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NOTES ON EXACT TRAVELLING WAVE SOLUTIONS FOR A LONG WAVE-SHORT WAVE MODEL*

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Abstract This paper considers a long wave-short wave model. It shows that under three different parameter conditions, this system has three types of exact explicit travelling wave solutions. Their parametric representations have been given.

Keywords Travelling wave solution, exact parametric representation, long wave-short wave model, nonlinear wave equation.

MSC(2010) 34C60, 35Q51, 35C05, 35C07.

1. Introduction

More recently, Li and Liu [2] considered the following long wave-short wave model:

$$\begin{aligned} A_t &= 2\sigma(|B|^2)_x, \\ B_t &= iB_{xx} - A_x B + iA^2 - 2i\sigma B|B|^2, \end{aligned} \tag{1.1}$$

where $A = A(x,t)$ represents the amplitude of the long wave and $B(x,t)$ the envelope of the short wave. This equation was studied by Newell [3,4], Chowdhury and Tataronis [1] et al from different points of view (see [2] and cited reference therein). The authors of [2] stated that "While this is an important model, to our knowledge, not much is known for its solutions." In this short article, we discuss the existence of the exact travelling wave solutions of system (1.1).

2. Three exact travelling wave solutions of system (1.1)

We consider the travelling wave solution of the form:

$$A = A(x,t) = A(x-ct) \equiv A(\xi), \quad B(x,t) = \phi(\xi)e^{i(\kappa x+\omega t)}, \ \ \xi = x - ct. \tag{2.1}$$

Substituting (2.1) into the first equation of system (1.1) and integrating the obtained equation once, we have

$$A(\xi) = -\frac{2\sigma}{c}|B|^2 + g = -\frac{2\sigma}{c}\phi^2(\xi) + g, \tag{2.2}$$

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where g is an integral constant. Substituting (2.2) into the second equation of (1.1) and decomposing the real and imaginary parts, we obtain

$$\phi' = \frac{2\sigma}{c(2\kappa - c)}\phi^3,$$
$$\phi'' = (\kappa^2 + \omega - g^2)\phi + 2\sigma\left(1 + \frac{2g}{c}\right)\phi^3 - \frac{4\sigma^2}{c^2}\phi^5, \tag{2.3}$$

where "$'$" is the derivative with respect to ξ.

Two equations of (2.3) implies that

$$\phi' = \frac{d\phi}{d\xi} = \frac{2\sigma(2\kappa - c)}{3c\phi}(\alpha + \beta\phi^2 - \phi^4), \tag{2.4}$$

where $\alpha = \frac{c^2(\kappa^2+\omega-g^2)}{4\sigma^2}$, $\beta = \frac{c(c+2g)}{2\sigma}$.

Thus, we have

$$\frac{2\sigma(2\kappa - c)}{3c}\xi = \int_0^{\phi} \frac{\phi d\phi}{\alpha + \beta\phi^2 - \phi^4} = \int_0^{\psi} \frac{d\psi}{2(\alpha + \beta\psi - \psi^2)}, \tag{2.5}$$

where $\psi = \phi^2$.

Let $q = -(4\alpha + \beta^2) = -\frac{c^2}{\sigma^2}\left(\kappa^2 + \frac{1}{4}c^2 + \omega + gc\right) \equiv -\frac{c^2}{\sigma^2}\Delta$. We see from (6) that the following conclusions hold.

1. When $\Delta > 0$,

$$\phi(\xi) = (\psi(\xi))^{\frac{1}{2}} = \left(\frac{1}{2}\beta + \frac{|c|}{2|\sigma|}\sqrt{\Delta}\tanh(\Omega\xi)\right)^{\frac{1}{2}}, \tag{2.6}$$

where $\Omega = \frac{2}{3}(2\kappa - c)\sqrt{\Delta}$.

2. When $\Delta < 0$,

$$\phi(\xi) = (\psi(\xi))^{\frac{1}{2}} = \left(\frac{1}{2}\beta - \frac{|c|}{2|\sigma|}\sqrt{|\Delta|}\tan(\Omega_1\xi)\right)^{\frac{1}{2}}, \tag{2.7}$$

where $\Omega_1 = \frac{2}{3}(2\kappa - c)\sqrt{|\Delta|}$.

3. When $\Delta = 0$, i.e., $\alpha < 0, \beta^2 = -4\alpha$, in this case, $\alpha + \beta\psi - \psi^2 = -\left(\frac{\beta}{2} - \psi\right)^2$. Hence, (6) follows that

$$\phi(\xi) = (\psi(\xi))^{\frac{1}{2}} = \left(\frac{\beta}{2}\left(1 - \frac{1}{\Omega_0\xi + 1}\right)\right)^{\frac{1}{2}}, \tag{2.8}$$

where $\Omega_0 = \frac{2\beta\sigma(2\kappa-c)}{3c}$.

To sum up, we obtain the following result.

Theorem 2.1. *The long wave-short wave model* (1.1) *has the exact travelling wave solutions as follows:*

(1) For $\Delta = \kappa^2 + \frac{1}{4}c^2 + \omega + gc > 0$,

$$A(x,t) = A(\xi) = -\left(\frac{1}{2}(c + 2g) + \sqrt{\Delta}\tanh(\Omega\xi)\right) + g,$$
$$B(x,t) = B(\xi) = \left(\frac{1}{2}\beta + \frac{|c|}{2|\sigma|}\sqrt{\Delta}\tanh(\Omega\xi)\right)^{\frac{1}{2}} e^{i(\kappa x + \omega t)}. \tag{2.9}$$

(2) For $\Delta = \kappa^2 + \frac{1}{4}c^2 + \omega + gc < 0$,

$$A(x,t) = A(\xi) = \left(-\frac{1}{2}(c+2g) + \sqrt{|\Delta|}\tan(\Omega_1\xi)\right) + g,$$

$$B(x,t) = B(\xi) = \left(\frac{1}{2}\beta - \frac{|c|}{2|\sigma|}\sqrt{|\Delta|}\tan(\Omega_1\xi)\right)^{\frac{1}{2}} e^{i(\kappa x+\omega t)}. \tag{2.10}$$

(3) For $\Delta = \kappa^2 + \frac{1}{4}c^2 + \omega + gc = 0$,

$$A(x,t) = A(\xi) = -\frac{1}{2}(c+2g)\left(1 - \frac{1}{\Omega_0\xi+1}\right) + g,$$

$$B(x,t) = B(\xi) = \left(\frac{\beta}{2}\left(1 - \frac{1}{\Omega_0\xi+1}\right)\right)^{\frac{1}{2}} e^{i(\kappa x+\omega t)}. \tag{2.11}$$

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