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THE EIGENVALUE PROBLEM FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATION WITH TWO-POINT NONLOCAL CONDITIONS

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Abstract We study the spectral problem for the system of difference equations of a two-dimensional elliptic partial differential equation with nonlocal conditions. A new form of two-point nonlocal conditions that involve interior points is proposed. The matrix of the difference system is nonsymmetric thus different types of eigenvalues occur. The conditions for the existence of the eigenvalues and their corresponding eigenvectors are presented for the one dimensional problem. Then, these relations are generalized to the two-dimensional problem by the separation of variables technique.

Keywords Elliptic partial differential equation; Two-point conditions; Eigenvalue problem; Finite difference method.

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1. Introduction

An essential part of any mathematical model is the prescribed conditions. The conditions complete the description of the nature of the considered phenomena. They also affect the choice of the appropriate method to be utilized for solving the mathematical model. The classical type of conditions is referred to as local conditions when the values of the unknown function or its derivative are specified only at the end points of the problem domain. Recently, new types of conditions called nonlocal conditions are proposed where the values of the unknown function at all or some points inside the problem domain take part in the condition formulation.

The problems with nonlocal boundary conditions were investigated in various fields of mathematical physics, biology, biotechnology. Many papers appeared since the work of Cannon [4] and Batten [3] in 1963. Models with nonlocal boundary conditions include elliptic equations [13,28,31] hyperbolic equations [9,32], difference equations [10, 22], and parabolic equations [6, 12].

One of the important problems related to the nonlocal boundary conditions is the eigenvalue problem. The analysis of eigenvalue problems of the difference operator with nonlocal conditions permits us to investigate the stability of difference schemes and check the conditions of convergence of iterative methods utilized to

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solve such problems [5, 7, 15, 27]. The eigenvalue problems for differential operators with nonlocal conditions, except of a few separate articles, has been systematically investigated only over the past decade. Articles [1, 2, 12, 16, 17, 19] deal with the eigenvalue problem subject to nonlocal condition including only boundary values (Ionkin-Samarsky conditions). The eigenvalue problem for one and two-dimensional differential operators subject to Bitsadze-Samarsky nonlocal condition are investigated in [20,23,29], while articles [8,14,21,24,30] deal with the same problem subject to integral nonlocal condition.

In this work, we consider the elliptic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \qquad 0 < x < 1, \qquad 0 < y < 1, \tag{1.1}$$

with the boundary conditions

$$u(x,0) = u_1(x), \tag{1.2}$$
$$u(x,1) = u_2(x), \tag{1.3}$$
$$u(0,y) = \gamma_1 u(1,y), \tag{1.4}$$
$$u(\xi,y) = \gamma_2 u(1-\xi,y), \tag{1.5}$$

where ξ, γ_1 and γ_2 are given constants such that $0 < \xi < 1-\xi < 1$.

Nonlocal boundary condition (1.5) can be considered as a generalization for the Bitsadze-Samarskii condition [20, 23, 29]. This type of nonlocal conditions relates the value of the unknown function at one point on the boundary to its value at one interior point whereas condition (1.5) relates the value of the unknown function at two interior points. Also, condition (1.5) is formulated in the same periodic pattern as condition (1.4).

We study the eigenvalue problem of the finite-difference operator corresponding to problem (1.1-1.5). For this purpose, we introduce two uniform grids Ω_h and Ω_k

$$\Omega_h = \{x_i : x_i = ih,\ i = 0,1,\dots,N\},$$
$$\Omega_k = \{y_j : y_j = jk,\ j = 0,1,\dots,M\},$$

with grid steps h and k defined by $h = \frac{1}{N}$, $k = \frac{1}{M}$, where N and M are positive integers that define the dimensions of the grids. Then, the two-dimensional grid $\Omega_{h\times k}$ is defined by

$$\Omega_{h\times k} = \Omega_h \times \Omega_k = \{(x_i, y_j) : x_i \in \Omega_h,\ y_j \in \Omega_k\}.$$

To incorporate condition (1.5) into the system of difference equations, we choose h such that ξ, and $1-\xi$ are points on the grid Ω_h, i.e. $\xi = sh$, and $1-\xi = (N-s)h$ for a positive integer s.

First, we consider the eigenvalue problem for one-dimensional finite difference operator with given nonlocal boundary conditions

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \tag{1.6}$$
$$u_0 = \gamma_1 u_N, \tag{1.7}$$
$$u_s = \gamma_2 u_{N-s}. \tag{1.8}$$

Then, the results obtained from this problem are utilized to study the two-dimensional difference eigenvalue problem of the form

$$\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} + \lambda\, u_i^j = 0, \tag{1.9}$$

$$u_i^0 = 0, \tag{1.10}$$

$$u_i^N = 0, \tag{1.11}$$

$$u_0^j = \gamma_1 u_N^j, \tag{1.12}$$

$$u_s^j = \gamma_2 u_{N-s}^j. \tag{1.13}$$

The values of λ for which the one-dimensional problem (1.6-1.8) or the two-dimensional problem (1.9-1.13) has non-trivial solutions are called eigenvalues, and the set of all eigenvalues is called the spectrum of the problem. Since conditions (1.7-1.8) and (1.12-1.13) are nonlocal, the corresponding finite-difference operators are non-self-adjoint. Therefore, the analysis of the spectra of these problems leads to the problems on the existence of both real and complex eigenvalues.

The aim of this work is to study the effect of the proposed nonlocal boundary conditions on the conditions for the existence of different types of eigenvalues and to provide the analytical expressions for them. We use techniques and arguments which are used, for example, in papers [18,23,25,26] to investigate similar problems with other types of nonlocal conditions.

2. The difference eigenvalue problem in one dimension

Here, we consider the case where $M = N$. Then, equations (1.6-1.8) generate an $(N-1) \times (N-1)$ linear system of equations. Define the square matrix A of order $(N-1)$ in the block matrix form as

$$A = \frac{1}{h^2}\begin{pmatrix} B \\ C \\ D \end{pmatrix}, \tag{2.1}$$

where

$$B = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \gamma_1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 & -1 & \cdots & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & \cdots & -1 & 2 & 0 & 0 & \cdots & -\gamma_2 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 2\gamma_2 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 2 & -1 & \cdots & -\gamma_2 & 0 & \cdots & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & \cdots & -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \end{pmatrix},$$

where matrices B, C, and D are of dimension $(s-2)\times(N-1)$, $3\times(N-1)$ and $(N-s-2)\times(N-1)$, respectively. Then, the finite-difference eigenvalue problem (1.6-1.8) is equivalent to the matrix eigenvalue problem

$$Au = \lambda\, u. \tag{2.2}$$

Matrix A of the difference system is nonsymmetric because of the nonlocal conditions. As some of the points involved in the nonlocal condition are inside the domain, the symmetry known for the matrix of the elliptic problem with classical conditions is altered in some rows and columns inside the matrix. Thus, matrix A can have zero, positive, negative or complex eigenvalues according to the parameters h, ξ, γ_1 and γ_2 and these cases are examined in the following analysis.

Equation (1.6) can be rewritten in the form

$$u_{i-1} - 2(1 - \frac{\lambda h^2}{2})u_i + u_{i+1} = 0, \tag{2.3}$$

which is useful in proving some of the following statements.

Lemma 2.1. *The difference eigenvalue problem (1.6-1.8) has zero eigenvalues, provided that they exist, in one of the following cases:*

(i) If $\gamma_1 = 1$ and $\gamma_2 = 1$. In this case, the corresponding difference eigenvector is given by $u_i = c$, where c is an arbitrary constant.

(ii) If $\gamma_1 \neq 1$ and $\gamma_2 = \frac{\xi+\gamma_1(1-\xi)}{\gamma_1\xi+(1-\xi)}$, and the corresponding difference eigenvector is given by $u_i = c(ih + \frac{\gamma_1}{1-\gamma_1})$.

Proof. If $\lambda = 0$, the difference eigenvalue problem (1.6) has a solution of the form $u_i = c_1 + c_2 ih$, $i = 0, 1, 2, \cdots, N$. Then, by applying condition (1.7) we have

$$(1-\gamma_1)c_1 - \gamma_1 c_2 = 0. \tag{2.4}$$

The second nonlocal boundary condition (1.8) yields

$$(1-\gamma_2)c_1 + (\xi - \gamma_2(1-\xi))c_2 = 0. \tag{2.5}$$

We have two cases: the first case is when $\gamma_1 = 1$ which yields $c_2 = 0$. Then, to obtain a nontrivial solution we get $\gamma_2 = 1$ and $u_i = c$. The second case if $\gamma_1 \neq 1$, by solving the system of two equations (2.4-2.5), we get $\xi + (1-\xi)\gamma_1 + (-1+\xi-\xi\gamma_1)\gamma_2 = 0$, and the lemma is proved. $\square$

Lemma 2.2. *The difference eigenvalue problem (1.6-1.8) has a unique negative eigenvalue, provided that it exists, given by $\lambda = \frac{-4}{h^2}\sinh^2(\frac{\alpha h}{2})$, where α is the positive parameter that satisfies the relation between γ_1 and γ_2 in one of the two following cases:*

(i) If $\gamma_1 = \frac{1}{\cosh(\alpha)}$ and $\gamma_2 = \frac{\cosh(\xi\alpha)}{\cosh((1-\xi)\alpha)}$. The corresponding difference eigenvector is given by $u_i = c\cosh(i\alpha h)$,

(ii) If $\gamma_1 \neq \frac{1}{\cosh(\alpha)}$ *and* $\gamma_2 = \frac{\sinh(\xi\alpha)+\gamma_1\sinh((1-\xi)\alpha)}{\gamma_1\sinh(\xi\alpha)+\sinh((1-\xi)\alpha)}$*, and the corresponding difference eigenvector is given by* $u_i = c(\sinh(\alpha i h) + \frac{\gamma_1\sinh(\alpha)}{1-\gamma_1\cosh(\alpha)}\cosh(\alpha i h))$.

Proof. If $\lambda < 0$, we have

$$1 - \frac{\lambda h^2}{2} > 1.$$

Denote

$$\cosh(\alpha h) = 1 - \frac{\lambda h^2}{2},$$

and rewrite the finite-difference equation (2.3) in the form

$$u_{i-1} - 2\cosh(\alpha h)\, u_i + u_{i+1} = 0.$$

The general solution of the latter equation is given by

$$u_i = c_1 \cosh(\alpha h i) + c_2 \sinh(\alpha h i).$$

By substituting this solution into nonlocal conditions (1.7) and (1.8), we obtain the following system of two linear algebraic equations with unknowns c_1 and c_2

$$c_1(1-\gamma_1(\cosh\alpha)) - c_2\gamma_1(\sinh\alpha) = 0, \tag{2.6}$$

$$c_1(\cosh(\xi\alpha) - \gamma_2\cosh((1-\xi)\alpha)) + c_2(\sinh(\xi\alpha) - \gamma_2\sinh((1-\xi)\alpha)) = 0. \tag{2.7}$$

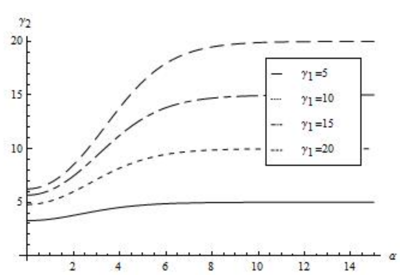
We have two cases: the first case is when $\gamma_1 = \frac{1}{\cosh(\alpha)}$ which yields $c_2 = 0$. This case yields a nontrivial solution only if

$$\gamma_2 = g_1(\alpha;\xi) = \frac{\cosh(\xi\alpha)}{\cosh((1-\xi)\alpha)}, \tag{2.8}$$

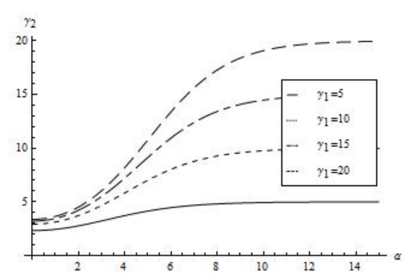
and $u_i = c\cosh(i\alpha h)$. The second case is when $\gamma_1 \neq \frac{1}{\cosh(\alpha)}$. By solving the system of two equations (2.6-2.7), we get

$$\gamma_2 = g_2(\alpha,\gamma_1;\xi) = \frac{\sinh(\xi\alpha) + \gamma_1\sinh((1-\xi)\alpha)}{\sinh((1-\xi)\alpha) + \gamma_1\sinh(\xi\alpha)}, \tag{2.9}$$

and the lemma is proved. □



(a) $\xi = 0.1$



(b) $\xi = 0.2$

Figure 1. Effect of changing γ_1 and α on the values of γ_2 (a) $\xi = 0.1$ (b) $\xi = 0.2$ as described by relation (2.9).

Figure **1** illustrates the relation between the parameters γ_2 and α as described by equation (2.9) at different values for γ_1 and for two different values of ξ. The figure indicates that α assumes higher values as the value of γ_2 asymptotically approaches the value of γ_1.

Lemma 2.3. *The positive eigenvalues $0 < \lambda < \frac{4}{h^2}$ for the difference eigenvalue problem (1.6-1.8), provided that they exist, take the form $\lambda_k = \frac{4}{h^2}\sin^2(\frac{\alpha_k h}{2})$, where the parameters $\alpha_k \in (0, \frac{\pi}{h})$ satisfy the relation between γ_1 and γ_2 in one of the two following cases:*

(i) If $\gamma_1 = \frac{1}{\cos(\alpha_k)}$ and $\gamma_2 = \frac{\cos(\xi\alpha_k)}{\cos((1-\xi)\alpha_k)}$. The corresponding difference eigenvector is given by $u_i = c\cos(i\alpha_k h)$.

(ii) If $\gamma_1 \neq \frac{1}{\cos(\alpha_k)}$ and $\gamma_2 = \frac{\sin(\xi\alpha_k)+\gamma_1\sin((1-\xi)\alpha_k)}{\gamma_1\sin(\xi\alpha_k)+\sin((1-\xi)\alpha_k)}$, and the corresponding difference eigenvector is given by $u_i = c(\sin(\alpha_k ih) + \frac{\gamma_1\sin(\alpha_k)}{1-\gamma_1\cos(\alpha_k)}\cos(\alpha_k ih))$.

Proof. If $0 < \lambda < \frac{4}{h^2}$, then we have

$$|1 - \frac{\lambda h^2}{2}| < 1.$$

Denote

$$\cos(\alpha h) = 1 - \frac{\lambda h^2}{2},$$

and rewrite the finite-difference equation (2.3) in the form

$$u_{i-1} - 2\cos(\alpha h)\, u_i + u_{i+1} = 0.$$

By substituting the general solution of the latter equation which is given by

$$u_i = c_1 \cos(\alpha h i) + c_2 \sin(\alpha h i),$$

into nonlocal conditions (1.7) and (1.8), the following system of equations is obtained

$$c_1(1 - \gamma_1\cos(\alpha)) - c_2\gamma_1\sin(\alpha) = 0, \tag{2.10}$$
$$c_1(\cos(\xi\alpha) - \gamma_2\cos((1-\xi)\alpha)) + c_2(\sin(\xi\alpha) - \gamma_2\sin((1-\xi)\alpha)) = 0. \tag{2.11}$$

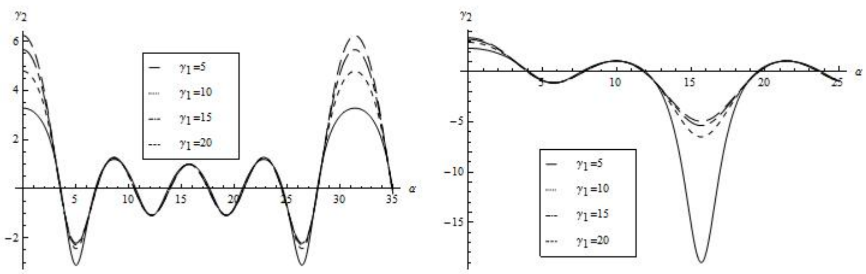
When $\gamma_1 = \frac{1}{\cos(\alpha)}$, then $c_2 = 0$. Then, to obtain a nontrivial solution we get $\gamma_2 = \frac{\cos(\xi\alpha)}{\cos((1-\xi)\alpha)}$, and $u_i = c\cos(i\alpha h)$. But when $\gamma_1 \neq \frac{1}{\cos(\alpha)}$, then by solving system (2.10-2.11), we get

$$\sin(\xi\alpha) + \gamma_1\sin((1-\xi)\alpha) - \gamma_2(\sin((1-\xi)\alpha) + \gamma_1\sin(\xi\alpha)) = 0, \tag{2.12}$$

and the lemma is proved. □

Figure **2** illustrates the relation between the parameters γ_2 and α as described by equation (2.12) at different values for γ_1 and for two different values of ξ. The periodic behavior of the graph indicates the fact that for given values for γ_1 and γ_2, several values of α satisfy equation (2.12) and yields different positive eigenvalues.

Lemma 2.4. *The eigenvalue $\lambda = \frac{4}{h^2}$, for the eigenvalue problem (1.6-1.8), provided it exists, occurs in one of the following cases:*



(a) $\xi = 0.1$ (b) $\xi = 0.2$

Figure 2. Effect of changing γ_1 and α on the values of γ_2 (a) $\xi = 0.1$ (b) $\xi = 0.2$ as described by relation (2.12).

(i) If $\gamma_1 = (-1)^N$ and $\gamma_2 = (-1)^N$, and the corresponding difference eigenvector is given by $u_i = (-1)^i c$, where c is an arbitrary constant.

(ii) If $\gamma_1 \neq (-1)^N$ and $\gamma_2 = \frac{\xi+(-1)^N(1-\xi)\gamma_1}{\gamma_1\xi+(-1)^N(1-\xi)}$, and the corresponding difference eigenvector is given by $u_i = (-1)^i\, c(\, i\, h + \frac{(-1)^N\gamma_1}{1-(-1)^N\gamma_1})$.

Proof. If $\lambda = \frac{4}{h^2}$, in this case the finite-difference equation (2.3) takes the form

$$u_{i+1} + 2\, u_i + u_{i-1} = 0,$$

and the general solution for this equation is given by

$$u_i = (-1)^i\, (c_1 + c_2\, (i\, h)).$$

Then, from nonlocal conditions (1.7) and (1.8), the system of two linear algebraic equations that relates c_1 and c_2 takes the form

$$(1-(-1)^N\gamma_1)c_1 - c_2(-1)^N\gamma_1 = 0, \tag{2.13}$$

$$c_1((-1)^s - \gamma_2(-1)^{N-s}) + c_2((-1)^s\xi - \gamma_2(-1)^{N-s}(1-\xi)) = 0. \tag{2.14}$$

If $\gamma_1 = (-1)^N$, we get $c_2 = 0$. Then, to obtain a nontrivial solution we get $\gamma_2 = (-1)^N$, and $u_i = (-1)^i c$.

If $\gamma_1 \neq (-1)^N$, by solving the system of two equations (2.13-2.14), we get

$$\xi + (-1)^N(1-\xi)\gamma_1 - \gamma_2((-1)^{(N)}(1-\xi) + \gamma_1\xi) = 0. \tag{2.15}$$

□

Lemma 2.5. *The positive eigenvalue $\lambda > \frac{4}{h^2}$ for the difference eigenvalue problem (1.6-1.8), provided that it exists, takes the form $\lambda = \frac{4}{h^2}\cosh^2(\frac{\alpha\, h}{2})$, where α is the positive parameter that satisfies the relation between γ_1 and γ_2 in one of the two following cases:*

(i) If $\gamma_1 = \frac{(-1)^N}{\cosh(\alpha)}$ and $\gamma_2 = \frac{(-1)^N\cosh(\xi\alpha)}{\cosh((1-\xi)\alpha)}$, and the corresponding difference eigenvector is given by $u_i = (-1)^i c\cosh(i\alpha h)$, where c is an arbitrary constant.

(ii) If $\gamma_1 \neq \frac{(-1)^N}{\cosh(\alpha)}$ and $\gamma_2 = \frac{\sinh(\xi\alpha)+(-1)^N\gamma_1\sinh((1-\xi)\alpha)}{(-1)^N\sinh((1-\xi)\alpha)+\gamma_1\sinh(\xi\alpha)}$, and the corresponding difference eigenvector is given by $u_i = (-1)^i c(\sinh(\alpha i h) + \frac{(-1)^N\gamma_1\sinh(\alpha)}{1-(-1)^N\gamma_1\cosh(\alpha)}\cosh(i\alpha h))$.

Proof. If $\lambda > \frac{4}{h^2}$, denote

$$1 - \frac{\lambda h^2}{2} = -\cosh(\alpha h),$$

and rewrite the finite difference equation (2.3) in the form

$$u_{i-1} + 2\cosh(\alpha h)\, u_i + u_{i+1} = 0.$$

By substitution the general solution of the latter equation which is given by

$$u_i = (-1)^i (c_1 \cosh(\alpha h i) + c_2 \sinh(\alpha h i)),$$

into nonlocal conditions (1.7) and (1.8), we obtain the following system

$$c_1(1 - (-1)^N \gamma_1 \cosh(\alpha)) - c_2(-1)^N \gamma_1 \sinh(\alpha) = 0, \tag{2.16}$$

$$\begin{aligned} &c_1(\cosh(\xi\alpha) - \gamma_2(-1)^N \cosh((1-\xi)\alpha)) \\ &+ c_2(\sinh(\xi\alpha) - \gamma_2 (-1)^N \sinh((1-\xi)\alpha)) = 0. \end{aligned} \tag{2.17}$$

We have two cases: the first case is when $\gamma_1 = \frac{(-1)^N}{\cosh(\alpha)}$ which yields $c_2 = 0$. Then to obtain a nontrivial solution we get $\gamma_2 = \frac{(-1)^N \cosh(\xi\alpha)}{\cosh((1-\xi)\alpha)}$, and $u_i = (-1)^i c \cosh(i\alpha h)$. The second case if $\gamma_1 \neq \frac{(-1)^N}{\cosh(\alpha)}$, by solving the system of two equations (2.16-2.17), we get

$$\sinh(\xi\alpha) + \gamma_1((-1)^N \sinh((1-\xi)\alpha)) - \gamma_2((-1)^N \sinh((1-\xi)\alpha) + \sinh(\xi\alpha)) = 0, \tag{2.18}$$

and the lemma is proved. □

The eigenvalues $\lambda = \frac{4}{h^2}$ in Lemma 2.4 and $\lambda > \frac{4}{h^2}$ in Lemma 2.5 are typical only for difference operators, but not for differential operators. The matter is that corresponding eigenvector has not analogue for differential operator.

Lemma 2.6. *The complex eigenvalues for the difference eigenvalue problem (1.6-1.8), provided that they exist, take the form* $\lambda_k = \frac{4}{h^2}\sin^2(\frac{q_k h}{2})$, *where* q_k *are the nontrivial complex numbers that satisfy*

$$\gamma_2 = \frac{\sin(q\xi) + \sin(q - q\xi)\gamma_1}{\gamma_1 \sin(q\xi) + \sin(q - q\xi)}. \tag{2.19}$$

The corresponding eigenvector is $u_i = \frac{c((1-\gamma_1\cos(q\xi)+\iota\gamma_1\sin(q\xi))\sin(i\,q\,h)}{(1-\gamma_1\cos(q\xi)-\iota\gamma_1\sin(q\xi)}$, *where* c *is an arbitrary constant and* $\iota = \sqrt{-1}$.

Proof. Let $q = \alpha + \iota\beta$, where $\iota = \sqrt{-1}$. We assume that $\alpha \neq 0, \beta \neq 0$. If $\alpha = 0$, $\beta \neq 0$ or $\alpha \neq 0$, $\beta = 0$ then this case coincides with lemma (2.3) or lemma (2.2), respectively. However, when, $\alpha = \beta = 0$, a situation is the same as in lemma (2.1).

Denote the notation $\cos(q h) = 1 - \frac{\lambda h^2}{2}$. Now, the finite-difference equation (2.3) takes the form

$$u_{i-1} - 2\cos(q h)\, u_i + u_{i+1} = 0.$$

By substituting the general solution of the latter equation which is given by

$$u_i = c_1 e^{\iota q\, h\, i} + c_2 e^{-\iota q\, h\, i},$$

into nonlocal conditions (1.7) and (1.8), we obtain the following system of two linear algebraic equations with unknowns c_1 and c_2

$$c_1(1 - \gamma_1 e^{\iota q}) + c_2(1 - \gamma_1 e^{-\iota q}) = 0, \tag{2.20}$$

$$c_1(e^{\iota \xi q} - \gamma_2 e^{\iota(1-\xi)q}) + c_2(e^{-\iota \xi q} - \gamma_2 e^{-\iota(1-\xi)q}) = 0. \tag{2.21}$$

This system has a non-trivial solution when its determinant is equal to zero which yields condition (2.19). The values of α and β can be obtained from the two equations that result from equating the real and imaginary parts of the determinant to zero. □

3. The difference eigenvalue problem in two dimensions

Let us consider the two-dimensional finite-difference eigenvalue problem (1.9-1.13). By separating variables, i.e., by representing the solution of problem (1.9-1.13) in the form

$$u_{ij} = v_i\, z_j, \; i, j = 0, 1, 2, \cdots, N,$$

we obtain two one-dimensional eigenvalue problems

$$\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} + \mu_i v_i = 0, \; v_0 = \gamma_1 v_N, \; v_{s_2} = \gamma_2 v_{N-s_2}, \tag{3.1}$$

and

$$\frac{z_{j-1} - 2z_j + z_{j+1}}{h^2} + \omega_j z_j = 0, \; z_0 = 0, \; z_N = 0, \tag{3.2}$$

where $\lambda = \lambda_{k,\ell} = \mu_k + \omega_\ell$. The eigenvalues of (3.2) are real, positive and can be computed by the formula [23]

$$\omega_\ell = \frac{4}{h^2} \sin^2(\frac{\ell\, \pi\, h}{2}), \; \ell = 1, 2, \cdots, N-1, \tag{3.3}$$

and the corresponding eigenvectors are given by

$$z_\ell = \sin(\frac{\ell\, \pi\, h}{2}), \; \ell = 1, 2, \cdots, N-1.$$

The value of the eigenvalue $\lambda_{k,\ell}$ is conditioned by the value of μ_k. In the following statements, we illustrate the cases for positive, zero, and negative eigenvalues for problem (1.9-1.13).

Corollary 3.1. *The positive eigenvalues* $0 < \lambda_{k,\ell} < \frac{8}{h^2}$ *of the problem (1.9-1.13), can be computed by the formula*

$$\lambda_{k,\ell} = \frac{4}{h^2} (\sin^2(\frac{\alpha_k\, h}{2}) + \sin^2(\frac{\ell\, \pi\, h}{2})), k = 1, 2, \cdots, N-1.$$

The corresponding eigenvectors in case $\gamma_1 = \frac{1}{\cos(\alpha_k)}$ *and* $\gamma_2 = \frac{\cos(\xi\alpha_k)}{\cos((1-\xi)\alpha_k)}$ *are given by*

$$(u_{k,\ell})_{ij} = c\cos(i\alpha_k h)\,\sin(\frac{\ell\,\pi\,h\,j}{2}),$$

whereas in the case $\gamma_1 \neq \frac{1}{\cos(\alpha_k)}$ *and* $\gamma_2 = \frac{\sin(\xi\alpha_k)+\gamma_1\sin((1-\xi)\alpha_k)}{\sin((1-\xi)\alpha_k)+\gamma_1\sin(\xi\alpha_k)}$ *are given by*

$$(u_{k,\ell})_{ij} = c(\sin(\alpha_k ih) + \frac{\gamma_1\sin\alpha_k}{1-\gamma_1\cos\alpha_k}\cos(\alpha_k ih))\,\sin(\frac{\ell\,\pi\,h\,j}{2}),$$

where α_k *are the roots of equation (2.12) and the indices* $i,\ j,\ \ell$ *and* k *are from* 1 *to* $N-1$.

Corollary 3.2. *The positive eigenvalues of the problem (1.9-1.13) of the form*

$$\tilde{\lambda_\ell} = \frac{4}{h^2}(1+\sin^2(\frac{\pi\ell h}{2})),\ell = 1,2,\cdots,N-1,$$

if they exist at all, occur in two cases: case $\gamma_1 = (-1)^N$ *and* $\gamma_2 = (-1)^N$ *and the corresponding eigenvectors are given by*

$$(\tilde{u_\ell})_{ij} = c(-1)^i\sin(\frac{\ell\,\pi\,h\,j}{2}),$$

whereas in the case $\gamma_1 \neq (-1)^N$ *and* $\gamma_2 = \frac{\xi+((-1)^N(1-\xi)\gamma_1)}{(-1)^N(1-\xi)+\gamma_1\xi}$, *the corresponding eigenvectors are given by*

$$(\tilde{u_\ell})_{ij} = (-1)^i\,c(\,i\,h + \frac{(-1)^N\gamma_1}{1-(-1)^N\gamma_1})\sin(\frac{\ell\,\pi\,h\,j}{2}),$$

where the indices $i,\ j$ *and* l *are from* 1 *to* $N-1$.

Corollary 3.3. *The positive eigenvalues of the problem (1.9-1.13) of the form*

$$\bar{\lambda_\ell} = \frac{4}{h^2}(\cosh^2(\frac{\alpha h}{2}) + \sin^2(\frac{\pi\ell h}{2})),\ \ \ell = 1,2,\cdots,N-1,$$

if they exist at all, occur in two cases: case $\gamma_1 = \frac{(-1)^N}{\cosh(\alpha)}$ *and* $\gamma_2 = \frac{(-1)^N\cosh(\xi\alpha)}{\cosh((1-\xi)\alpha)}$ *and the corresponding eigenvectors are given by*

$$(\bar{u_\ell})_{ij} = (-1)^i c\cosh(i\alpha h)\sin(\frac{\ell\,\pi\,h\,j}{2}),$$

whereas in the case $\gamma_1 \neq \frac{(-1)^N}{\cosh(\alpha)}$ *and* $\gamma_2 = \frac{\sinh(\xi\alpha)+(-1)^N\gamma_1\sinh((1-\xi)\alpha)}{(-1)^N\sinh((1-\xi)\alpha)+\gamma_1\sinh(\xi\alpha)}$, *the corresponding eigenvectors are given by*

$$(\bar{u_\ell})_{ij} = ((-1)^i c(\sinh(\alpha ih) + \frac{(-1)^N\gamma_1\sinh(\alpha)}{1-(-1)^N\gamma_1\cosh(\alpha)}\cosh(i\alpha h))\sin(\frac{\ell\,\pi\,h\,j}{2}),$$

where α *are the roots of equation (2.18) and the indices* $i,\ j$ *and* l *are from* 1 *to* $N-1$.

Let us investigate the existence of zero and negative eigenvalues of problem (1.9-1.13). We know that there exists a unique negative eigenvalue of the problem (3.1) with the form

$$\mu = -\frac{4}{h^2}\sinh^2(\frac{\alpha h}{2}),$$

where α is the positive root of equation (2.9). Then, since the numbers

$$\alpha_\ell^* = \frac{2}{h}\log(\sin(\frac{\pi\ell h}{2}) + \sqrt{\sin^2(\frac{\pi\ell h}{2}) + 1}),\ \ell = 1, 2, \cdots, N-1,$$

are the positive roots of the equations

$$\sinh^2(\frac{\alpha h}{2}) = \sin^2(\frac{\pi\ell h}{2}),\ \ell = 1, 2, \cdots, N-1,$$

the following statement is valid.

Corollary 3.4. *Problem (1.9-1.13) has an algebraically simple zero eigenvalue* $\lambda_{k,\ell} = 0$ *in one of the two following cases*

(i) If $\gamma_1 = \frac{1}{\cosh(\alpha_\ell^*)}$ *and* $\gamma_2 = g_1(\alpha_\ell^*;\xi) = \frac{\cosh(\xi\alpha_\ell^*)}{\cosh((1-\xi)\alpha_\ell^*)}$*, and the corresponding difference eigenvector is given by* $(u_\ell)_{ij} = c\cosh(i\alpha_\ell^* h)\sin(\frac{\ell\pi h j}{2})$*, where* c *is an arbitrary constant.*

(ii) If $\gamma_1 \neq \frac{1}{\cosh(\alpha_\ell^*)}$ *and* $\gamma_2 = g_2(\alpha_\ell^*;\gamma_1;\xi) = \frac{\sinh(\xi\alpha_\ell^*)+\gamma_1\sinh((1-\xi)\alpha_\ell^*)}{\sinh((1-\xi)\alpha_\ell^*)+\gamma_1\sinh(\xi\alpha_\ell^*)}$*, and the corresponding difference eigenvector are given by* $(u_\ell)_{ij} = c(\sinh(\alpha_\ell^* ih) + \frac{\gamma_1\sinh(\alpha_\ell^*)}{1-\gamma_1\cosh(\alpha_\ell^*)}\cosh(\alpha_\ell^* ih))\sin(\frac{\ell\pi h j}{2})$.

If either of conditions (i) or (ii) is satisfied with α_p^* *for a positive integer* p, $1 \le p \le N-1$*, then problem (1.9-1.13) has* $p-1$ *negative eigenvalues*

$$\lambda_{p,\ell} = -\frac{4}{h^2}(\sinh^2(\frac{\alpha^* h}{2}) - \sin^2(\frac{\pi\ell h}{2})),\ \ell = 1, 2, \cdots, p-1,$$

and an algebraically simple eigenvalue $\lambda_{p,p} = 0$.

4. Conclusion

We studied the eigenvalue problem of an elliptic partial differential equation with nonlocal boundary conditions that involve points interior to the problem domain. The two nonlocal boundary conditions change the classical form for the matrix of the system of difference equations as they change the first row and cause a shift in the tridiagonal elements. The position of the points and the coefficients of the nonlocal conditions affect the type and the value of the eigenvalue, hence the corresponding eigenvector. Finally, by using the separation of variables technique, the properties and relations of one-dimensional problems can be combined together to obtain the corresponding ones of the two-dimensional case.

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