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THE HILBERT NUMBER OF A CLASS OF DIFFERENTIAL EQUATIONS*

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Abstract The notion of Hilbert number from polynomial differential systems in the plane of degree n can be extended to the differential equations of the form

$$\frac{dr}{d\theta} = \frac{a(\theta)}{\displaystyle\sum_{j=0}^{n} a_j(\theta) r^j} \tag{$*$}$$

defined in the region of the cylinder $(\theta, r) \in \mathbb{S}^1 \times \mathbb{R}$ where the denominator of $(*)$ does not vanish. Here $a, a_0, a_1, \ldots, a_n$ are analytic 2π–periodic functions, and the *Hilbert number* $\mathbb{H}(n)$ is the supremum of the number of limit cycles that any differential equation $(*)$ on the cylinder of degree n in the variable r can have. We prove that $\mathbb{H}(n) = \infty$ for all $n \geq 1$.

Keywords Periodic orbit, averaging theory, trigonometric polynomial, Hilbert number.

MSC(2000) 34C29, 37C27.

1. Introduction

In the article [6] Lins Neto studied the following problem posed by Charles Pugh.

Problem 1. *Let $a_0, a_1, \ldots, a_n : \mathbb{S}^1 \to \mathbb{R}$ be continuous 2π–periodic functions and consider the differential equation*

$$\frac{dr}{d\theta} = a_0(\theta) + a_1(\theta) r + \ldots + a_n(\theta) r^n, \tag{1.1}$$

on the cylinder $(\theta, r) \in \mathbb{S}^1 \times \mathbb{R}$. Then the problem is to know the number of isolated periodic solutions (i.e. limit cycles) of the differential equation (1.1) *in function of n.*

Problem 1 was motivated by the Hilbert's 16–th problem (see for instance [3–5]), because some polynomial differential systems in the plane can be reduced to equations (1.1) as all the polynomial differential systems of degree 2 (see for instance the proposition of [6]), all polynomial differential systems with the linear center

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$\dot{x} = -y$, $\dot{y} = x$ with nonlinearities given by homogeneous polynomials of degree n for all positive integer n (see for instance [7]), all polynomial differential systems such that in polar coordinates (r, θ) have $\dot{\theta} = 1$, ... See also [1] for more details on the differential equations (1.1).

For polynomial differential systems in the plane it is defined the *Hilbert number* $H(n)$, i.e. the supremum of the number of limit cycles that a polynomial differential system in the plane of degree n can have. For the moment it is unknown if the Hilbert number is finite or infinite when $n > 1$. We can extend the notion of Hilbert number to the differential equations (1.1) defined on the cylinder as follows. The *Hilbert number* $\mathcal{H}(n)$ is the supremum of the number of limit cycles that a differential equation (1.1) on the cylinder of degree n in the variable r can have.

The Hilbert number for the Problem 1 has the following answer. For the differential equations of the form

(i) $\dfrac{dr}{d\theta} = a_0(\theta) + a_1(\theta)r$ (*periodic linear differential equations*) it is known that $\mathcal{H}(1) = 1$.

(ii) $\dfrac{dr}{d\theta} = a_0(\theta) + a_1(\theta)r + a_2(\theta)r^2$ (*periodic Riccati differential equations*) we have that $\mathcal{H}(2) = 2$, see for instance Theorem 1 of [6].

(iii) $\dfrac{dr}{d\theta} = a_0(\theta) + a_1(\theta)r + a_2(\theta)r^2 + a_3(\theta)r^3$ (*periodic Abel differential equations*) can have k limit cycles for all positive k, see the example of section 3 of [6]. So $\mathcal{H}(3) = \infty$.

(iv) $\dfrac{dr}{d\theta} = a_0(\theta) + a_1(\theta)r + ... + a_s(\theta)r^s$ can have k limit cycles for all positive k. We have the same conclusion than for the periodic Abel differential equation and the proof follows easily modifying the proof of (iii). Hence $\mathcal{H}(n) = \infty$ for $n > 3$.

In this paper we consider the following problem:

Problem 2. *Let $a, a_0, a_1, \ldots, a_n : \mathbb{S}^1 \to \mathbb{R}$ be continuous 2π–periodic functions and consider the differential equation*

$$\frac{dr}{d\theta} = \frac{a(\theta)}{a_0(\theta) + a_1(\theta)r + \ldots + a_n(\theta)r^n}, \tag{1.2}$$

on the region of the cylinder $(\theta, r) \in \mathbb{S}^1 \times \mathbb{R}$ where the denominator of (1.2) *does not vanish. Then the problem is to know the number of limit cycles of the differential equation* (1.2) *in function of n.*

Again we can extend the notion of Hilbert number to the differential equations (1.2) defined on the cylinder as follows. The *Hilbert number* $\mathbb{H}(n)$ is the supremum of the number of limit cycles that a differential equation (1.2) on the cylinder of degree n in the variable r can have.

The main result of this paper is to compute the Hilbert number for the Problem 2.

Theorem 1.1. *For all positive integer k there are analytic differential equations* (1.2) *with $n = 1$ having at least k limit cycles. So $\mathbb{H}(1) = \infty$.*

Theorem 1.1 is proved in section 3 using the averaging theory of first order for studying the periodic solutions. We present the results of this theory that we need in section 2.

A corollary of Theorem 1.1 is the following.

Corollary 1.1. *For all positive integers n and k there are analytic differential equations* (1.2) *having at least k limit cycles. So $\mathbb{H}(n) = \infty$ for $n > 1$.*

Corollary 1.1 is also proved in section 3.

2. The averaging theory

Now we summarize the basic results from averaging theory that we need for proving the results of this paper. The following result provides a first order approximation for the periodic solutions of a periodic differential equation.

We deal with the differential equation

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.1}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Suppose that the functions $F_1(t, \mathbf{x})$ and $F_2(t, \mathbf{x}, \varepsilon)$ are $T-$periodic in t. Then consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{2.2}$$

where

$$f(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt.$$

The next result shows that under convenient conditions, the equilibrium solutions of the averaged equation correspond with $T-$periodic solutions of the differential equation (2.1).

Theorem 2.1. *Consider the two differential equations* (2.1) *and* (2.2)*. Assume:*

(i) the functions F_1, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, F_2 and its Jacobian $\partial F_2/\partial x$ are continuous and bounded by a constant independent of ε in the sets $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$.

(ii) the functions F_1 and F_2 are $T-$periodic in t (T independent of ε).

Then the next statements hold.

(a) If p is an equilibrium point of the averaged equation (2.2) *and*

$$\det \left(\frac{\partial f}{\partial \mathbf{y}} \right) \bigg|_{\mathbf{y}=p} \neq 0,$$

then there is a $T-$periodic solution $\varphi(t, \varepsilon)$ of equation (2.1) *such that $\varphi(0, \varepsilon) \to p$ as $\varepsilon \to 0$.*

(b) The kind of stability or instability of the limit cycle $\varphi(t, \varepsilon)$ is given by the kind of stability or instability of the equilibrium point p of the averaged system (2.2)*. Indeed, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t, \varepsilon)$.*

For a proof of Theorem 2.1 see Theorems 11.5 and 11.6 of Verhulst [8].

3. Proof of Theorem 1.1

Consider the subclass of differential equations (1.2) with $n = 1$ given by

$$\frac{dr}{d\theta} = \varepsilon \frac{a(\theta)}{a_0(\theta) + a_1(\theta) r}, \tag{3.1}$$

where ε is a small parameter, and

$$a(\theta) = \sum_{j=0}^{k} \alpha_j \cos(j\theta), \quad a_0(\theta) = 1, \text{ and } a_1(\theta) = \cos\theta, \tag{3.2}$$

being $\alpha_0, \alpha_1, \ldots, \alpha_k$ arbitrary constants.

Clearly the differential equation (3.1) is defined in the open cylinder $\{(\theta, r) \in \mathbb{S}^1 \times (0,1)\}$. This differential equation satisfies the assumptions of Theorem 2.1, so we shall apply this theorem to it.

The averaged differential equation (2.2) corresponding to equation (3.1) is

$$\dot{r} = \varepsilon f(r), \tag{3.3}$$

where

$$f(r) = \sum_{j=0}^{k} \alpha_j \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(j\theta)}{1 + r\cos\theta} d\theta = \sum_{j=0}^{k} \alpha_j f_j(r).$$

The function $f_j(r)$ for $r \in (0,1)$ can be computed, and we get

$$f_j(r) = \frac{1}{\sqrt{1-r^2}} \left(\frac{\sqrt{1-r^2} - 1}{r} \right)^j. \tag{3.4}$$

In fact this integral was computed in the formula 3.613 of [2]. Therefore

$$f(r) = \sum_{j=0}^{k} \alpha_j f_j(r) = \sum_{j=0}^{k} \alpha_j \frac{1}{\sqrt{1-r^2}} \left(\frac{\sqrt{1-r^2} - 1}{r} \right)^j.$$

The equilibrium points of the averaged equation (3.3) are the zeros of the function $f(r)$.

Let I be an interval of $\mathbb{R}$, and let $f_0, f_1, \ldots, f_k : I \to \mathbb{R}$ be $\mathcal{C}^1$ functions *linearly independent*, i.e. if $\sum_{j=0}^{k} \beta_j f_j(r) = 0$ then $\beta_0 = \beta_1 = \ldots = \beta_k = 0$. The following result is well known, for a proof see for instance the Proposition 1 of the Appendix A of [7].

Proposition 3.1. *If the functions $f_0, f_1, \ldots, f_k : I \to \mathbb{R}$ are linearly independent, then there exist $\alpha_0, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and $r_1, \ldots, r_k \in I$ such that for every r_ℓ with $\ell \in \{1, \ldots, k\}$ we have that*

$$\sum_{j=0}^{k} \alpha_j f_j(r_\ell) = 0.$$

Clearly our functions $f_j(r)$ for $j = 0, 1, \ldots, k$ given in (3.4) are linearly independent. So we can apply Proposition 3.1 to them, and consequently we know that there are values of $\alpha_0, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and values $r_1, \ldots, r_k \in (0,1)$ such that $f(r_\ell) = 0$ for $\ell = 1, \ldots, k$, being the r_ℓ simple zeros of $f(r)$. Hence, by Theorem 2.1, since the averaged equation (3.3) has k simple zeros $r_1, \ldots, r_k \in (0,1)$ we conclude that the differential equation (3.1) has k limit cycles. This completes the proof of Theorem 1.1.

4. Proof of Corollary 1.1

We consider for a given integer $n > 1$ the differential equation

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon \frac{a(\theta)}{a_0(\theta) + a_1(\theta) r + \varepsilon\big(a_2(\theta) r^2 + \ldots + a_n(\theta) r^n\big)}, \\ &= \varepsilon \frac{a(\theta)}{a_0(\theta) + a_1(\theta) r} + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{4.1}$$

Taking again the expressions (3.2) for the functions $a(\theta)$, $a_0(\theta)$ and $a_1(\theta)$, we can apply Theorem 2.1 to the differential equation (4.1) as we have done for the differential equation (3.1), and we also obtain that the differential equation (4.1) has k limit cycles. This completes the proof of the Corollary 1.1.

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