regardless of what polynomial in a, b, c, d it multiplies, $(\det M)\,\overline{V}_k$ is always in $\mathfrak{B}$. So, in order to analyze the expression more fully, we need to know something of the behaviour of the coefficient functions α_k, β_k, γ_k which is not possible for general values of k. In particular, we would have to show that each of these coefficients is in $\mathfrak{B}$ as well.

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