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TOPOLOGICAL HORSESHOE IN A FRACTIONAL-ORDER QI FOUR-WING CHAOTIC SYSTEM*

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Abstract A fractional-order Qi four-wing chaotic system is present based on the Grünwald-Letnikov definition. The existence of topological horseshoe in a fractional chaotic system is analyzed by utilizing topological horseshoe theory. A Poincaré section is properly chosen to obtain the Poincaré map which is proved to be semi-conjugate to a 2-shift map, implying that the fractional-order Qi four-wing chaotic system exhibits chaos.

Keywords Fractional-order system, topological horseshoe, Poincaré map, topologcal entropy, Qi four-wing chaotic system.

MSC(2000) 37B40, 65P20.

1. Introduction

Since the chaotic Lorenz system was discovered in 1963 [9], chaotic systems have been a focal subject of renewed interest in the past few decades. Many chaotic systems have been proposed such as Chua circuits [3], Chen system [4], Lü system [10] and Qi system [15, 16]. Fractional calculus has been known since the early seventeenth century [2, 5, 13]. Although it has a long history, applications of fractional calculus to physics and engineering are just a recent focus of interest. At present, the number of applications of fractional calculus rapidly grows. These mathematical phenomena allow us to describe and model a real object more accurately than the classical integer methods. Recently, by utilizing fractional calculus techniques, many investigations have been devoted to the chaotic behaviours and chaos control of dynamical systems involved fractional derivatives, called fractional-order chaotic systems [1, 11, 12].

In the mathematics of chaos theory, a horseshoe map is a member of a class of chaotic maps of the square into itself. It is a core example in the study of dynamical systems. The map was introduced by Smale while studying the behaviour of the orbits of the van der Pol oscillator [17]. The action of the map is defined geometrically by squishing the square, then stretching the result into a long strip,

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and finally folding the strip into the shape of a horseshoe. Historically, the most basic results about horseshoe theory are perhaps the Smale horseshoe first studied by Smale [18]. After then, great efforts have been made to find sufficient conditions for the existence of a horseshoe. The topological horseshoe is well recognized as one of the most rigorous approaches to study chaos with computer. Many chaotic systems have been proved to contain a horseshoe [6, 7, 18, 20–23].

In this paper, a fractional-order Qi system is studied. It comes from the Qi four-wing chaotic system proposed by Qi *et al.* [16]. Based on the Grünwald-Letnikov definition, the numerical solution of fractional-order Qi four-wing chaotic system is presented. Some phase portraits of the system is given to verify the chaotic dynamics of the fractional-order Qi system. Based on the topological horseshoe theory, the existence of horseshoe in this system is proved with a computer assisted method. A suitable Poincaré section is selected to get the corresponding Poincaré map, which is semi-conjugate to a 2-shift map, proving that the system is chaotic from the fact that it has the topological horseshoe.

2. The fractional Qi system description

Qi *et al.* proposed a new three-dimensional continuous quadratic autonomous chaotic system [16], which is different from the Lorenz system family. The system is described as follows:

$$\begin{aligned} \dot{x} &= a(y-x)+eyz, \\ \dot{y} &= cx+dy-xz, \\ \dot{z} &= -bz+xy. \end{aligned} \tag{2.1}$$

Here, $a,\ b,\ d,\ e \in R^+$ and $c \in R$ are constant parameters of the system. In Ref. [16], it has been reported that the model is simple and produces a four-wing attractor for $a = 14, b = 43, c = -1, d = 16$ and $e = 4$.

For getting the fractional-order Qi four-wing chaotic system, the fractional definition must be given. The fractional order derivatives have many definitions; one of them is the Grünwald-Letnikov definition [13] which is given by

$$D_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(t-jh), \tag{2.2}$$

where

$$\binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}, \tag{2.3}$$

$f(t)$ is a continuous function, α is α-th derivative $0 < \alpha < 1$, $\Gamma(n)$ is Euler's Gamma function: $\Gamma(n) = (n-1)!$.

Then the corresponding fractional-order Qi system can be written in the form

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= a(y-x)+eyz, \\ \frac{d^\alpha y}{dt^\alpha} &= cx+dy-xz, \\ \frac{d^\alpha x}{dt^\alpha} &= -bz+xy. \end{aligned} \tag{2.4}$$

The relation to the explicit numerical approximation of α-th derivative at the points $kh, (k = 1, 2, \cdots)$ has the following form [14].

$$ {}_{(k-L_m/h)}D_t^\alpha f(t) \approx h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(t_{k-j}), \tag{2.5} $$

where L_m is is the "memory length", $t_k = kh$, h is time step of calculation and $(-1)^j \binom{\alpha}{j}$ are binomial coefficients $c_j^{(\alpha)} (j = 0, 1, \cdots)$. It is the following expression [14].

$$ c_0^{(\alpha)} = 1, c_j^{(\alpha)} = (1 + \frac{1+\alpha}{j}) c_{j-1}^{(\alpha)}. \tag{2.6} $$

Then, general numerical solution of the fractional differential equation

$$ {}_aD_t^\alpha x(t) = f(x(t), t) $$

can be expressed as

$$ x(t_k) = f(x(t_k), t_k) h^\alpha - \sum_{j=v}^{k} c_j^{(\alpha)} x(t_{k-j}). \tag{2.7} $$

For simulation purposes, a numerical solution of fractional-order Qi system (2.4) is obtained by using the relationship (2.7) derived from the Grünwald-Letnikov definition which leads to equations in the form (2.8):

$$ \begin{aligned} x(t_k) &= f_x h^{\alpha_1} - \sum_{j=v}^{k} c_j^{(\alpha_1)} x(t_{k-j}), \\ y(t_k) &= f_y h^{\alpha_2} - \sum_{j=v}^{k} c_j^{(\alpha_2)} y(t_{k-j}), \\ z(t_k) &= f_z h^{\alpha_3} - \sum_{j=v}^{k} c_j^{(\alpha_3)} z(t_{k-j}), \end{aligned} \tag{2.8} $$

where

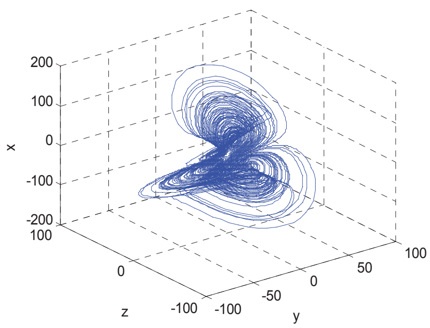
$$ \begin{aligned} f_x &= a[y(t_{k-1}) - x(t_{k-1})] + e y(t_{k-1}) z(t_{k-1}), \\ f_y &= c x(t_k) + d y(t_{k-1}) - x(t_k) z(t_{k-1}), \\ f_z &= -b z(t_{k-1}) + x(t_k) y(t_k). \end{aligned} $$

When $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 0.99$ and $(a, b, c, d, e) = (14, 43, -1.1, 16, 4)$, system (2.4) has the four-wing chaotic attractor shown in Fig.1.

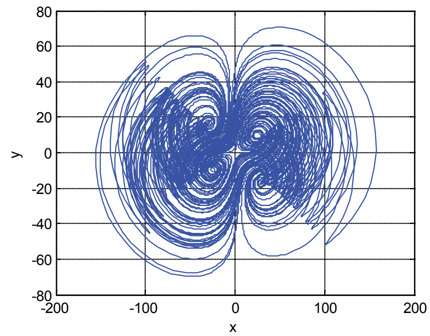
3. Smale horseshoe theorem

3.1. Symbolic Dynamics

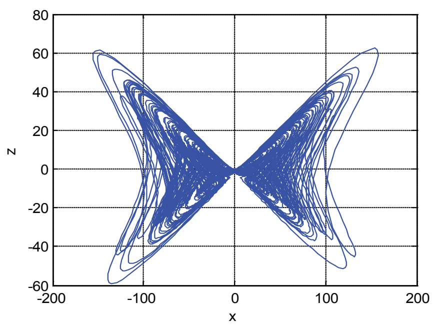
In order to making the Smale horseshoe be understood easily, Some aspects of symbolic dynamics are recalled [19].



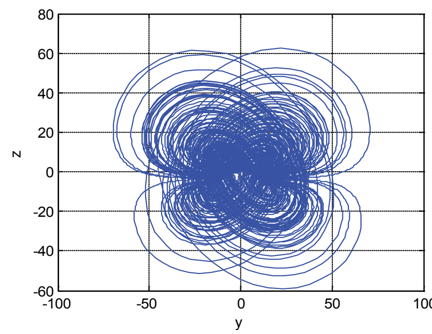
(a) 3D view on the $y-z-x$ plane.



(b) Projection on the $x-y$ plane.



(c) Projection on the $x-z$ plane.



(d) Projection on the $y-z$ plane.

Figure 1. Chaotic attractor of the fractional order Qi system with $\alpha = 0.99$ and $a = 14, b = 43, c = -1.1, d = 16, e = 4$.

Let $S = \{0, 1, \cdots, N\}, N \geq 2$ be the set of non-negative successive integer. Let Σ be the collection of all bi-infinite sequences with their elements of S, *i.e.* every element s of Σ implies

$$s = \{\ldots, s_{-n}, \ldots, s_{-1}, s_0, s_1, \ldots, s_n, \ldots\}, \; s_i \in S.$$

Now consider another sequence $\overline{s} \in \Sigma$

$$\overline{s} = \{\ldots, \overline{s}_{-n}, \ldots, \overline{s}_{-1}, \overline{s}_0, \overline{s}_1, \ldots, \overline{s}_n, \ldots\}, \; \overline{s}_i \in S.$$

The distance between s and $\overline{s}$ defined as

$$d(s, \overline{s}) = \sum_{-\infty}^{\infty} \frac{1}{2^i} \frac{|s_i - \overline{s}_i|}{1 + |s_i - \overline{s}_i|}. \tag{3.1}$$

With the distance defined as (3.1), Σ is a metric space, and there is a well known theorem [8, 19, 24].

Theorem 3.1. *the space Σ is*
i) compact; ii) totally disconnected; iii) perfect.

A set having the three properties in the above proposition is often defined as a Cantor set, such a Cantor set frequently appears in characterization of complex structure of invariant set in a chaotic dynamical system [8, 19, 24].

3.2. m-shift map

Now define a m-shift map $\sigma : \Sigma \to \Sigma$ follows:

$$\sigma(s_i) = s_{i+1}. \tag{3.2}$$

Then there are the following results:

(a) $\sigma(\Sigma) = \Sigma$ and σ is continuous,

(b) the shift map σ as a dynamical system defined on Σ has the following properties:

i) σ has a countable infinity of periodic orbits consisting of orbits of all periods,

ii) σ has an uncountable infinity of nonperiodic orbits and,

iii) σ has a dense orbit.

A consequence of statement (b) is that the dynamics generated by the shift map σ display sensitive dependence on initial conditions on a closed invariant set, and thus are chaotic(see ref. [19] for a proof of the above statements).

3.3. The horseshoe theorem

Let X be a metric space, D is a compact subset of X, $f : D \to X$ is a map satisfying the assumption that there exist m mutually disjoint subsets $D_1, D_2, \ldots, D_m$ of D. the restriction of f to each $D_i, i.e., f|D_i$ is continuous [19].

Definition 3.1 ([8, 19, 24]). Let γ be a compact subset of D, such that for each $1 \le i \le m, \gamma_i = \gamma \cap D_i$ is nonempty and compact, then γ is called a connection with respect to $D_1, D_2, \ldots, D_m$. Let F be a family of connections γs with respect to $D_1, D_2, \ldots, D_m$ satisfying the following property:

$$\gamma \in F \Longrightarrow f(\gamma_i) \in F,$$

then F is said to be a f-connected family with respect to $D_1, D_2, \ldots, D_m$.

Theorem 3.2 ([8, 19]). *Suppose that there exists a f-connected family F with respect to $D_1, D_2, \ldots, D_m$. Then there exists a compact invariant set $K \subset D$, such that $f|D$ is semi-conjugate to m-shift.*

Theorem 3.3 ([8, 19, 24]). *There are two dynamical systems (X, f) and (Y, g). If (X, f) is semi-conjugate to (Y, g), then the topological entropy of f is not less than that of g, i.e.*

$$ent(f) \ge ent(g). \tag{3.3}$$

If g is a m-shift map, *i.e.* $g = \sigma$, then

$$ent(f) \ge ent(g) = \log m. \tag{3.4}$$

The topological entropy is a nonnegative real number. Thus, if system topological entropy $ent(f)$ is not zero, it is a chaotic system. That is to say, if $m > 1$, the system is a chaotic system.

4. Horseshoe in Fractional-order Qi four-wing chaotic system

In this section, a computer assisted verification of chaos in system (2.4) is given by utilizing the above topological horseshoe theorem.

Consider Poincaré section $\Gamma = \{(x, y, z)|0 \leq x \leq 100, 5 \leq z \leq 60, y = 0$ and $\dot{y} < 0\}$ as shown in Fig. 2. The Poincaré map H is chosen as follow: for each point $(x, y, z) \in \Gamma, H(x, y, z)$ is taken to be the first return point in Γ under the flow of system (4) with the initial condition (x, y, z). In this plane Γ, after a great deal of computer simulations, the quadrangle K with four vertices is selected as:

$$A = [50, 0, 12.2], \quad B = [53, 0, 10.3],$$
$$C = [47, 0, 8.6], \quad D = [46.1, 0, 10.1].$$

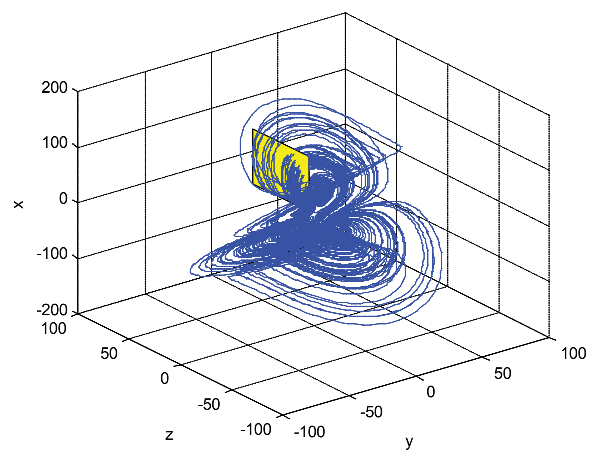


Figure 2. The attractor of system (2.4) and the Poincaré section

Under the first return Poincaré map H, the image of block K_1 is like a very thin hook wholly across the quadrangle regions of K as shown in Fig. 3, in which

$$A' = H(A), \quad B' = H(B), \quad C' = H(C), \quad D' = H(D).$$

According to the topological theory, two disjointed subsets L_1 and L_2 of K_1 need be found. After many trial attempts, two mutually disjoint quadrangles are found as shown in Fig. 4. The first one is block1 $|ABC_1D_1|$ with four vertices being:

$$A = [50, 0, 12.2], \qquad B = [53, 0, 10.3],$$
$$C_1 = [52.7, 0, 10.215], \quad D = [49.805, 0, 12.094].$$

The second one is block2 $|A_1B_1CD|$ with four vertices being:

$$A_1 = [49.21, 0, 11.77], \quad B = [51.8, 0, 9.96],$$
$$C = [47, 0, 8.6], \qquad D = [46.1, 0, 10.1].$$

Under the first return Poincaré map H, for the first subset, the image of $|A_1'B_1'C'D'| = H(|A_1B_1CD|)$ is shown in Fig.5.

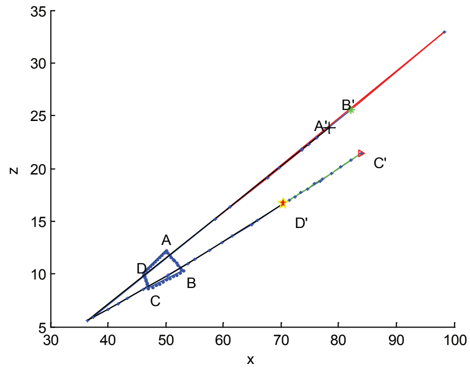


Figure 3. The quadrangle K_1 and its image under the first return Poincaré map H.

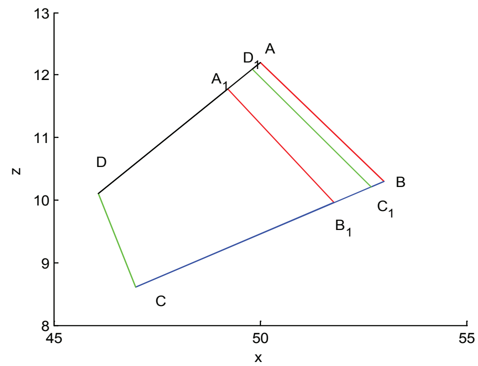


Figure 4. Two disjointed compact subsets $|ABC_1D_1|$ and $|A_1B_1CD|$.

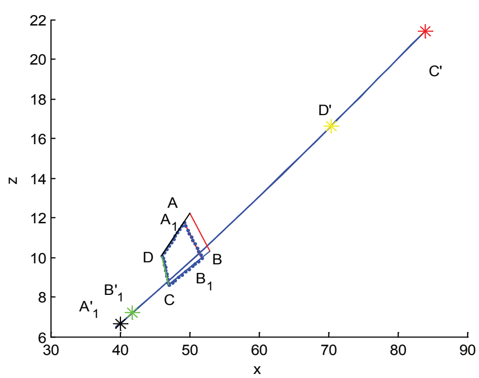


Figure 5. The compact subset $|A_1B_1CD|$ with $A_1' = H(A_1), B_1' = H(B_1), C' = H(C), D' = H(D)$.

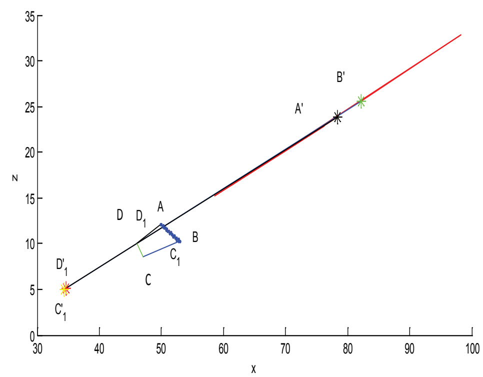


Figure 6. he compact subset $|ABC_1D_1|$ with $A' = H(A), B_1' = H(B), C_1' = H(C_1), D_1' = H(D_1)$.

Then the image $A_1'B_1' = H(A_1B_1)$ is on the left side of the edge CD and the image $C'D' = H(CD)$ is on the right side of edge AB.

For the second subset, the image of $|A'B'C_1'D_1'| = H(|ABC_1D_1|)$ is shown in Fig. 6. AB and C_1D_1 are mapped to $A'B'$ and $C_1'D_1'$, respectively. Similarly, $A'B'$ is on the right side of the edge AB, and C_1D_1 is on the left side of the edge CD.

Upon the above simulation results, it follows that for every connection γ lying in $|ABCD|$ with respect to $|A_1B_1CD|$ and $|ABC_1D_1|$, the images $H(\gamma \cap |A_1B_1CD|)$ and $H(\gamma \cap |ABC_1D_1|)$ lie wholly across the quadrangle $|ABCD|$, that is to say, if $\gamma \in F, H(\gamma \cap |A_1B_1CD|) \in F$ and $H(\gamma \cap |ABC_1D_1| \in F$, thus F is called a connected family with respect $|A_1B_1CD|$ and $|ABC_1D_1|$ based on Definition 3.1. According to Theorem 3.1 and Theorem 3.2, there exists a H-connected family, which means that the Poincar— map H is semi-conjugate to the 2-shift map. These facts prove that the topological entropy of the fractional-order Qi system is no less than $\log 2$, and hence it is chaotic.

5. Conclusion

In this paper, the dynamics of fractional-order Qi four-wing chaotic system was studied. The existence of a topological horseshoe in the system was proved based on the first return Poincar— map. The first return Poincare map defined for the system was proved to be semi-conjugate to 2-shift map, so it has the entropy no less than $\log 2$, which obviously shows the system has chaotic dynamics.

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