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SOLVING NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS USING THE NDM

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Abstract In this research paper, we examine a novel method called the Natural Decomposition Method (NDM). We use the NDM to obtain exact solutions for three different types of nonlinear ordinary differential equations (NLODEs). The NDM is based on the Natural transform method (NTM) and the Adomian decomposition method (ADM). By using the new method, we successfully handle some class of nonlinear ordinary differential equations in a simple and elegant way. The proposed method gives exact solutions in the form of a rapid convergence series. Hence, the Natural Decomposition Method (NDM) is an excellent mathematical tool for solving linear and nonlinear differential equation. One can conclude that the NDM is efficient and easy to use.

Keywords Natural transform, Sumudu transform, Laplace transform, Adomian decomposition method, ordinary differential equations.

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1. Introduction

Nonlinear differential equations have received a considerable amount of interest due to its broad applications. Nonlinear ordinary differential equations play an important role in many branches of applied and pure mathematics and their applications in engineering, applied mechanics, quantum physics, analytical chemistry, astronomy and biology. From last decade, researcher pay attentions towards analytical and numerical solutions of nonlinear ordinary differential equations. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving linear and nonlinear ordinary differential equations.

We present a new integral transform method called the Natural Decomposition Method (NDM) [29], and apply it to find exact solutions to nonlinear ODEs. There are many integral transform methods [3, 13–19] exists in the literature to solve ODEs. The most used one is the Laplace transformation [30]. Other methods used recently to solve PDEs and ODEs, such as, the Sumudu transform [6], the Reduced Differential Transform Method (RDTM) [25–28] and the Elzaki transform [14–19]. Fethi Belgacem and R. Silambarasan [11, 12], used the N–Transform to solve the Maxwell's equation, Bessel's differential equation and linear and nonlinear Klein Gordon Equations and more. Also, Zafar H. Khan and Waqar A. Khan [21], used

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the N–Transform to solve linear differential equations and they presented a table with some properties of the N–Transform of different functions.

We present several applications in the fields of Physics and Engineering to show the efficiency and the accuracy of the NDM. The Adomian decomposition method (ADM) [1,2], proposed by George Adomian, has been applied to a wide class of linear and nonlinear PDEs. For the nonlinear models, the NDM shows reliable results in supplying exact solutions and analytical approximate solutions that converges rapidly to the exact solutions.

Our aim in this paper is to develop an efficient algorithm for numerical computation by natural decomposition method for such problems. The natural decomposition method provides solution as rapidly convergent series.

In this paper, we solve the following NLODEs:

First, consider the nonlinear second order differential equation of the form:

$$\frac{d^2v}{dt^2} + \left(\frac{dv}{dt}\right)^2 + v^2(t) = 1 - \sin(t), \tag{1.1}$$

subject to the initial conditions

$$v(0) = 0, \qquad v'(0) = 1. \tag{1.2}$$

Second, the first order nonlinear ordinary differential equation of the form:

$$\frac{dv}{dt} - 1 = v^2(t), \tag{1.3}$$

subject to the condition

$$v(0) = 0. \tag{1.4}$$

Third, the nonlinear Riccati differential equation of the form:

$$\frac{dv}{dt} = 1 - t^2 + v^2(t), \tag{1.5}$$

subject to the condition

$$v(0) = 0. \tag{1.6}$$

The rest of this paper is organized as follows: In Section 2 and 3, we give some background materials about the NDM. In section 4, we explain the methodology of the NDM. In section 5, we apply the NDM to three test problems to show the effectiveness of our method. Section 6 is for discussion and conclusion of this paper.

2. Basic Idea of The Natural Transform Method

In this section, we present some background about the nature of the Natural Transform Method (NTM). Assume we have a function $f(t),\ t \in (-\infty, \infty)$, and then the general integral transform is defined as follows [11,12]:

$$\Im\left[f(t)\right](s) = \int_{-\infty}^{\infty} K(s,t)\, f(t)\, dt, \tag{2.1}$$

where $K(s,t)$ represent the kernel of the transform, s is the real (complex) number which is independent of t. Note that when $K(s,t)$ is e^{-st}, $t\, J_n(st)$ and $t^{s-1}(st)$, then Eq. (2.1) gives, respectively, Laplace transform, Hankel transform and Mellin transform.

Now, for $f(t),\ t \in (-\infty, \infty)$ consider the integral transforms defined by:

$$\Im\left[f(t)\right](u) = \int_{-\infty}^{\infty} K(t)\, f(ut)\, dt, \tag{2.2}$$

and

$$\Im\left[f(t)\right](s,u) = \int_{-\infty}^{\infty} K(s,t)\, f(ut)\, dt. \tag{2.3}$$

It is worth mentioning here when $K(t) = e^{-t}$, Eq. (2.2) gives the integral Sumudu transform, where the parameter s replaced by u. Moreover, for any value of n the generalized Laplace and Sumudu transform are respectively defined by [11, 12]:

$$\ell\left[f(t)\right] = F(s) = s^n \int_0^{\infty} e^{-s^{n+1}t}\, f(s^n t)\, dt, \tag{2.4}$$

and

$$\mathbb{S}\left[f(t)\right] = G(u) = u^n \int_0^{\infty} e^{-u^n t}\, f(tu^{n+1})\, dt. \tag{2.5}$$

Note that when $n = 0$, Eq. (2.4) and Eq. (2.5) are the Laplace and Sumudu transform, respectively.

3. Definitions and Properties of the N–Transform

The natural transform of the function $f(t)$ for $t \in (-\infty, \infty)$ is defined by [11, 12]:

$$\mathbb{N}\left[f(t)\right] = R(s,u) = \int_{-\infty}^{\infty} e^{-st}\, f(ut)\, dt; \quad s,\, u \in (-\infty, \infty), \tag{3.1}$$

where $\mathbb{N}\left[f(t)\right]$ is the natural transformation of the time function $f(t)$ and the variables s and u are the natural transform variables. Note that Eq. (3.1) can be written in the form [4, 5]:

$$\begin{aligned}
\mathbb{N}\left[f(t)\right] &= \int_{-\infty}^{\infty} e^{-st}\, f(ut)\, dt; \quad s,\, u \in (-\infty, \infty) \\
&= \Big[\int_{-\infty}^{0} e^{-st}\, f(ut)\, dt;\ s,\, u \in (-\infty, 0)\Big] + \Big[\int_{0}^{\infty} e^{-st}\, f(ut)\, dt;\ s,\, u \in (0, \infty)\Big] \\
&= \mathbb{N}^-\left[f(t)\right] + \mathbb{N}^+\left[f(t)\right] \\
&= \mathbb{N}\left[f(t)H(-t)\right] + \mathbb{N}\left[f(t)H(t)\right] \\
&= R^-(s,u) + R^+(s,u),
\end{aligned}$$

where $H(.)$ is the Heaviside function.

It should be mentioned here, if the function $f(t)H(t)$ is defined on the positive real axis, with $t \in \mathbb{R}$, then we define the Natural transform (N–Transform) on the set

$$A = \left\{ \begin{matrix} f(t) : \exists\ M,\, \tau_1,\, \tau_2 > 0, \text{ such that } |f(t)| < M e^{\frac{|t|}{\tau_j}}, \\ \text{if } t \in (-1)^j \times [0, \infty),\ j \in \mathbb{Z}^+ \end{matrix} \right\}$$

as:

$$\mathbb{N}\left[f(t)H(t)\right] = \mathbb{N}^{+}\left[f(t)\right] = R^{+}(s,u) = \int_{0}^{\infty} e^{-st} f(ut)\, dt; \quad s,\, u \in (0,\infty)\,, \qquad (3.2)$$

where $H(.)$ is the Heaviside function. Note if $u = 1$, then Eq. (3.2) can be reduced to the Laplace transform and if $s = 1$, then Eq. (3.2) can be reduced to the Sumudu transform. Now we give some of the N–Transforms and the conversion to Sumudu and Laplace [11, 12].

Table 1. Special N–Transforms and the conversion to Sumudu and Laplace

$f(t)$	$\mathbb{N}\left[f(t)\right]$	$\mathbb{S}\left[f(t)\right]$	$\ell\left[f(t)\right]$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
$\frac{t^{n-1}}{(n-1)!}, n{=}1,2,...$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$
$\sin(t)$	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{1+s^2}$

Remark 3.1. The reader can read more about the Natural transform in [11, 12].

Now we give some important properties of the N–Transforms are given as follows [11, 12, 20, 21]:

Table 2. Properties of N–Transforms

Functional Form	Natural Transform
$y(t)$	$Y(s,u)$
$y(at)$	$\frac{1}{a}Y(s,u)$
$y'(t)$	$\frac{s}{u}Y(s,u) - \frac{y(0)}{u}$
$y''(t)$	$\frac{s^2}{u^2}Y(s,u) - \frac{s}{u^2}y(0) - \frac{y'(0)}{u}$
$\gamma\, y(t) \pm \beta\, v(t)$	$\gamma\, Y(s,u) \pm \beta\, V(s,u)$

4. The Natural Decomposition Method

In this section, we illustrate the applicability of the Natural Decomposition Method to some nonlinear ordinary differential equations.

Methodology of the NDM:

Consider the general nonlinear ordinary differential equation of the form:

$$Lv + R(v) + F(v) = g(t), \tag{4.1}$$

subject to the initial condition

$$v(0) = h(t), \tag{4.2}$$

where L is an operator of the highest derivative, R is the remainder of the differential operator, $g(t)$ is the nonhomogeneous term and $F(v)$ is the nonlinear term.

Suppose L is a differential operator of the first order, then by taking the N–Transform of Eq. (4.1), we have:

$$\frac{sV(s,u)}{u} - \frac{V(0)}{u} + \mathbb{N}^+ [R(v)] + \mathbb{N}^+ [F(v)] = \mathbb{N}^+ [g(t)]. \tag{4.3}$$

By substituting Eq. (4.2) into Eq. (4.3), we obtain:

$$V(s,u) = \frac{h(t)}{s} + \frac{u}{s}\mathbb{N}^+ [g(t)] - \frac{u}{s}\mathbb{N}^+ [R(v) + F(v)]. \tag{4.4}$$

Taking the inverse of the N–Transform of Eq. (4.4), we have:

$$v(t) = G(t) - \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+ [R(v) + F(v)]\right], \tag{4.5}$$

where $G(t)$ is the source term.

We now assume an infinite series solution of the unknown function $v(t)$ of the form:

$$v(t) = \sum_{n=0}^{\infty} v_n(t). \tag{4.6}$$

Then by using Eq. (4.6), we can re-write Eq. (4.5) in the form:

$$\sum_{n=0}^{\infty} v_n(t) = G(t) - \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[R\sum_{n=0}^{\infty} v_n(t) + \sum_{n=0}^{\infty} A_n(t)\right]\right], \tag{4.7}$$

where $A_n(t)$ is an Adomian polynomial which represent the nonlinear term.

Comparing both sides of Eq. (4.7), we can easily build the recursive relation as follows:

$$\begin{aligned}
v_0(t) &= G(t),\\
v_1(t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+ [Rv_0(t) + A_0(t)]\right],\\
v_2(t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+ [Rv_1(t) + A_1(t)]\right],\\
v_3(t) &= -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+ [Rv_2(t) + A_2(t)]\right].
\end{aligned}$$

Eventually, we have the general recursive relation as follows:

$$v_{n+1}(t) = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+ [Rv_n(t) + A_n(t)]\right], \qquad n \geq 0. \tag{4.8}$$

Hence, the exact or approximate solution is given by:

$$v(t) = \sum_{n=0}^{\infty} v_n(t). \tag{4.9}$$

5. Worked Examples

In this section, we employ the NDM to three physical applications and then compare our solutions to existing exact solutions.

Example 5.1. Consider the first order nonlinear differential equation of the form:

$$\frac{d^2v}{dt^2} + \left(\frac{dv}{dt}\right)^2 + v^2(t) = 1 - \sin(t), \tag{5.1}$$

subject to the initial condition

$$v(0) = 0, \qquad v'(0) = 1. \tag{5.2}$$

We begin by taking the N–transform to both sides of Eq. (5.1), we obtain:

$$\frac{s^2V(s,u)}{u^2} - \frac{sV(0)}{u^2} - \frac{v'(0)}{u} + \mathbb{N}^+\left[\left(\frac{dv}{dt}\right)^2\right] + \mathbb{N}^+\left[v^2(t)\right] = \frac{1}{s} - \frac{u}{s^2+u^2}. \tag{5.3}$$

By substituting Eq. (5.2) into Eq. (5.3) we obtain:

$$V(s,u) = \frac{u^2}{s^3} + \frac{u}{s^2+u^2} - \frac{u^2}{s^2}\mathbb{N}^+\left[\left(\frac{dv}{dt}\right)^2 + v^2(t)\right]. \tag{5.4}$$

Then by taking the inverse N–Transform of Eq. (5.4), we have:

$$v(t) = \frac{t^2}{2!} + \sin(t) - \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[\left(\frac{dv}{dt}\right)^2 + v^2(t)\right]\right]. \tag{5.5}$$

We now assume an infinite series solution of the unknown function $v(t)$ of the form:

$$v(t) = \sum_{n=0}^{\infty} v_n(t). \tag{5.6}$$

By using Eq. (5.6), we can re-write Eq. (5.5) as follows:

$$\sum_{n=0}^{\infty} v_n(t) = \frac{t^2}{2!} + \sin(t) - \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n\right]\right], \tag{5.7}$$

where A_n and B_n are the Adomian polynomials of the nonlinear terms $\left(\frac{dv}{dt}\right)^2$ and $v^2(t)$ respectively.

Then by comparing both sides of Eq. (5.7), we can drive the general recursive relation as follows:

$$\begin{aligned}
v_0(t) &= \frac{t^2}{2!} + \sin(t),\\
v_1(t) &= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[A_0 + B_0\right]\right],\\
v_2(t) &= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[A_1 + B_1\right]\right],\\
v_3(t) &= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[A_2 + B_2\right]\right].
\end{aligned}$$

Therefore, the general recursive relation is given by:

$$v_{n+1}(t) = -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[A_n + B_n\right]\right], \qquad n \geq 0. \tag{5.8}$$

Then by using the recursive relation derived in Eq. (5.8), we can easily compute the remaining components of the unknown function $v(t)$ as follows:

$$\begin{aligned} v_1(t) &= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[A_0 + B_0\right]\right] \\ &= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[\left(v_0'\right)^2 + v_0^2\right]\right] \\ &= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[\left(v_0'\right)^2 + v_0^2\right]\right] \\ &= -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}^+\left[1\right]\right] + \cdots \\ &= -\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right] + \cdots \\ &= -\frac{t^2}{2!} + \cdots . \end{aligned}$$

Hence, by canceling the noise terms that appears between $v_0(t)$ and $v_1(t)$, one can see that the non-canceled term of $v_0(t)$ still satisfies the given differential equation which lead to an exact solution of the form:

$$v(t) = \sin(t).$$

The exact solution is in closed agreement with the result obtained by (ADM) [31].

Example 5.2. Consider the first order nonlinear ordinary differential equation of the form [31]:

$$\frac{dv}{dt} - 1 = v^2(t), \tag{5.9}$$

subject to the initial condition

$$v(0) = 0. \tag{5.10}$$

Taking the Natural transform to both sides of Eq. (5.9), we obtain:

$$\frac{s}{u}V(s,u) - \frac{1}{u}V(s,u) - \frac{1}{s} = \mathbb{N}^+\left[v^2(t)\right]. \tag{5.11}$$

Substituting Eq. (5.10), we obtain:

$$V(s,u) = \frac{u}{s^2} + \frac{u}{s}\left[\mathbb{N}^+\left[v^2(t)\right]\right]. \tag{5.12}$$

Taking the inverse Natural transform of Eq. (5.12), we obtain:

$$v(t) = t + \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^+\left[v^2(t)\right]\right]\right]. \tag{5.13}$$

We now assume an infinite solution of the unknown function $v(t)$ of the form:

$$v(t)=\sum_{n=0}^{\infty} v_n(t). \tag{5.14}$$

Using Eq. (5.14), we can re-write Eq. (5.13) in the form:

$$\sum_{n=0}^{\infty} v_n(t)=t+\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} A_n(t)\right]\right]\right], \tag{5.15}$$

where $A_n(t)$ is the Adomian polynomial representing the nonlinear term $v^2(t)$.

Then from Eq. (5.15), we can generate the recursive relation as follows:

$$\begin{aligned}
v_0(t) &= t,\\
v_1(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[A_0(t)\right]\right]\right],\\
v_2(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[A_1(t)\right]\right]\right],\\
v_3(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[A_2(t)\right]\right]\right].
\end{aligned}$$

Thus, the general recursive relation is given by:

$$v_{n+1}(t)=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[A_n(t)\right]\right]\right], \qquad n\geq 0. \tag{5.16}$$

Using Eq. (5.16), we can easily compute the remaining components of the unknown function $v(t)$ as follows:

$$\begin{aligned}
v_1(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[A_0(t)\right]\right]\right]=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[v_0^2(t)\right]\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[t^2\right]\right]\right]=\mathbb{N}^{-1}\left[\frac{2u^3}{s^4}\right]=\frac{1}{3}t^3,\\
v_2(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[A_1(t)\right]\right]\right]=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[2v_0(t)v_1(t)\right]\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[\frac{2t^4}{3}\right]\right]\right]=\mathbb{N}^{-1}\left[\frac{48u^5}{3s^6}\right]=\frac{2t^5}{15},\\
v_3(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[A_2(t)\right]\right]\right]=\mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[2v_0(t)v_2(t)+v_1^2(t)\right]\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\left[\mathbb{N}^{+}\left[\frac{17t^6}{45}\right]\right]\right]=\mathbb{N}^{-1}\left[\frac{12240u^7}{45s^8}\right]=\frac{17t^7}{315}.
\end{aligned}$$

Then the approximate solution of the unknown function $v(t)$ is given by:

$$\begin{aligned}
v(t) &= \sum_{n=0}^{\infty} v_n(t)\\
&= v_0(t)+v_1(t)+v_2(t)+v_3(t)+\cdots\\
&= t+\frac{1}{3}t^3+\frac{2t^5}{15}+\frac{17t^7}{315}+\cdots
\end{aligned}$$

Hence, the exact solution of Eq. (5.9) is given by:

$$v(t) \;=\; \tan(t).$$

The exact solution is in closed agreement with the result obtained by (ADM) [31].

Example 5.3. Consider the Riccati differential equation of the form [31]:

$$\frac{dv}{dt} = 1 - t^2 + v^2(t), \tag{5.17}$$

subject to the initial condition

$$v(0) = 0. \tag{5.18}$$

Taking the N–Transform to both sides of Eq. (5.17), we obtain:

$$\frac{sV(s,u)}{u} - \frac{v(0)}{u} = \frac{1}{s} - \frac{2u^2}{s^3} + \mathbb{N}^+\left[v^2(t)\right]. \tag{5.19}$$

By substituting Eq. (5.18) into Eq. (5.19), we obtain:

$$v(s,u) = \frac{u}{s^2} - \frac{2u^3}{s^4} + \frac{u}{s}\mathbb{N}^+\left[v^2(t)\right]. \tag{5.20}$$

Taking the inverse N–Transform of Eq. (5.20), we have:

$$v(t) = t - \frac{t^3}{3} + \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[v^2(t)\right]\right]. \tag{5.21}$$

We now assume an infinite series solution of the unknown function $v(t)$ of the form:

$$v(t) = \sum_{n=0}^{\infty} v_n(t). \tag{5.22}$$

Then by using Eq. (5.22), we can re-write Eq. (5.21) in the form:

$$\sum_{n=0}^{\infty} v_n(t) = t - \frac{t^3}{3} + \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[\sum_{n=0}^{\infty} A_n(t)\right]\right], \tag{5.23}$$

where A_n is the Adomian polynomial which represent the nonlinear term $v^2(t)$.

By comparing both sides of Eq. (5.23), we can easily build the general recursive relation as follows:

$$\begin{aligned}
v_0(t) &= t - \frac{t^3}{3},\\
v_1(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[A_0(t)\right]\right],\\
v_2(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[A_1(t)\right]\right],\\
v_3(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[A_2(t)\right]\right].
\end{aligned}$$

Then the general recursive relation is given by:

$$v_{n+1}(t) = \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[A_n(t)\right]\right]. \tag{5.24}$$

By using Eq. (5.24), we can easily compute the remaining components of the unknown function $v(t)$ as follows:

$$\begin{aligned}
v_1(t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[A_0(t)\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_0^2(t)\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[\left(t-\frac{t^3}{3}\right)^2\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t^2\right]\right] - \frac{2}{3}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t^4\right]\right] + \frac{1}{9}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t^6\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{2u^3}{s^4}\right] - \frac{2}{3}\mathbb{N}^{-1}\left[\frac{4!u^5}{s^6}\right] + \frac{1}{9}\mathbb{N}^{-1}\left[\frac{6!u^7}{s^8}\right] \\
&= \frac{t^3}{3} - \frac{2t^5}{15} + \frac{t^7}{63}.
\end{aligned}$$

From $v_1(t)$ it is obvious that one noise term appear in the components $v_0(t)$. Then by canceling the noise term from $v_0(t)$, the remaining non-canceled term of $v_0(t)$ provide us with the exact solution. This can easily be verified by substitution.

Therefore, the exact solution of the given problem is given by:

$$v(t) = t. \tag{5.25}$$

The exact solution is in closed agreement with the result obtained by (ADM) [31].

6. Conclusion

In this paper, the Natural Decomposition Method (NDM) was proposed for solving the Riccati differential equation and two nonlinear ordinary differential equations. We successfully found exact solutions to all three applications. The NDM introduces a significant improvement in the fields over existing techniques. Our goal in the future is to apply the NDM to other linear nonlinear differential equations (PDEs, ODEs) that arise in other areas of science and engineering.

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