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NEW IDENTIFICATION AND CONTROL METHODS OF SINE-FUNCTION JULIA SETS*

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Abstract In this paper, we propose two new methods to realize drive-response system synchronization control and parameter identification for two kinds of sine-function Julia sets. By means of these two methods, the zero asymptotic sliding variables and the stability theory in difference equations are applied to control the fractal identification. Furthermore, the problem of synchronization control is solved in the case of a drive system with unknown parameters, where the unknown parameters of the drive system can be identified in the asymptotic synchronization process. The results of simulation examples demonstrate the effectiveness of the new methods.

Keywords Julia set, synchronization, parameter identification.

MSC(2000) 34H05, 93C28.

1. Introduction

The fractals theory which describes fractal properties and corresponding applications was proposed by Mandelbort [9]. As a forefront nonlinear science theory, the fractals has successfully explained a lot of nonlinear phenomena (Mandelbrot [10] and Liu et al. [7]). At present, this theory has a variety of applications in the meteorology, cancer cell growth, image processing, geography and so on (Wang et al. [12], Wang et al. [13], Yang et al. [21], Yu and Chen [22], Wang and He [14], Ding and Jiang [2]). In recent years, researchers have paid much attention to fractal sets which come from an analytic mapping iteration on the complex plane. The analytic mapping iteration divides the complex plane into two parts: Fatou set and Julia set. It has been found that the generalized Mandelbrot-Julia (M-J) sets which enjoy an important role in the fractals with a fine and complex structure. In 2004 and 2007, Wang et al. [15, 16] studied the application of M-J sets in physics. That is a typical Langevin problem, i.e. the analysis about the dynamics of a charged particle which is under the continuous influence of a constant impulse at one-dimensional discrete-time points in a double-well potential and a time-dependent magnetic field. Nowadays, the M-J fractal system research is mainly based on a complex mapping as a representative of the polynomial function $f(z, c) = z^{\alpha} + c, c \in C$, but for a

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non-polynomial function it is rarely studied (Fu et al. [3]). Moreover, many mathematicians are interested in properties and dimensions of the polynomial-function Julia set in theoretical studies (Wu and Chen [17], Wang and Shi [18], Gao [5], Wang and Sun [19], Ashish et al. [1], Huang and Wang [6]). Importantly, people often have actual requirements for the non-linear domain-range and parameters of the fractal collection. For example, some significant achievements about the fractal synchronous control have been reported (Zhang et al. [23], Liu and Liu [8], Wang and Liu [20]). It is pointed out that these methods are feasible only in the case of knowing the drive system parameters. However, the drive system parameters are usually unknown, so synchronization control of the drive-response system is difficult to be solved by the existing methods. In this paper, for two kinds of sine-function Julia sets, we propose two new methods to realize drive-response system synchronization control and parameter identification. By applying the zero-asymptotical sliding variable discrete control method and the stability theory in difference equations to sine-function Julia sets, synchronous control of the drive-response system and parameter identification of the drive system are achieved simultaneously under the conditions that the drive system has unknown parameters.

2. Design of synchronization controller and parameter identifier for sine-function Julia sets

2.1. Preliminaries

A Julia set $J(f)$ is created by the iteration of a complex variable function f, which is defined to be the closure of the repelling periodic points of f. For the complex polynomial f, its Julia set has the following properties.

(i) $J(f)$ is nonempty and bounded;

(ii) $J(f)$ is fully invariant, i.e., $J(f) = f(J(f)) = f^{-1}(J(f))$;

(iii) $J(f)$=$J(f^p)$, for any positive integer p;

(iv) If ω is an attractive fixed point of f, then $\partial A(\omega) = J(f)$, where $A(\omega)$ is the attractive domain of the attractive fixed point ω. That is, $A(\omega) = \{z \in C : f^k(z) \to \omega, (k \to \infty)\}$, i.e., $\omega \to \infty$.

From the definition of Julia set and its properties, we can control the trajectory of iterative points and achieve the objective of controlling Julia set. Now, we consider two complex systems with the same structure:

$$x_{n+1} = f(x_n, a_i, c), \tag{2.1}$$

$$y_{n+1} = f(y_n, a_i, c_n). \tag{2.2}$$

We design a controller for system (2.2) in order to correlate with system (2.1), that is,

$$y_{n+1} = f(y_n, a_i, c_n) + u_n(x_n, y_n, a_i), \tag{2.3}$$

where a_i $(i = 1, 2, \dots)$ are given complex numbers, and c is an unknown complex parameter which needs to be identified. Then, system (2.1) is a drive system and system (2.3) is a response system. The controller u_n of system (2.3) control drive system (2.1) and response system (2.3) to achieve asymptotic synchronization. If the parameters a_i, c, c_n are given, their corresponding Julia sets are also identified, which are written as J^*, J_n respectively. If

$$\lim_{n\to\infty}(J_n \cup J^* - J_n \cap J^*) = \emptyset \tag{2.4}$$

then systems (2.1) and (2.3) can achieve synchronization.

In fact, we know that the orbits of Julia set $J(f)$ and f are closely related, according to the definition of Julia set. Therefore, we design the following controller based on Julia sets synchronization as soon as their orbits synchronized. When $n \to \infty$, systems (2.1) and (2.3) gradually synchronize with the same initial iterative value, in the meantime, the unknown parameter c of system (2.3) is identified, that is, $c_n \to c$.

2.2. Design of synchronous controller and parameter identifier for sine-function Julia sets

For a class of sine-function Julia sets,

$$x_{n+1} = c\sin(x_n), \tag{2.5}$$

where c is a complex constant, we discuss the synchronization problem and identification of parameter c.

We can obtain the Julia set by iterative calculation of the points in a bounded region using the above property (i). Therefore, suppose, in the bounded region D, we only consider the iteration of the mid-point of D. Using the above property (ii), we only need to calculate the points, whose trajectories are all in the D, because if there is an n_0 to make $f^{n_0}(z) \notin D$, using the property (ii), we know $z \notin J$.

System (2.5) is the drive system, and the other system with the same structure of sine-function Julia sets is the response system:

$$y_{n+1} = c_n \sin(y_n) + u_n, \tag{2.6}$$

where u_n is the synchronous controller to be designed.

We assume that the drive system and the response system are uncertainty fractal sets, i.e., the parameter c is unknown. Then, we design an appropriate synchronous controller for the drive system in the case of the response system with unknown parameters. Thus, we will realize system (2.5) and system (2.6) completely fully synchronizing and c is identified at the same time.

We take the error variables between the drive system and the response system as follows:

$$e_1(n) = x_n - y_n, \tag{2.7}$$

$$e_2(n) = c_n - c. \tag{2.8}$$

Next, we design the synchronous controller

$$u_n = x_{n+1} - c_n \sin(y_n) + (k_1 - 1)e_1(n), \tag{2.9}$$

where the constant k_1 satisfies $0 < k_1 < 2$. It should be remarked that the controller u_n is not a truly real-time controller, which needs to be implemented with one-step time delay, namely, the controller needs to be kept for one step by a zero-order-holder and then takes the actual control action.

We design the parameter identification law

$$c_{n+1} = (1-k_2)c_n + k_2 x_{n+1}/\sin(x_n), \tag{2.10}$$

where the constant k_2 satisfies $0 < k_2 < 2$.

Lemma 2.1 (Furuta [4]). *If $\sigma(n)$ is a sliding variable of a single-input and single-output discrete variable structure control system, then the discrete control variable satisfies*

$$\sigma(n)\Delta\sigma(n) < -\frac{1}{2}[\Delta\sigma(n)]^2, \quad \sigma(n) \neq 0, \tag{2.11}$$

where $\Delta\sigma(n) = \sigma(n+1) - \sigma(n)$ and $|\sigma(n+1)| < |\sigma(n)|$, in which $\sigma(n)$ tends to zero when $n \to \infty$.

Theorem 2.1. *If the synchronous controller of system (2.6) is described by (2.9) and its parameter identifier is described by (2.10), then drive system (2.5) and response system (2.6) with arbitrary initial value can achieve global asymptotic synchronization, and the parameter of sine-function Julia sets can be identified.*

Proof. From the previous discussion, if systems (2.5) and (2.6) realize track synchronization, then the Julia sets of systems (2.5) and (2.6) realize global asymptotic synchronization with arbitrary initial value. We only prove that systems (2.5) and (2.6) realize track synchronization, that is, $|x_n - y_n| \to 0$ $(n \to \infty)$. This is equivalent to that $|e_1(n)| \to 0$ $(n \to \infty)$.

From (2.5)-(2.7), we have

$$e_1(n+1) - e_1(n) = c\sin(x_n) - c_n\sin(y_n) - u_n - x_n + y_n. \tag{2.12}$$

Substituting (2.9) into (2.12), we obtain

$$\begin{aligned} & e_1(n+1) - e_1(n) \\ =& c\sin(x_n) - c_n\sin(y_n) - x_n + y_n - (k_1-1)e_1(n) + x_{n+1} - c_n\sin(y_n) \\ =& -k_1 e_1(n). \end{aligned} \tag{2.13}$$

According to the sliding variable property, we take $e_1(n)$ as the sliding variable. If $0 < k_1 < 2$ is satisfied, then it follows from (2.13) that

$$\begin{aligned} e_1(n)[e_1(n+1) - e_1(n)] &= -k_1[e_1(n)]^2 \\ &< -\frac{1}{2}k_1^2[e_1(n)]^2 = -\frac{1}{2}[e_1(n+1) - e_1(n)]^2. \end{aligned} \tag{2.14}$$

That is,

$$e_1(n)[e_1(n+1) - e_1(n)] < -\frac{1}{2}[e_1(n+1) - e_1(n)]^2. \tag{2.15}$$

Hence, we have $|e_1(n+1)| < |e_1(n)|$ according to Lemma 2.1, that is, $|e_1(n)| \to 0$ as $n \to \infty$. So, Julia sets of systems (2.5) and (2.6) have achieved global asymptotic synchronization with arbitrary initial value.

Similarly, according to the sliding variable property and (2.10), if $e_2(n)$ is the sliding variable and $0 < k_2 < 2$, we get

$$\begin{aligned} c_{n+1} - c_n &= c_{n+1} - c - (c_n - c) \\ &= e_2(n+1) - e_2(n) \\ &= -k_2 e_2(n). \end{aligned} \tag{2.16}$$

If $0 < k_2 < 2$ is satisfied, then it follows from (2.16) that

$$e_2(n)[e_2(n+1) - e_2(n)] = -k_2[e_2(n)]^2 < -\frac{1}{2}k_2^2[e_2(n)]^2 = -\frac{1}{2}[e_2(n+1) - e_2(n)]^2. \quad (2.17)$$

That is,

$$e_2(n)[e_2(n+1) - e_2(n)] < -\frac{1}{2}[e_2(n+1) - e_2(n)]^2. \quad (2.18)$$

According to the Lemma 2.1, we obtain $|e_2(n+1)| < |e_2(n)|$, i.e., $|e_2(n)| \to 0$. It ensures that c_n asymptotically tends to the actual value of c as $n \to \infty$. □

Example 2.1. Let $c = i$, $c_0 = 0.6$ for drive system (2.5) and response system (2.6). After n steps, the Julia sets of drive system (2.5) and response system (2.6) are shown in Figure 1, where $k_1 = 1.8, k_2 = 1.5$.

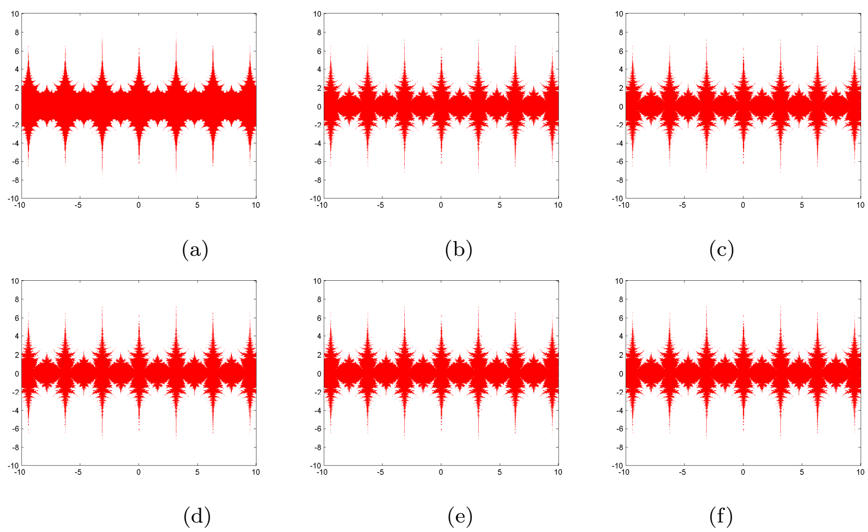


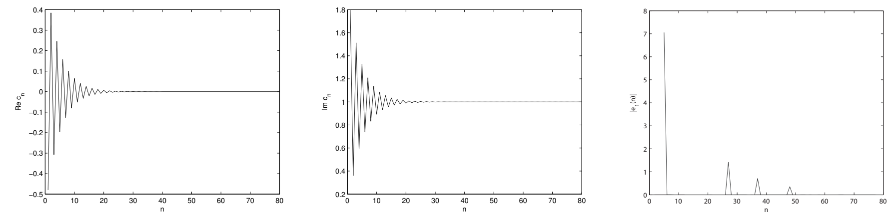
Figure 1. The Julia sets of response system (2.6) after (a) 5 steps, (b) 20 steps, (c) 50 steps, (e) 80 steps, and the Julia sets of drive system (2.5) after (d) 50 steps, (f) 80 steps, where $k_1 = 1.8$, $k_2 = 1.5$.

The figures show that response system (2.6) synchronizes drive system (2.5) after 50 steps. Moreover, response system (2.6) keeps synchronous with drive system (2.5) after 80 steps. Besides, the identification process of c_n in response system (2.6) and $|e_1(n)|$ changing with n are shown in Figure 2.

From the simulation results, in the iterative process the real and imaginary parts of c_n are stable at 0 and 1 and the synchronization error $|e_1(n)|$ tends to zero. Hence, response system (2.6) and drive system (2.5) achieve synchronization, where the unknown c of drive system (2.5) can be identified.

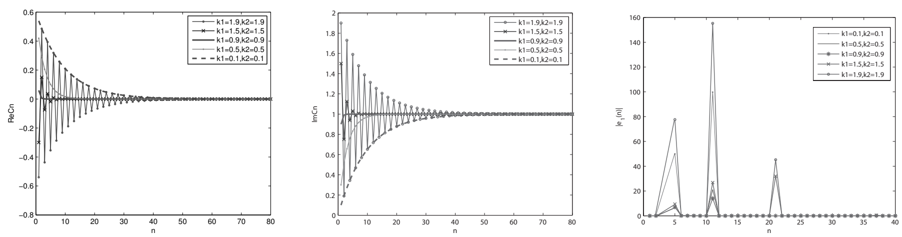
As the values of k_1 and k_2 increase from 0 to 2, the changing processes of c_n and $|e_1(n)|$ are shown in Figure 3.

From the results, we know that as the values of k_1 and k_2 increase from 0 to 1, the identification process of c_n is getting better and $|e_1(n)|$ tends to 0 more quickly. When the values of k_1 and k_2 continue to increase from 1 to 2, the identification process of c_n is getting worse, and $|e_1(n)|$ tends to 0 slower. So, when the values



(a) (b) (c)

Figure 2. The (a) real, (b) imaginary parts of c_n of system (2.6) and (c) $|e_1(n)|$ between systems (2.5) and (2.6) changing with n, where $k_1 = 1.8$ and $k_2 = 1.5$.

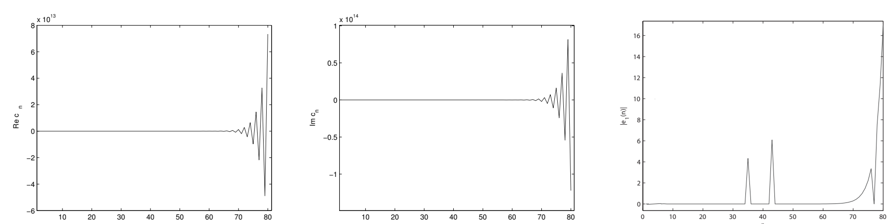


(a) (b) (c)

Figure 3. The (a) real, (b) imaginary parts of c_n and (c) $|e_1(n)|$ change with n for different k_1 and k_2.

of k_1 and k_2 are near 1, the identification effect of c_n is the best, and the speed of $|e_1(n)|$ tending to 0 gets the fast. That is to say, the speed of the response system and the drive systems achieving synchronization is the fastest.

However, if $k_1 = 2.5, k_2 = 3$, the identification process of c_n of response system (2.6) and the changing process of $|e_1(n)|$ are shown in Figure 4.



(a) (b) (c)

Figure 4. The (a) real, (b) imaginary parts of c_n of system (2.6) and $|e_1(n)|$ changing with n, where $k_1 = 2.5$ and $k_2 = 3$.

From Figure 4, we know that $|e_1(n)|$ between the two synchronous systems and the c_n of system (2.6) tend to infinity gradually. Therefore, the synchronization of the two systems can not be achieved and the unknown c of drive system (2.5) can not be identified.

2.3. Design of synchronous controller and parameter identifier for another sine-function Julia set

In this section, we discuss the problem of synchronous controller and parameter identifier for another sine-function Julia set:

$$z_{n+1} = \sin(z_n^2) + c.$$

In this case, the drive system and the response system can be written as

$$x_{n+1} = \sin(x_n^2) + c, \tag{2.19}$$

$$y_{n+1} = \sin(y_n^2) + c_n + u_n. \tag{2.20}$$

The error between drive system (2.19) and response system (2.20) is

$$e_n = x_n - y_n. \tag{2.21}$$

In order to design an adaptive synchronization controller for sine-function Julia sets, the following lemma [11] is firstly introduced.

Lemma 2.2. *Consider a liner equation*

$$x_{n+2} - x_{n+1} + kx_n = 0,$$

where x_n is complex. If the real k satisfies $0 < k < 1$, then the equation root is stable as $n \to \infty$, that is, $x_n \to 0$.

Theorem 2.2. *If the synchronous controller of system (2.20) is designed by*

$$u_n = \sin(x_n^2) - \sin(y_n^2)\ , \tag{2.22}$$

and if the parameter identifier is

$$c_{n+1} - c_n = k_3 e_n, \tag{2.23}$$

where the constant k_3 satisfies $0 < k_3 < 1$, then drive system (2.19) and response system (2.20) with arbitrary initial value can achieve global asymptotic synchronization, and the parameters of the sine-function Julia set can be identified.

Proof. If systems (2.19) and (2.20) realize track synchronization with arbitrary initial value, then the Julia sets of systems (2.19) and (2.20) realize global asymptotic synchronization. We only prove that systems (2.19) and (2.20) realize track synchronization, that is, $|x_n - y_n| \to 0 (n \to \infty)$. This is equivalent to that $|e_n| \to 0$ $(n \to \infty)$.

From (2.19)-(2.21), we obtain

$$\begin{aligned} e_{n+1} &= x_{n+1} - y_{n+1} \\ &= \sin(x_n^2) + c - \sin(y_n^2) - c_n - u_n. \end{aligned} \tag{2.24}$$

Next, substituting (2.22) into (2.24), we have

$$e_{n+1} = \sin(x_n^2) + c - \sin(y_n^2) - c_n - \sin(x_n^2) + \sin(y_n^2).$$

That is,

$$e_{n+1} = c - c_n. \tag{2.25}$$

Hence,

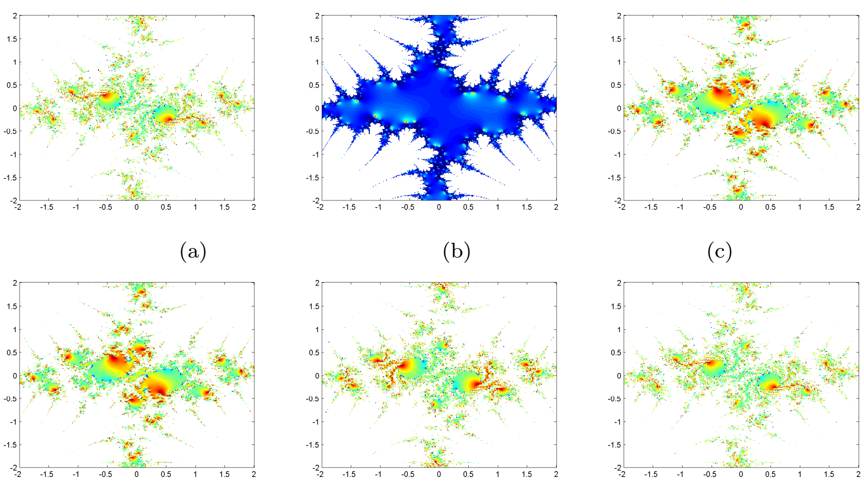
$$\begin{aligned} e_{n+1} - e_n &= c - c_n - (c - c_{n-1}) \\ &= c_{n-1} - c_n \\ &= -k_3 e_{n-1}. \end{aligned} \tag{2.26}$$

That is,

$$e_{n+1} - e_n + k_3 e_{n-1} = 0. \tag{2.27}$$

If k_3 satisfies $0 < k_3 < 1$, then the equation root is stable as $n \to \infty$ according to Lemma 2.2, that is, $e_n \to 0$. So, the Julia sets of systems (2.19) and (2.20) achieve global asymptotic synchronization with arbitrary initial value. □

Example 2.2. Let $c = -0.7 + 0.3i$, $c_0 = 1$, for drive system (2.19) and response system (2.20). After n steps, the Julia set of drive system (2.19) and response system (2.20) are shown in Figure 5, where $k_3 = 0.1$.



(a) (b) (c)

(d) (e) (f)

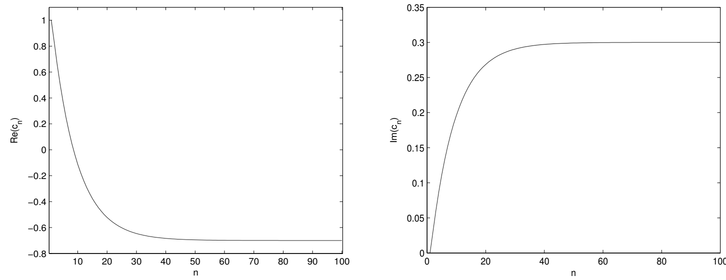
Figure 5. The Julia sets of drive system (2.19) after (a) 100 steps; and the Julia sets of response system (2.20) after (b) 5 steps, (c) 20 steps, (d) 50 steps, (e) 80 steps, (f) 100 steps; where $k_3 = 0.1$.

The figures show that response system (2.20) synchronizes drive system (2.19) after 100 steps. Besides, the identification process of c_n of response system (2.20) is shown in Figure 6. The changing process of the e_n between drive system (2.19) and response system (2.20) is shown in Figure 7.

From the simulation results, the real and imaginary parts of the c_n are stable at -0.7 and 0.3 in the iterative process and the real and imaginary parts of error e_n tend to zero. Hence, response system (2.20) and drive system (2.19) achieve synchronization, where the unknown c of drive system (2.19) has also been identified.

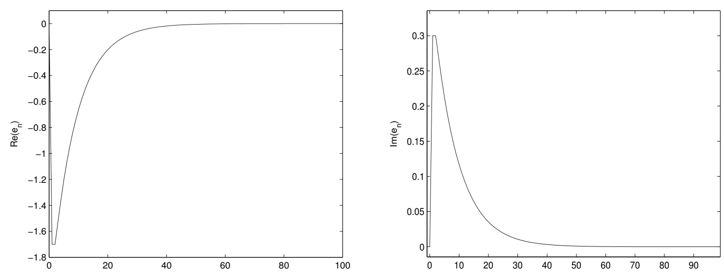
As the value of k_3 increase from 0 to 1, the changing process of c_n and e_n are shown in Figure 8 and Figure 9.

From the simulation results, we know that as the value of k_3 increase from 0 to 0.5, the identification process of c_n is getting better and e_n tends to 0 more quickly. When the value of k_3 continues to increase from 0.5 to 1, the identification process



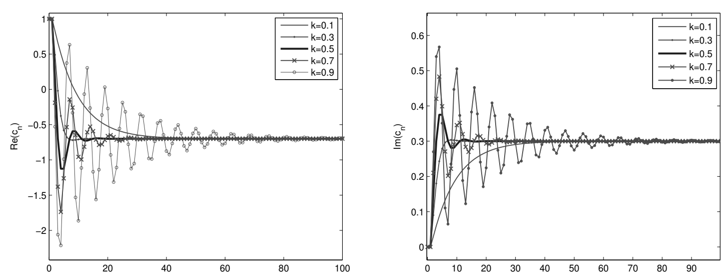
(a) (b)

Figure 6. The (a) real and (b) imaginary parts of c_n of system (2.20) changing with n, where $k_3 = 0.1$.



(a) (b)

Figure 7. The (a) real and (b) imaginary parts of e_n between systems (2.19) and (2.20) changing with n, where $k_3 = 0.1$.



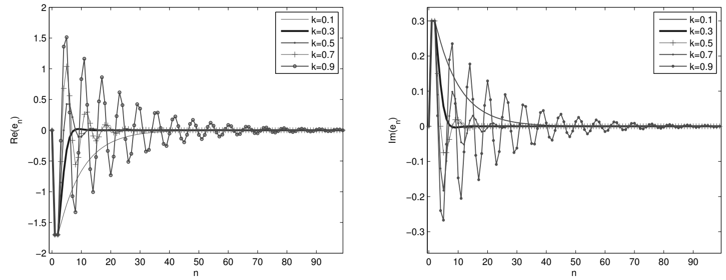
(a) (b)

Figure 8. Identification processes of (a) real and (b) imaginary parts of c_n for different k_3.

of c_n is getting worse, and e_n tends to 0 slower. When the value of k_3 is near 0.3 the identification effect of c_n is the best, and the speed of e_n tending to 0 gets the fastest. Thus, the speed of the response system and the drive system achieving synchronization is the fastest.

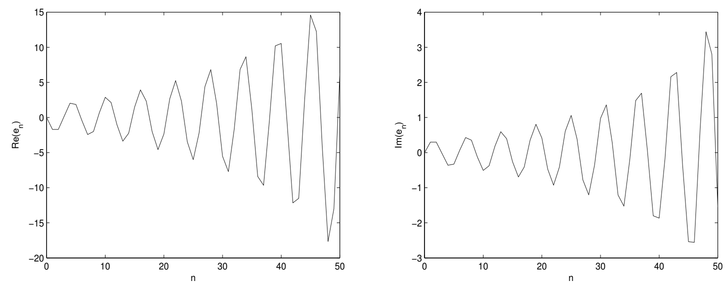
However, if $k_3 = 1.1$, the identification process of e_n between drive system (2.19) and response system (2.20) is shown in Figure 10.

From Figure 10, we know that e_n between the two synchronous systems tends to infinity gradually. Therefore, synchronization of the two systems can not be



(a) (b)

Figure 9. The (a) real and (b) imaginary parts of e_n between systems (2.19) and (2.20) changing with n for different k_3.



(a) (b)

Figure 10. The (a) real and (b) imaginary parts of e_n between systems (2.19) and (2.20) changing with n, where $k_3 = 1.1$.

achieved and the unknown c of drive system (2.19) can not be identified.

3. Conclusions

In this work, two new methods are put forward to realize the synchronization control of a drive-response system and parameter identification of sine-function Julia sets. The zero asymptotic property of the sliding variable of the discrete control system and the stability theory in difference equations are applied to realize identification and control of sine function Julia sets. Then, we successfully solved the problem of synchronization control of the drive-response system and parameter identification, in the case of the drive system having unknown parameters. Meanwhile, we designed an adaptive synchronization controller and parameter identifier. These results are significant to future important applications of Julia sets.

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