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A PRIORI BOUNDS AND WELL-POSEDNESS OF A SYSTEM ASSOCIATED WITH UNSTEADY BOUNDARY LAYER FLOWS *

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Abstract An integral equation with singularities is introduced to characterize unsteady laminar boundary layer flows and some properties of solutions of this integral equation are investigated. Utilizing these properties, a priori bounds are obtained for the skin friction function and the similarity stream function and the well-posedness of solutions is proved.

Keywords Boundary layer, unsteady flows, nonautonomous, a priori bounds, well-posedness.

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1. Introduction

The following nonautonomous system

$$f''' + (f - \frac{A\eta}{2})f'' - (A + f')f' = 0 \qquad \text{on } (0,\infty), \tag{1.1}$$

$$\frac{1}{P_r}\theta'' + (f - \frac{A\eta}{2})\theta' - (A + f')\theta = 0 \qquad \text{on } (0,\infty) \tag{1.2}$$

subject to the boundary conditions

$$f(0) = f_0, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{1.3}$$

and

$$\theta(0) = 1, \quad \theta(\infty) = 0, \tag{1.4}$$

has been used to study unsteady laminar boundary layer flows [5, 9] and is reduced from the governing unsteady two-dimensional Navier-Stokes equations and energy equation via the similarity transformations [5, 18]

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\\
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2},\\
&\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}
\end{aligned}$$

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subject to the boundary conditions

$$u = U_w, \quad v = V_w, \quad T = T_w \quad at \quad y = 0.$$

For more details, one may refer to [5, 18].

In the problem (1.1)-(1.4), f is the similarity stream function, f'' is the skin friction function, $A > 0$ is the unsteady parameter, $f_0 > 0$ (respectively, $f_0 < 0$) corresponds to suction (respectively, injection) of fluid at the surface, and Pr is the Prandtl number. Ishak et al. [5] investigated the problem numerically and the nature of the solutions as the physical parameters are varied. Recently, Paullet [9] studied the existence and uniqueness of the solutions for some (but not all) values of the parameters and obtained a priori bounds for the skin friction coefficient and local Nusselt number.

This paper extends and strengthens the study [5, 9] in two aspects:

(i) A priori bounds are presented for the skin friction function f'' and the similarity stream function f.

(ii) The solutions obtained in [9] are well-posed related to the parameters involved in the system.

It is well-known that numerical and analytical study of similarity solutions is very important in many fields and can provide a standard of comparison without introducing the complication of non-similar solutions. Much attention is always focused on this subject. One may refer to some recent research achievements such as boundary layer flows [3, 8], magnetohydrodynamic(MHD) [2, 4, 8], heat transfer [5, 6, 9, 11, 18], manufacturing polymer sheets, processing paper products [5], designing heat exchangers and chemical processing equipment [1] and the references therein. Also, one may refer to the review and extension of similarity solutions [12, 13].

This paper is organized as follows: in section 2, we establish the relation between the BVP (1.1), (1.3) and an integral equation with singularities and study some properties of solutions of this integral equation. Utilizing these properties, a priori bounds are obtained. In section 3, the well-posedness of solutions is proved.

2. A priori bounds for the skin friction function and the similarity stream function

Notation:

$$P = \{f \in C^3[0, \infty) : f'' < 0 \text{ on } [0, \infty)\},$$
$$Q = \{z \in C[0, 1] : z(t) < 0 \text{ on } (0, 1] \text{ and is strictly decreasing } [0, 1]\}.$$

In this section, we first establish the relation between the BVP (1.1), (1.3) and an integral equation with singularities. Next, we study some properties of solutions of this integral equation and present a priori bounds for the skin friction function f'' and the similarity stream function f.

Lemma 2.1. *Let $f \in P$ satisfy the BVP (1.1), (1.3). Then f'' is strictly increasing on $[0, \infty)$ and $\lim_{\eta \to \infty} f''(\eta) = 0$.*

Proof. From $f''(\eta) < 0$ on $[0, \infty)$ and (1.3), we know that

$$f' \text{ is strictly decreasing on } (0, \infty)$$

and

$$0 < f'(\eta) < 1 \text{ on } (0, \infty).$$

By $f'(\infty) = 0$, we know

$$\sup \lim_{\eta \to \infty} f''(\eta) = 0 \tag{2.1}$$

and there exists $\{\eta_n\}$ such that

$$\eta_n \to \infty, f''(\eta_n) < 0 \text{ and } f''(\eta_n) \to 0.$$

We are going to prove that f'' is strictly increasing on $[0, \infty)$. In fact, if there exist $0 \leq \eta^{(1)} < \eta^{(2)}$ satisfying $f''(\eta^{(1)}) > f''(\eta^{(2)})$. Then, by $f''(\eta_n) \to 0$, there exists n_0 such that

$$\eta_{n_0} > \eta^{(2)} \text{ and } f''(\eta_{n_0}) > f''(\eta^{(2)}).$$

Let $\eta_* \in (\eta^{(1)}, \eta_{n_0})$ satisfy

$$f''(\eta_*) = \min\{f''(\eta) : \eta \in [\eta^{(1)}, \eta_{n_0}]\} < 0.$$

This implies that

$$f'''(\eta_*) = 0 \text{ and } f^{(4)}(\eta_*) \geq 0. \tag{2.2}$$

On the other hand, by (1.1), we have

$$f^{(4)} = -(f - \frac{A\eta}{2})f''' + (f' + \frac{3A}{2})f''. \tag{2.3}$$

It follows from this and $A > 0$ that

$$f^{(4)}(\eta_*) = (f'(\eta_*) + \frac{3A}{2})f''(\eta_*) < 0, \tag{2.4}$$

then we have a contradiction between (2.2) and (2.4). Hence, f'' is increasing on $(0, \infty)$.

If there exist $0 \leq \eta^{(1)} < \eta^{(2)}$ satisfying $f''(\eta^{(1)}) = f''(\eta^{(2)})$, then

$$f''(\eta) = \mathit{constant} \text{ on } \ (\eta^{(1)}, \eta^{(2)}) \ \text{ and } \ f'''(\eta) = 0 \ \text{ on } \ (\eta^{(1)}, \eta^{(2)}).$$

By (2.3), we see

$$f^{(4)} = (f' + \frac{3A}{2})f'' < 0 \text{ on } (\eta^{(1)}, \eta^{(2)}),$$

which is a contradiction. Hence, f'' is strictly increasing on $(0, \infty)$. This, together with (2.1), implies $\lim_{\eta \to \infty} f''(\eta)$ exists and $\lim_{\eta \to \infty} f''(\eta) = 0$. □

Lemma 2.2. *Let $f \in P$ be a solution of the BVP* (1.1), (1.3). *Then the following integral equation*

$$z(t) = Az(t) + tBz(t) - f_0 t \tag{2.5}$$

has a solution $z \in Q$ satisfying

$$z(t) = f''(\eta), \quad -\int_{f'(\eta)}^{1} \frac{1}{z(s)} ds = \eta \text{ and } -\int_t^1 \frac{s}{z(s)} ds + f_0 = f(\eta), \tag{2.6}$$

where

$$Az(t) = \int_0^t \frac{(2s + \frac{A}{2})s}{z(s)} ds, \quad Bz(t) = \int_t^1 \frac{s - \frac{A}{2}}{z(s)} ds.$$

Proof. Let $t = f'(\eta)$. $f''(\eta) < 0$ on $[0, \infty)$ shows that f' is strictly decreasing on $[0, \infty)$, we conclude therefore that the inverse function $\eta = (f')^{-1}(t)$ of $f'(\eta)$ exists, $(f')^{-1}(0) = \infty$ and $(f')^{-1}(1) = 0$.

Let $z(t) = f''(\eta) = f''\big[(f')^{-1}(t)\big]$. Then $z(t)$ is continuous and $z(t) < 0$ on $(0, 1]$.

It follows from Lemma 2.1 that $z(t)$ is strictly decreasing on $[0, 1]$ and

$$\lim_{t \to 0^+} z(t) = \lim_{\eta \to \infty} f''(\eta) = f''(\infty) = z(0) = 0,$$

that is, $z(t)$ is the right continuous at 0 and $z \in C[0, 1]$.

By $t = f'(\eta)$, we have

$$1 = f''(\eta) \frac{d\eta}{dt}, \quad \frac{d\eta}{dt} = \frac{1}{f''(\eta)} = \frac{1}{z(t)}.$$

Integrating the last equality from t to 1, we have

$$\eta = -\int_t^1 \frac{1}{z(s)} ds.$$

By setting $s = f'(\sigma)$, we obtain

$$f(\eta) = \int_0^\eta f'(\sigma) d\sigma + f_0 = -\int_{f'(\eta)}^1 \frac{s}{z(s)} ds + f_0 = -\int_t^1 \frac{s}{z(s)} ds + f_0.$$

Since $f''(\eta) = z(t)$, we see $f'''(\eta) = z'(t) \frac{dt}{d\eta} = z'(t)z(t)$. Substituting η, $f(\eta)$, $f'(\eta)$, $f''(\eta)$ and $f'''(\eta)$ into (1.1) implies

$$z'(t) = \frac{(A + t)t}{z(t)} + \int_t^1 \frac{s - \frac{A}{2}}{z(s)} ds - f_0 \tag{2.7}$$

and $z(0) = 0$.

Integrating the previous equality from 0 to t, noticing that $z(0) = 0$ and

$$\begin{aligned} \int_0^t \int_\sigma^1 \frac{s - \frac{A}{2}}{z(s)} ds d\sigma &= \int_0^t \int_0^s \frac{s - \frac{A}{2}}{z(s)} d\sigma\, ds + \int_t^1 \int_0^t \frac{s - \frac{A}{2}}{z(s)} d\sigma\, ds \\ &= \int_0^t \frac{(s - \frac{A}{2})s}{z(s)} ds + t \int_t^1 \frac{s - \frac{A}{2}}{z(s)} ds, \end{aligned}$$

we have that z satisfies (2.5). □

Since (2.5) contains the improper integrals $Az(t)$ and $Bz(t)$, the following results provide some properties of z, which will be applied to prove the main results of this paper.

Lemma 2.3. *Let $z \in Q$ be a solution of* (2.5). *Then the following facts hold:*

(P_1) *(i)* $z(0) = 0$ *and* $Az(t)$ *converges for* $t \in [0, 1]$.

(ii) $\int_0^1 \frac{1}{z(s)} ds = -\infty$.

(iii) $\lim_{t\to 0^+} Bz(t) = +\infty$ *and* $\lim_{t\to 0^+} tBz(t) = 0$.

(P_2) z *is a solution of* (2.5) *if and only if*

(i) $z(0) = 0$ *and*

$$z'(t) = \frac{(A+t)t}{z(t)} + \int_t^1 \frac{s - \frac{A}{2}}{z(s)} ds - f_0, 0 < t \le 1. \tag{2.8}$$

(ii) $z(0) = 0$, $z'(1) = \dfrac{A+1}{z(1)} - f_0$ *and*

$$z''(t) = -\frac{(A+t)tz'(t)}{z^2(t)} + \frac{3A+2t}{2z(t)}, 0 < t \le 1. \tag{2.9}$$

(P_3)

$$c(f_0, A) \le z(1) \le d(f_0, A), \tag{2.10}$$

where

$$c(f_0, A) = \begin{cases} -\sqrt{A + \frac{4}{3}} & \text{if } f_0 < 0, \\ -f_0 - \sqrt{f_0^2 + A + \frac{4}{3}} & \text{if } f_0 \ge 0. \end{cases}$$

$$d(f_0, A) = \frac{-f_0 - \sqrt{f_0^2 + \frac{8+3A}{3}}}{2}.$$

(P_4) $z(t) \le \sigma(f_0, A)(t)$ *on* $[0, 1]$, *where* $\sigma(f_0, A)(t) = d(f_0, A)t^{\frac{3}{2}}$.

Proof. (P_1) (i) If $z(0) \ne 0$, then

$$z(t) > 0 \quad \text{for } t \in [0, 1].$$

Since both integrands in the right hand in (2.5) are continuous in $[0, 1]$. This implies

$$z(0) = 0,$$

so we have a contradiction.

Since $z(1) = Az(1) - f_0$ and $\dfrac{(2s + \frac{A}{2})s}{z(s)} < 0$ for $s \in (0, 1)$, we know that the Lebesgue integral $Az(1)$ converges and $Az(t)$ exists and is finite for $t \in [0, 1]$. Hence, (i) holds.

(ii) Let $\gamma = \int_0^\infty \frac{1}{z(s)} ds$. If the conclusion is false, then $-\infty < \gamma < 0$.

Since

$$\frac{(2s + \frac{A}{2})s}{z(s)} \ge \frac{(2s + \frac{A}{2})t}{z(s)} \ge \frac{(2 + \frac{A}{2})t}{z(s)} \text{ for } 0 \le s \le t \le 1$$

and

$$\frac{s}{z(s)} \geq \frac{1}{z(s)} \text{ for } 0 \leq s \leq 1,$$

we see

$$\int_0^t \frac{(2s+\frac{A}{2})s}{z(s)}ds \geq (2+\frac{A}{2})t\int_0^t \frac{1}{z(s)}ds \geq (2+\frac{A}{2})t\int_0^\infty \frac{1}{z(s)}ds = (2+\frac{A}{2})t\gamma$$

and

$$\int_t^1 \frac{s}{z(s)}ds \geq \int_t^1 \frac{1}{z(s)}ds \geq \int_0^\infty \frac{1}{z(s)}ds = \gamma.$$

Hence

$$\begin{aligned} 0 > z(t) &\geq \int_0^t \frac{(2s+\frac{A}{2})s}{z(s)}ds + t\int_t^1 \frac{s}{z(s)}ds - f_0 t \\ &\geq (2+\frac{A}{2})t\gamma + t\gamma - f_0 t \\ &= (3\gamma + \frac{A}{2}\gamma - f_0)t := Kt, \end{aligned}$$

where $K = 3\gamma + \frac{A}{2}\gamma - f_0 < 0$. From this, we obtain

$$\int_0^1 \frac{1}{z(s)}ds \leq \int_0^1 \frac{1}{Ks}ds = \frac{1}{K}\int_0^1 \frac{1}{s}ds = -\infty,$$

which contradicts $\gamma > -\infty$. Hence, (ii) holds.

(iii) Since $Az(1)$ converges, we know that $\int_0^1 \frac{s}{z(s)}ds$ converges. This, together with (ii) and $Bz(t) = \int_t^1 \frac{s}{z(s)}ds - \frac{A}{2}\int_t^1 \frac{1}{z(s)}ds$ for $t \in (0,1]$, implies $\lim_{t\to 0^+} Bz(t) = +\infty$. $z(0) = 0$ and $A(0) = 0$ imply $\lim_{t\to 0^+} tBz(t) = 0$.

(P_2) Let z be a solution of (2.5). Then $z(0) = 0$ by (P_1) (i).

(i) Differentiating (2.5) with respect to t, we know (2.8) holds. Conversely, integrating (2.8) from 0 to t and utilizing $z(0) = 0$, we have that (2.5) holds.

(ii) Differentiating (2.5) with respect to t twice, we know that (2.9) holds and $z'(1) = \frac{A+1}{z(1)} - f_0$ by (2.8). Conversely, integrating (2.9) from t to 1 and utilizing $z'(1) = \frac{A+1}{z(1)} - f_0$, we obtain that (2.8) holds and z satisfies (2.5) by (i).

(P_3) Since $z(t)$ is strictly decreasing $[0,1]$, we know $0 > z(t) \geq z(1)$ for $t \in (0,1]$. Notice that $2s^2 + \frac{A}{2} \geq 0$ for $s \in [0,1]$, we see

$$z(1) = \int_0^1 \frac{(2s+\frac{A}{2})s}{z(s)}ds - f_0 \leq \frac{1}{z(1)}\int_0^1 (2s+\frac{A}{2})sds - f_0$$

and $z^2(1) + f_0 z(1) - (\frac{2}{3}+\frac{A}{4}) \geq 0$. This implies that the right side of (2.10) holds.

Since $z(t) < 0$ for $t \in (0,1]$ and $A > 0$, we obtain therefore by (2.7)

$$z'(t) = \frac{(A+t)t}{z(t)} + \int_t^1 \frac{s-\frac{A}{2}}{z(s)}ds - f_0 \geq \frac{(A+t)t}{z(t)} + \int_t^1 \frac{s}{z(s)}ds - f_0$$

and

$$z'(t) \geq \frac{(A+t)t}{z(t)} + \frac{1}{z(t)}\int_t^1 sds - f_0.$$

From this, we see

$$z(t)z'(t) \leq (A+t)t + \frac{1-t^2}{2} - f_0 z(t) = At + \frac{1+t^2}{2} - f_0 z(t).$$

Integrating this inequality from 0 to 1 and utilizing $z(0) = 0$, we have

$$z^2(1) \leq \frac{3A+4}{3} - 2f_0\int_0^1 z(s)ds.$$

If $f_0 < 0$, by $\int_0^1 z(s)ds < 0$, we obtain

$$z^2(1) \leq \frac{3A+4}{3}$$

and

$$z(1) \geq -\sqrt{\frac{3A+4}{3}}.$$

If $f_0 \geq 0$, by $0 > \int_0^1 z(s)ds \geq \int_0^1 z(1)ds = z(1)$, we have

$$z^2(1) \leq \frac{3A+4}{3} - 2f_0 z(1),$$

and

$$z(1) \geq c(f_0, A).$$

Hence, the left side of (2.10) holds.

(P_4) We proceed by contradiction. If there exists $t \in [0,1]$ such that $z(t) > \sigma(f_0, A)(t)$, then $t \in (0,1)$ by $z(0) = 0 = \sigma(f_0, A)(0)$ and $z(1) \leq d(f_0, A) = \sigma(f_0, A)(1)$.

Let $\varphi(t) = z(t) - \sigma(f_0, A)(t)$. Then $\varphi(0) = 0$, $\varphi(1) \leq 0$. Let $\xi \in (0,1)$ satisfy

$$\varphi(\xi) = \max\{\varphi(t), t \in [0,1]\} > 0.$$

Then $z(\xi) > \sigma(f_0, A)(\xi)$, $\varphi'(\xi) = 0$ and $\varphi''(\xi) \leq 0$. This implies $z'(\xi) = \frac{3}{2}d(f_0, A)\xi^{\frac{1}{2}}$ and $z''(\xi) \leq \sigma''(f_0, A)(\xi) = \frac{3}{4}d(f_0, A)\xi^{-\frac{1}{2}} < 0$. On the other hand, by (2.9), we see

$$\begin{aligned} z''(\xi) &= -\frac{(A+\xi)\xi z'(\xi)}{z^2(\xi)} + \frac{3A+2\xi}{2z(\xi)} \\ &= \frac{(3A+2\xi)z(\xi) - 2(A+\xi)\xi z'(\xi)}{2z^2(\xi)} := \frac{M(\xi)}{z^2(\xi)}, \end{aligned}$$

where $M(t) = (3A+2t)z(t) - 2(A+t)tz'(t)$.

It follows from $z(\xi) > \sigma(f_0, A)(\xi)$ that

$$\begin{aligned} M(\xi) &= (3A+2\xi)z(\xi) - 2(A+\xi)\xi z'(\xi) \\ &> (3A+2\xi)\sigma(f_0, A)(\xi) - 3(A+\xi)\xi d(f_0, A)\xi^{\frac{1}{2}} = -d(f_0, A)\xi^{\frac{5}{2}} > 0, \end{aligned}$$

which contradicts $z''(\xi) \le 0$. □

Lemma 2.2 shows that if the BVP (1.1), (1.3) has a solution $f \in P$, then (2.5) has a solution $z \in Q$. Conversely, we may construct a solution $f \in P$ of the BVP (1.1), (1.3) via a solution $z \in Q$ of (2.5).

Lemma 2.4. *Let $z \in Q$ be a solution of* (2.5). *Then the BVP* (1.1), (1.3) *has a solution $f \in P$ satisfying*

$$\eta = -\int_{f'(\eta)}^{1} \frac{1}{z(s)} ds, \quad f(\eta) = -\int_{t}^{1} \frac{s}{z(s)} ds + f_0 \tag{2.11}$$

and

$$|f''(\eta)| \le |c(A, f_0)|, \quad |f(\eta)| \le \frac{2}{|d(A, f_0)|} + |f_0| \text{ on } [0, \infty). \tag{2.12}$$

Proof. Let

$$\eta = \eta(t) = -\int_{t}^{1} \frac{1}{z(s)} ds \text{ on } (0, 1]. \tag{2.13}$$

Then $\eta(t)$ is continuous and strictly increasing on $(0, 1]$. By Lemma 2.3 (P_1) (ii), we have $\eta(0) = \infty$ and $\eta(1) = 0$.

Let $t = h(\eta)$ be the inverse function of $\eta = \eta(t)$. Then $h(0) = 1$ and $h(\infty) = 0$. We define a function

$$f(\eta) = \int_0^{\eta} h(s) ds + f_0.$$

Then

$$f(0) = f_0, \; f'(\eta) = h(\eta) = t, \; f'(0) = h(0) = 1 \text{ and } f'(\infty) = h(\infty) = 0.$$

Differentiating (2.13) with respect to t, we have $\dfrac{d\eta}{dt} = \dfrac{1}{z(t)}$ and by $s = f'(\sigma)$

$$f(\eta) - f(0) = \int_0^{\eta} f'(\sigma) d\sigma = \int_1^{t} \frac{s}{z(s)} ds.$$

Hence

$$\eta = -\int_{f'(\eta)}^{1} \frac{1}{z(s)} ds$$

and

$$f(\eta) = \int_0^{\eta} f'(\sigma) d\sigma + f_0 = -\int_t^1 \frac{s}{z(s)} ds + f_0.$$

By the differentiation of $f'(\eta) = t$ with respect to η, we have

$$f''(\eta) = \frac{dt}{d\eta} = z(t), \quad f'''(\eta) = \frac{dt}{d\eta} = z'(t)\frac{dt}{d\eta} = z'(t)z(t).$$

Substituting η, $f(\eta)$, $f'(\eta)$, $f''(\eta)$, $f'''(\eta)$ into (1.1) and utilizing (2.8), we obtain

$$\begin{aligned} & f''' + (f - \frac{A\eta}{2})f'' - (A + f')f' \\ =& z'(t)z(t) + \big[-\int_t^1 \frac{s}{z(s)} ds + f_0 + \frac{A}{2}\int_t^1 \frac{1}{z(s)} ds\big] z(t) - (A + t)t \\ =& z(t)\big[z'(t) - \frac{(A + t)t}{z(t)} - \int_t^1 \frac{s - \frac{A}{2}}{z(s)} ds + f_0\big] = 0. \end{aligned}$$

It follows from Lemma 2.3 (P_3) and the decrease in z that $z(1) \leq z(t) \leq 0$ on $[0,1]$. This implies $|f''(\eta)| = |z(t)| \leq |z(1)| \leq |c(f_0, A)|$ for $\eta \in [0, \infty)$.

By Lemma 2.3 (P_4),

$$|f(\eta)| \leq \int_0^1 \frac{s}{|z(s)|} ds + |f_0| \leq \int_0^1 \frac{s}{|\sigma(f_0, A)(s)|} ds + |f_0|$$

and

$$|f(\eta)| \leq \int_0^1 \frac{1}{|d(f_0, A)s^{\frac{1}{2}}|} ds + |f_0| = \frac{2}{|d(A, f_0)|} + |f_0|.$$

Hence, (2.12) holds. □

By Lemmas 2.2 and 2.4, we establish the relation between the BVP (1.1), (1.3) and (2.5) as follows.

Theorem 2.1. *(i) If the BVP* (1.1), (1.3) *has a solution $f \in P$, then* (2.5) *has a solution $z \in Q$ satisfying* (2.6).

(ii) If (2.5) *has a solution $z \in Q$, then the BVP* (1.1), (1.3) *has a solution $f \in P$ satisfying* (2.11) *and* (2.12).

Remark 2.1. By Theorem 2.1, for $f \in P$ satisfying the BVP (1.1), (1.3), we obtain a priori bounds (2.12) for the skin friction function f'' and the stream function f. Hence, Theorem 2.1 extends the study on f and f'' in [5, 9].

3. The well-posedness of the system (1.1)-(1.4)

Notation:

$$\Theta = \{\theta \in C^2[0, \infty) : \theta' < 0 \text{ and } \theta'' \geq 0 \text{ on } [0, \infty)\}.$$

The following Lemma can be found in [9](see, Theorems 1, 2 and 5).

Lemma 3.1. *For any $A > 0$, $-\infty < f_0 < \infty$ and $P_r > 0$, the problem* (1.1)-(1.4) *has a unique solution $(f, \theta) \in P \times \Theta$.*

In order to prove the well-posedness, we need to prove that the solutions in $P \times \Theta$ of (1.1)-(1.4) depend on parameters A, f_0 and P_r continuously. For this, we need the following Helly selection principle (see [10, Corollary 3.2]), where $BV[a, b]$ is the space of all the bounded variation functions defined on $[a, b]$ and V_u is the total variation of $u \in BV[a, b]$.

Lemma 3.2. *Let $\{u_n(t)\} \subset BV[a, b]$ be an infinite sequence. Assume that $\{V_{u_n}\}$ is bounded and there exists $K > 0$ such that $|u_n(t)| \leq K$ for $t \in [a, b]$ and $n \in \mathbb{N}$. Then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in BV[a, b]$ such that $u_{n_k}(t) \to u(t)$ for each $t \in [a, b]$.*

Let $A_n > 0$, $A_n \to A > 0$, $f_0^{(n)} \to f_0$, $P_r^{(n)} > 0$, $P_r^{(n)} \to P_r > 0$, $(f_n, \theta_n) \in P \times \Theta$ denote the solutions of (1.1)-(1.4) when $A = A_n$, $f_0 = f_0^{(n)}$ and $P_r = P_r^{(n)}$, and $(f, \theta) \in P \times \Theta$ denote the solutions of (1.1)-(1.4). We prove the following theorem.

Theorem 3.1. *(i) $\|f_n - f\| \to 0$ and (ii) $\|\theta_n - \theta\| \to 0$ $(n \to \infty)$, where $\|f\| = \sup\{|f(\eta)| : \eta \in [0, \infty)\}$.*

Proof. It follows from Theorem 2.1 (i) that there exists $z_n \in Q$ which is a solution of (2.5) satisfying (2.6), that is,

$$\eta = -\int_{f'_n(\eta)}^{1} \frac{1}{z_n(s)} ds, \quad f_n(\eta) = -\int_{f'_n(\eta)}^{1} \frac{s}{z_n(s)} ds + f_0^{(n)}, \quad f''_n(\eta) = z_n(t).$$

By Lemma 2.3 (P_3), there exist constants $a > 0$, $c < d < 0$ and $N > 0$ such that $c \le c(A_n, f_0^{(n)}) < d(A_n, f_0^{(n)}) \le d$, $a < A_n < 2a$ for $n \ge N$.

We first prove the following fact:

$$(P) \quad \text{if } \lim_{n\to\infty} f'_n(\eta) = f'(\eta) \text{ for } \eta \in [0, \infty), \text{ then } \|f'_n - f'\| \to 0 (n \to \infty).$$

If it is false, then there exist $\varepsilon > 0$ and $\{\eta_{n_k}\}$ such that

$$|f'_{n_k}(\eta_{n_k}) - f'(\eta_{n_k})| \ge \varepsilon \text{ for all } k.$$

Lemma 2.3 (P_3), together with the decrease in z, implies that

$$|f''(\eta)| = |z(t)| \le |z(1)| \le |c| \quad \text{for} \quad n \ge N,$$

that is, $\{f''(\eta)\}$ is bounded on $[0, \infty)$. This shows that

$$\{f'_n(\eta)\} \quad \text{is compact set} \quad \text{on} \quad [0, \widetilde{\eta}] \text{ for each fixed } \widetilde{\eta} \in [0, \infty)$$

and

$$|f'_{n_k}(\eta) - f'(\eta)| \to 0 \quad \text{uniformly on } [0, \widetilde{\eta}].$$

From this, we obtain $\eta_{n_k} \to \infty (n_k \to \infty)$.

Without loss of generality, we may assume $f'_{n_k}(\eta_{n_k}) - f'(\eta_{n_k}) \ge \varepsilon$ for all k. By the decrease in f'_{n_k}, we know when $\eta_{n_k} \ge \eta$

$$f'_{n_k}(\eta) \ge f'_{n_k}(\eta_{n_k}) \ge f'(\eta_{n_k}) + \varepsilon \ge \varepsilon.$$

Taking limit when $k \to \infty$, we have $f'(\eta) \ge \varepsilon$.

Letting $\eta \to \infty$, we obtain $0 = f'(\infty) \ge \varepsilon$, which is a contradiction. Hence, (P) holds.

Next, we start to prove the well-posedness.

(i) If $\|f_n - f\| \to 0$ is false, then there exist $\varepsilon_0 > 0$ and $\eta_{n_k} \in (0, \infty)$ such that $|f_{n_k}(\eta_{n_k}) - f(\eta_{n_k})| \ge \varepsilon_0$. Without loss of generality, we may assume $|f_n(\eta_n) - f(\eta_n)| \ge \varepsilon_0$.

By Lemmas 3.2 and 2.3 (P_4), there exist a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ and $z \in BV[0, 1]$ such that

$$\lim_{n_i\to\infty} z_{n_i}(t) = z(t) \quad \text{for} \quad t \in [0, 1],$$

$$z_{n_i}(t) \le dt^{\frac{3}{2}} \quad \text{on } [0, 1] \quad \text{for } n_i \ge N$$

and

$$z(t) \le dt^{\frac{3}{2}} \text{ on } [0, 1].$$

Noticing that $\dfrac{1}{|z_{n_i}(s)|} \le \dfrac{1}{|d| s^{\frac{3}{2}}}$ on $(0, 1]$, then for $n_i \ge N$ and $s > 0$

$$\Big| \frac{(2s + \frac{A_{n_i}}{2})s}{z_{n_i}(s)} \Big| \le \frac{2s + a}{|d| s^{\frac{1}{2}}},$$

$$t\Big|\frac{s-\frac{A_{n_i}}{2}}{z_{n_i}(s)}\Big| \le t\Big|\frac{s+\frac{A_{n_i}}{2}}{z_{n_i}(s)}\Big| \le s\Big|\frac{(s+\frac{A_{n_i}}{2})}{z_{n_i}(s)}\Big| \le \frac{|s+a|}{|d|s^{\frac{1}{2}}} \quad \text{for} \quad t \le s,$$

the Lebesgue's Dominated Convergence Theorem with the dominated function $F(s) = \dfrac{2+a}{|d|s^{\frac{1}{2}}}$ for $s \in [0,1]$ implies

$$\lim_{i\to\infty} Az_{n_i}(t) = Az(t) \text{ and } \lim_{i\to\infty} tBz_{n_i}(t) = tBz(t) \text{ for } t \in [0,1].$$

Hence z is a solution of (2.5) and $z \in Q$.

By Theorem 2.1 (ii), the BVP (1.1), (1.3) has a solution $\widetilde{f}$ in P and

$$\eta = -\int_{\widetilde{f}'(\eta)}^1 \frac{1}{z(s)}ds, \quad \widetilde{f}(\eta) = -\int_{\widetilde{f}'(\eta)}^1 \frac{s}{z(s)}ds + \widetilde{f}_0.$$

It follows from Lemma 3.1 that $\widetilde{f} = f$, we get therefore that

$$\eta = -\int_{f'(\eta)}^1 \frac{1}{z(s)}ds \text{ and } f(\eta) = -\int_{f'(\eta)}^1 \frac{s}{z(s)}ds + f_0.$$

Since

$$-\int_{f'(\eta)}^1 \frac{1}{z(s)}ds = \eta = -\int_{f'_{n_i}(\eta)}^1 \frac{1}{z_{n_i}(s)}ds = -\int_{f'(\eta)}^1 \frac{1}{z_{n_i}(s)}ds - \int_{f'_{n_i}(\eta)}^{f'(\eta)} \frac{1}{z_{n_i}(s)}ds$$

and

$$\lim_{i\to\infty}\int_{f'(\eta)}^1 \frac{1}{z_{n_i}(s)}ds = \int_{f'(\eta)}^1 \frac{1}{z(s)}ds \quad \text{for} \quad \eta \in [0,\infty),$$

we have $\lim_{i\to\infty}\int_{f'_{n_i}(\eta)}^{f'(\eta)} \frac{1}{z_{n_i}(s)}ds = 0$ for $\eta \in [0,\infty)$.

By

$$\frac{1}{|c|}|f'(\eta) - f'_{n_i}(\eta)| = \Big|\int_{f'_{n_i}(\eta)}^{f'(\eta)} \frac{1}{c}ds\Big| \le \Big|\int_{f'_{n_i}(\eta)}^{f'(\eta)} \frac{1}{z_n(s)}ds\Big|,$$

we obtain $f'_{n_i}(\eta) \to f'(\eta)(i\to\infty)$ and $\|f'_{n_i} - f'\| \to 0(i\to\infty)$ by (P).

Noticing that $\dfrac{s}{|z_{n_i}(s)|} \le \dfrac{1}{|d|s^{\frac{1}{2}}} := F_0(s)(n_i \ge N)$ and $\int_0^1 F_0(s)ds < \infty$, the absolutely continuity of the Lebesgue integral implies there exists $\delta > 0$ such that $|\int_{t_1}^{t_2} F_0(s)ds| < \dfrac{\varepsilon_0}{6}$ when $|t_2 - t_1| < \delta$. The Lebesgue's Dominated Convergence Theorem with the dominated function $F_0(s)$ implies $\lim_{n_i\to\infty}\int_0^1 \big|\frac{s}{z_{n_i}(s)} - \frac{s}{z(s)}\big|ds = 0$. We choose $N_0 \ge N$ such that when $n_i \ge N_0$

$$|f_0^{(n_i)} - f_0| < \frac{\varepsilon_0}{6}, |f'_{n_i}(\eta) - f'(\eta)| \le \|f'_{n_i} - f'\| < \delta \text{ and } \int_0^1 \Big|\frac{s}{z_{n_i}(s)} - \frac{s}{z(s)}\Big|ds < \frac{\varepsilon_0}{6}.$$

By

$$f_{n_i}(\eta) = -\int_{f'_{n_i}(\eta)}^1 \frac{s}{z_{n_i}(s)}ds + f_0^{(n_i)} = -\int_0^1 \frac{s}{z_{n_i}(s)}ds + \int_0^{f'_{n_i}(\eta)} \frac{s}{z_{n_i}(s)}ds + f_0^{(n_i)},$$

$$f(\eta) = -\int_{f'(\eta)}^1 \frac{s}{z(s)}ds + f_0 = -\int_0^1 \frac{s}{z(s)}ds + \int_0^{f'(\eta)} \frac{s}{z(s)}ds + f_0,$$

we have for $n_i \geq N_0$

$$\begin{aligned}
&|f_{n_i}(\eta) - f(\eta)| \\
\leq& \Big| \int_0^1 \Big(\frac{s}{z_{n_i}(s)} - \frac{s}{z(s)}\Big) ds \Big| + \Big| \int_0^{f'_{n_i}(\eta)} \frac{s}{z_{n_i}(s)} ds - \int_0^{f'(\eta)} \frac{s}{z(s)} ds \Big| + |f_0^{(n_i)} - f_0| \\
=& \Big| \int_0^1 \Big(\frac{s}{z_{n_i}(s)} - \frac{s}{z(s)}\Big) ds \Big| + \Big| \int_0^{f'(\eta)} \Big(\frac{s}{z_{n_i}(s)} - \frac{s}{z(s)}\Big) ds + \int_{f'(\eta)}^{f'_{n_i}(\eta)} \frac{s}{z_{n_i}(s)} ds \Big| \\
&+ |f_0^{(n_i)} - f_0| \\
\leq& \int_0^1 \Big|\frac{s}{z_{n_i}(s)} - \frac{s}{z(s)}\Big| ds + \int_0^{f'(\eta)} \Big|\frac{s}{z_{n_i}(s)} - \frac{s}{z(s)}\Big| ds + \Big| \int_{f'(\eta)}^{f'_{n_i}(\eta)} \frac{s}{z_{n_i}(s)} ds \Big| \\
&+ |f_0^{(n_i)} - f_0| \\
\leq& 2\int_0^1 \Big|\frac{s}{z_{n_i}(s)} - \frac{s}{z(s)}\Big| ds + \int_{f'(\eta)}^{f'_{n_i}(\eta)} \frac{s}{|z_{n_i}(s)|} ds + |f_0^{(n_i)} - f_0| \\
\leq& 2\int_0^1 \Big|\frac{s}{z_{n_i}(s)} - \frac{s}{z(s)}\Big| ds + \Big| \int_{f'(\eta)}^{f'_{n_i}(\eta)} F_0(s) ds \Big| + |f_0^{(n_i)} - f_0| \\
<& \frac{\varepsilon_0}{3} + \frac{\varepsilon_0}{6} + \frac{\varepsilon_0}{6} = \frac{2\varepsilon_0}{3} < \varepsilon_0,
\end{aligned}$$

which contradicts $|f_n(\eta_n) - f(\eta_n)| \geq \varepsilon_0$. Hence, (i) holds.

(ii) If $\|\theta_n - \theta\| \to 0$ is false, then there exist $\varepsilon_0 > 0$ and $\eta_{n_k} \in (0, \infty)$ such that $|\theta_{n_k}(\eta_{n_k}) - \theta(\eta_{n_k})| \geq \varepsilon_0$. Without loss of generality, we may assume $\theta_n(\eta_n) - \theta(\eta_n) \geq \varepsilon_0$.

It follows from $\theta'' \geq 0$ that θ' is increasing on $[0, \infty)$. This implies that $\theta'(\infty)$ exists and $\theta'(\infty) = 0$ by $\theta(\infty) = 0$.

Since

$$0 \leq \frac{1}{P_r}\theta'' = -f\theta' + \frac{A\eta}{2}\theta' + (A + f')\theta,$$

and the boundedness of f(see (2.12)), we have $\underline{\lim}_{\eta\to\infty} \frac{A\eta}{2}\theta' \geq 0$, *i.e.*,

$$\underline{\lim}_{\eta\to\infty} \eta\theta' \geq 0.$$

On the other hand, $\theta' \leq 0$ on $[0, \infty)$ implies $\eta\theta' \leq 0$ for $\eta \geq 0$. Then

$$\overline{\lim}_{\eta\to\infty} \eta\theta' \leq 0.$$

Hence, $\lim_{\eta\to\infty} \eta\theta' = 0$. From this and (1.2), we know that $\eta\theta'$ and θ'' are bounded on $[0, \infty)$. This, together with the boundedness of f, θ, θ' and (i), implies that there exists $N > 0$ such that when $n \geq N$

$$\Big|[(f - f_n) + \frac{A_n - A}{2}\eta]\theta' + [(A_n - A) + (f'_n - f')]\theta\Big| < \frac{A\varepsilon_0}{6} \text{ for } \eta \geq 0 \tag{3.1}$$

and

$$\Big|\frac{1}{P_r^{(n)}} - \frac{1}{P_r}\Big|\theta'' < \frac{A\varepsilon_0}{6} \text{ for } \eta \geq 0, \quad A_n > \frac{A}{2} > 0. \tag{3.2}$$

Let $\widetilde{\theta}_n = \theta_n - \theta$. Then

$$\begin{aligned}
&(\frac{1}{P_r^{(n)}} - \frac{1}{P_r})\theta_n'' + \frac{1}{P_r}\widetilde{\theta}_n'' = \frac{1}{P_r^{(n)}}\theta_n'' - \frac{1}{P_r}\theta'' \\
&= - \big[(f_n - \frac{A_n\eta}{2})\theta_n' - (A_n + f_n')\theta_n\big] + \big[(f - \frac{A\eta}{2})\theta' - (A + f')\theta\big] \\
&= - (f_n - \frac{A_n\eta}{2})\widetilde{\theta}_n' + (A_n + f_n')\widetilde{\theta}_n \\
&\quad + \big[(f - f_n) + \frac{A_n - A}{2}\eta\big]\theta' + \big[(A_n - A) + (f_n' - f)\big]\theta. \qquad (3.3)
\end{aligned}$$

By $\widetilde{\theta}_n(0) = 0$ and $\widetilde{\theta}_n(\infty) = 0$, we know that there exists $\widetilde{\eta}_n > \eta_n$ such that $\theta_n(\widetilde{\eta}_n) < \varepsilon_0$.

Let $\xi_n \in (0, \widetilde{\eta}_n)$ such that

$$\widetilde{\theta}_n(\xi_n) = \max\{\widetilde{\theta}_n(\eta), \eta \in [0, \widetilde{\eta}_n]\}.$$

Then $\widetilde{\theta}_n(\xi_n) \geq \widetilde{\theta}_n(\eta_n) \geq \varepsilon_0$, $\widetilde{\theta}_n'(\xi_n) = 0$ and $\widetilde{\theta}_n''(\xi_n) \leq 0$.

On the other hand, by (3.1), (3.2), (3.3), $f_n'(\xi_n) > 0$ and $\theta_n'(\xi_n) = 0$, we have $n \geq N$

$$\begin{aligned}
(\frac{1}{P_r^{(n)}} - \frac{1}{P_r})\theta_n''(\xi_n) + \frac{1}{P_r}\widetilde{\theta}_n''(\xi_n) &\geq [A_n + f_n'(\xi_n)]\widetilde{\theta}_n(\xi_n) - \frac{A\varepsilon_0}{6} \\
&> A_n\widetilde{\theta}_n(\xi_n) - \frac{A\varepsilon_0}{6}.
\end{aligned}$$

Since $\widetilde{\theta}_n''(\xi_n) \leq 0$, we know $0 \leq \theta_n''(\xi_n) \leq \theta''(\xi_n)$. Then by (3.2)

$$\begin{aligned}
\frac{1}{P_r}\widetilde{\theta}_n''(\xi_n) &> \frac{A\varepsilon_0}{2} - \frac{A\varepsilon_0}{6} - (\frac{1}{P_r^{(n)}} - \frac{1}{P_r})\theta_n''(\xi_n) \\
&\geq \frac{A\varepsilon_0}{3} - \Big|\frac{1}{P_r^{(n)}} - \frac{1}{P_r}\Big|\theta''(\xi_n) \\
&> \frac{A\varepsilon_0}{3} - \frac{A\varepsilon_0}{6} = \frac{A\varepsilon_0}{6} > 0,
\end{aligned}$$

which contradicts $\widetilde{\theta}_n''(\xi_n) \leq 0$. □

Remark 3.1. Theorem 3.1 shows the solutions of the problem (1.1)-(1.4) obtained in [9] are well-posed related to the parameters involved in the system. Hence, Theorem 3.1 strengthens the results obtained in [9].

In this paper, utilizing integral methods, we treat some nonautonomous boundary layer problems analytically. Integral methods are used to treat autonomous boundary layer problems, one may refer to [7, 14–17].

References

[1] H.I. Andersson, J.B. Aarseth and B.S. Dandapat, *Heat transfer in a liquid film on an unsteady stretching surface*, Int. J. Heat Mass Tran., 43(2000), 69-74.

[2] A.H. Fatheah and H. Majid, *Analytic solution for MHD Falkner-Skan flow over a Pourous Surface*, J. Appl. Math., 2012, Article ID123185.

[3] M. Guedda, Emad H. Aly and A. Ouahsine, *Analytical and ChPDM analysis of MHD mixed convection over a vertical flat plate embedded in a porous medium filled with water at* $4^{\circ}\ C$, Appl. Math. Model., 35(2011), 5182-5197.

[4] M. Guedda and A. Ouahsine, *Similarity solutions of MHD flow in a saturated porous medium*, Eur. J. Mech. B-Fluid, 33(2012), 87-94.

[5] A. Ishak, R. Nazar and I. Pop, *Heat transfer over an unsteady stretching permeable surface with prescribed wall temperature*, Nonlinear Analysis(RWA), 10(2009), 2909-2913.

[6] S.M. Imran, S. Asghar and M. Mushtaq, *Mixed convection flow over an unsteady stretching surface in a porous medium with heat source*, Math. Probl. Eng., 2012, Article ID485418.

[7] K.Q. Lan and G.C. Yang, *Positive solutions of the Falkner-Skan equation arising in the boundary layer theory*, Canad. Math. Bull., 51(3)(2008), 386-398.

[8] O.D. Makinde and A. Aziz, *MHD mixed convection from a vertical plate embedded in a porous medium with a convective boundary condition*, Int. J. Therm. Sci., 49(2010), 1813-1820.

[9] J.E. Paullet, *Analysis of fluid flow and heat transfer over an unsteady stretching surface*, Nonlinear Analysis(TMA), 75(2012), 4079-4089.

[10] K. Schrader, *A generalization of the Helly selection theorem*, Bull. Amer. Math. Soc., 78(3)(1972), 415-419.

[11] K. Vajravel, K.V. Prasad and Chiu-On Ng, *Unsteady convective boundary layer flow of a viscous fluid at a vertical surface with variable fluid properties*, Nonlinear Analysis(RWA), 14(2013), 455-464.

[12] C.Y. Wang, *Similarity stagnation point solutions of the Navier-Stokes equations-review and extension*, Eur. J. Mech. B-Fluid, 27(2008), 678-683.

[13] C.Y. Wang, *Review of similarity stretching exact solutions of the Navier-Stokes equations*, Eur. J. Mech. B-Fluid, 30(2011), 475-479.

[14] G.C. Yang and K.Q. Lan, *Nonexistence of the reversed flow solutions of the Falkner-Skan equations*, Nonlinear Analysis(TMA), 74(2011), 5327-5339.

[15] G.C. Yang and K.Q. Lan, *Systems of singular intergal equations and applications to existence of revised flow solutions of Falkner-Skan equations*, Comm. Pure Appl. Anal., 12(2013), 2465-2495.

[16] G.C. Yang, L. Zhang and L.F. Dang, *Existence and nonexistence of solutions on opposing mixed convection problems in boundary layer theory*, Eur. J. Mech. B-Fluid, 43(2014), 148-153.

[17] G.C. Yang and K.Q. Lan, *The velocity and shear stress functions of the Falkner-Skan equation arising in boundary layer theory*, J. Math. Anal. Appl., 328(2)(2007), 1297-1308.

[18] Z. Ziabakhsh, G. Domairry, M. Mozaffari and M. Mahbobifar, *Analytical solution of heat transfer over an unsteady stretching surface with prescribed wall temperature*, J. Taiwan Inst. Chem. Eng., 41(2010), 169-177.