BAYESIAN MODELING OF FORESTRY DATA WITH R USING OPTIMIZATION AND SIMULATION TOOLS

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Abstract Data generated in forestry biometrics are not normal in statistical sense as they rarely follow the normal regression model. Hence, there is a need to develop models and methods in forest biometric applications for non-normal models. Due to generality of Bayesian methods it can be implemented in the situations when Gaussian regression models do not fit the data. Data on diameter at breast height (dbh), which is a very important characteristic in forestry has been fitted to Weibull and gamma models in Bayesian paradigm and comparisons have also been made with its classical counterpart. It may be noted that MCMC simulation tools are used in this study. An attempt has been made to apply Bayesian simulation tools using **R** software.

Keywords Bayesian inference, optim, LaplacesDemon, sampling importance resampling, LaplaceApproximation, model comparison.

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1. Introduction

Diameter distributions are important tools for the study of various features of a tree. Once we know the diameter at breast height, 1.37m, on all the sample of trees we can predict the volume, height etc. A plot specific height diameter relationship was used to determine height of any remaining trees in the plot.The comprehension of diameter distribution will affect the decision of growing and cultivation of trees as well as harvesting decisions such as where and when to harvest. In various growth models diameter distribution are used as input variable (Robinson and Hamann, 2011). As a result, statistical study of diameter is required in the form of its model fitting and related model prediction. Most common model that is normal model does not provide good fit in fitting diameter of a tree and hence, other models like Weibull and gamma distributions which are skewed in nature may be tried and model comparison studies tell the better goodness of fit of these models as compared to normal. We have the model of the form

$$y = \beta_0 + \beta_1 x + e,$$

where y is the response variable, x is the input variable, β_0 is the intercept and β_1 is the regression coefficient of y on x. Our goal is to construct a relationship between height (y) and diameter (x). As height of tree follows normal distribution,

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then obviously the fitting of height given diameter i.e., height as response variable and diameter as input variable will also follow normal distribution. On the other hand, if the fitting of regression model is done with volume (as response variable) and diameter (as input variable) then, the skewed distribution either Weibull or gamma would be the better fit and normal distribution will not be our choice. For estimating the parameter of a linear regression model, we have two approaches; one is the classical approach and second is the Bayesian approach. In classical approach, maximum likelihood method is the most common method of inference, which has certain drawbacks like most of its properties hold only for large sample size and things are not clear if sample size is small. For example, asymptotic normality of MLE forces the analyst to keep sampling distribution in a symmetric form. On the other hand, there are instances in which Bayesian approach, which is free from such limitations shows that the posterior density is not symmetric even if sample size is large. Moreover, Bayesian simulation tools provide exact method of inference even if sampler size is very small. Thus, Bayesian methods are superior over the classical methods. That is why, in this paper, we have adopted this method of inference for the modeling of forestry data. The software package used in this paper for summarization of posterior inference is **LaplacesDemon** package. This and related package will be discussed in Section 4. In Bayesian paradigm, data and prior are combined together to make inference about the parameter of interest. The formal basis of this approach is provided by the Bayes' rule, which is simply expressed as

$$p(\theta|y) \propto p(y|\theta)p(\theta).$$

Here, $p(\theta)$ is the prior distribution, $p(y|\theta)$ is the likelihood, and $p(\theta|y)$ is the posterior distribution. For fitting of data we use weakly informative prior distributions as we feel that we have very little prior knowledge about the model parameters.

2. Upper Flat Creek Forest Cruise tree data (UFC)

This data is available in `FAwR` package in **R** software. It is in the form of data frame with 336 rows and 5 columns. This was generated in 1991 from the Upper Flat Creek unit of the University of Idaho Experimental Forest. This data frame consists of 5 variables, namely, plot, trees, species, dbh.cm and height.m. However, diameter at breast height measured in centimeter, dbh.cm is the most important characteristic for analysis. To load the library `FAwR` which is meant for forest analysis with **R**, we use the function `library` as

```
library(FAwR)
```

To load the data `ufc`, use the function `data` as

```
data(ufc)
```

To print the names of the variables

```
names(ufc)
 "plot"     "tree"      "species"  "dbh.cm"    "height.m"
```

The data frame `ufc` has 336 rows and 5 columns but we can display only the few rows by using the function `head`

```
head(ufc)
  plot tree species dbh.cm height.m
1    2    1      DF     39     20.5
2    2    2      WL     48     33.0
3    3    2      GF     52     30.0
4    3    5      WC     36     20.7
5    3    8      WC     38     22.5
6    4    1      WC     46     18.0
```

3. Fitting statistical distribution

We have already discussed in Section 1 about the importance of diameter distribution. Diameter distributions are sometimes used as the subject of growth modeling themselves. Now, we will fit three distributions one by one and will see which one of them will be the good fit on the basis of model comparison methods, deviance, Bayesian information criterion (BIC) and Akaike information criterion (AIC).

3.1. Fitting of Weibull distribution

Weibull distribution has two parameters, shape and scale. It can be denoted as

$$y \sim W(\alpha, \beta)$$

thus, the likelihood is

$$p(y|\alpha, \beta) = \prod_{i=1}^{n} \frac{\alpha}{\beta} \left(\frac{y_i}{\beta} \right)^{\alpha-1} \exp \left[- \left(\frac{y_i}{\beta} \right)^{\alpha} \right],$$

which implies log-likelihood as

$$\log p(y|\alpha, \beta) = \sum_{i=1}^{n} \log \left(\frac{\alpha}{\beta} \left(\frac{y_i}{\beta} \right)^{\alpha-1} \exp \left[- \left(\frac{y_i}{\beta} \right)^{\alpha} \right] \right).$$

This loglikelihood is expressed in **R** as

```
loglikelihood<-sum(dweibull(x=y,shape=alpha,scale=beta,log=T))
```

Since both the parameters are positive, therefore, priors for both shape and scale parameters of Weibull is half-Cauchy with scale parameter 25. Consequently, log-prior for shape and scale parameters are $\log \left(\frac{2 \times 25}{\pi(\text{shape}^2 + 25^2)} \right)$ and $\log \left(\frac{2 \times 25}{\pi(\text{scale}^2 + 25^2)} \right)$, respectively, which can be expressed in **R** as

```
shape.prior<-dhalfcauchy(shape,25,log=T)
scale.prior<-dhalfcauchy(scale,25,log=T)
```

Thus,

$$\text{logposterior} = \text{loglikelihood} + \text{shape.prior} + \text{scale.prior}$$

Bayesian fitting can be done in **R** by using the function `optim` which is a general purpose function for unrestricted optimization. It requires negative of log of the posterior as an argument and a vector of values to start iterations in the optimization

algorithm. The default algorithm in `optim` is that of Nelder and Mead (1965), which is essentially a simplex algorithm and does not require derivatives of the objective function. Contrary to Newton Raphson method of optimization this works well even if guess values are far from the optimum. In Weibull model, since both shape and scale are restricted parameters, therefore, a logarithmic transformation is required to make them unrestricted.

Now, we write a function, namely `nlwp` which returns negative of the logposterior as defined above

```
nlwp<-function(theta,data)
{
shape<-exp(theta[1])
scale<-exp(theta[2])
ll<-dweibull(data,shape,scale,log=T)
ll<-sum(ll)
lp.shape<-dhalfcauchy(shape,25,log=T)
lp.scale<-dhalfcauchy(scale,25,log=T)
lpost<-ll+lp.shape+lp.scale
return(-lpost)
}
```

For Bayesian fitting of Weibull distribution the function `optim` is used as

```
M1<-optim(par=c(shape=log(1),scale=log(180)),fn=nlwp,hessian=TRUE,
data=ufc$dbh.cm)
```

The outputs obtained from the fitted object `M1` are summarized in Table 1. However, **R** codes for fitting normal and gamma models with the same priors are not reported just to save the space.

Table 1. Posterior summaries for each parameter of the distribution.

	Weibull		Normal		Gamma	
	shape	scale	mean	sd	shape	scale
Posterior mode	2.30	42.3	37.3	17.3	4.63	0.12
Posterior sd	1.04	1.02	0.94	1.04	1.07	1.08

3.2. Model comparison

To compare the three models; namely Weibull, gamma and normal, the model selection criterion preferred by the Bayesians and likelihoodists are Deviance, AIC (Akaike information criterion) and BIC (Bayesian information criterion), which are defined as

$$\text{Deviance} = -2 \text{ loglikelihood},$$
$$\text{AIC} = -2 \text{ loglikelihood} + 2k,$$
$$\text{BIC} = -2 \text{ loglikelihood} + k \log n,$$

where k is the number parameters in the model and n stands for number of observations. According to this criterion the distribution having the lesser value of BIC and AIC is the best fit. We have the following Deviance, BIC and AIC values:

Table 2. It may be noted that gamma is the best fit as it has least value of the model comparison
criteria.

Model	Deviance	BIC	AIC
Weibull	2833	2837	2862
Normal	2868	2872	2908
Gamma	2819	2823	2846

These results suggest that the two parameter gamma distribution has the lesser
value so, is a better fit to our data than the Weibull and normal distributions.

3.3. Graphic comparison of the three distributions

The graphical summary which is shown in Figure 1 clearly shows that gamma
distribution provides a better fit than the Weibull and Weibull provides better
fit than the normal. This fact is supported by the BIC values also. The gamma
distribution has the smallest BIC (2823) whereas the normal distribution has largest
BIC (2872). Thus, on the basis of graphics as well as numeric evidences gamma
distribution provides the best fit followed by Weibull and normal is the last choice
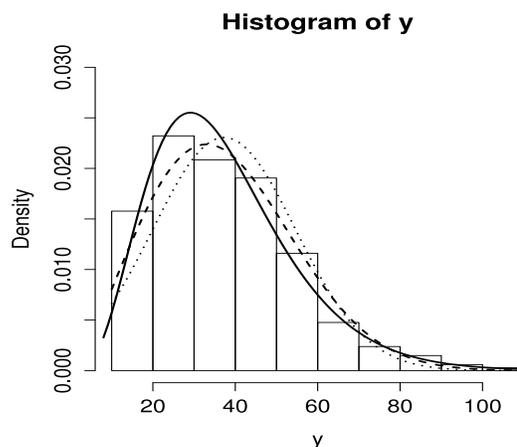in model selection.

Figure 1. Plots of fitted models, namely, Weibull, normal and gamma show that gamma is the best
choice for fitting diameter followed by Weibull and normal.

So far, only some aspects of posterior density are studied. A more compre-
hensive study follows in the next section using the `LaplacesApproximation` and
`LaplacesDemon` functions of **LaplacesDemon** package.

4. Bayesian fitting with LaplacesDemon

LaplacesDemon package can be used for Bayesian fitting. `LaplaceApproximation`
and `LaplacesDemon` are the two main functions of this package. The function

`LaplaceApproximation` deterministically maximizes the logarithm of the unnormalized joint posterior density with one of several optimization algorithms. Moreover, it has the implementation of sampling importance resampling (SIR) algorithm to simulate observation from posterior. It is capable with any Bayesian model for which the likelihood is specified. `LaplacesDemon` is the implementation of Markov chain Monte Carlo tools. It is a very comprehensive function which implements around 25 MCMC algorithms. Most efficient among them is the Independent Metropolis algorithm as it has the high acceptance rate, but requires posterior mode and modal variance which are obtained from `LaplaceApproximation`. To use these two functions the user must specify a model, a prior for parameters and a data object which is required for fitting.

4.1. Fitting of diameter with normal distribution

The study of diameter is already emphasized in Section 1. Its fitting details which includes code for creation of data, definition of model and its fitting with `LaplaceApproximation` are reported as

```
## Creation of data
library(LaplacesDemon)
library(FAwR)
data(ufc)
y<-ufc$dbh.cm
X<-matrix(1,nrow=length(y))
J<-1
mon.names<-c("LP","sigma")
parm.names<-as.parm.names(list(beta=rep(0,J),log.sigma=0))
MyData<-list(J=J,X=X,mon.names=mon.names,parm.names=parm.names,y=y)


## Initial values
Initial.Values<-c(rep(0,J),log(1))


## Model specification
Model <- function(parm, Data)
{
beta<-parm[1:Data$J]
sigma<-exp(parm[Data$J+1])
beta.prior<-sum(dnormv(beta,0,10000,log=T))
sigma.prior<-dhalfcauchy(sigma,25,log=T)
theta<-tcrossprod(Data$X,t(beta))
LL<-sum(dnorm(Data$y,theta,sigma,log=T))
LP<-LL+beta.prior+sigma.prior
Modelout<-list(LP=LP,Dev=-2*LL,Monitor=c(LP,sigma),
    yhat=rnorm(length(theta),theta,sigma),parm=parm)
return(Modelout)
}


## Fitting with LaplaceApproximation
M1<-LaplaceApproximation(Model,Initial.Values,Data=MyData,
Iterations=10000)
```

Fitting of diameter with Weibull and gamma models is not shown in this document to save the space but their posterior summary is reported in Table 3 and Table 4 and also their graphical posterior summary plots are reported in Figure 2.

Table 3. Posterior summaries obtained from fitted object `M1` are reported for the three distributions. These summaries are based on SIR algorithm which is implicit in the `LaplaceApproximation`. It may be noted that LB and UB are quantiles (0.025 and 0.975)

	Weibull		Normal		Gamma	
	beta	shape	beta	sigma	beta	shape
Mean	3.74	2.29	37.40	17.30	2.10	4.60
SD	0.03	0.10	0.99	0.67	0.08	0.34
LB	3.69	2.10	35.30	16.10	1.90	4.00
Median	3.74	2.29	37.40	17.30	2.10	4.60
UB	3.79	2.48	39.20	18.60	2.20	5.30

```
## Fitting with LaplacesDemon
Initial.Values<-as.initial.values(M1)
M10<-LaplacesDemon(Model,Data=MyData,Initial.Values,
    Covar=M1$Covar,Algorithm="IM",Iterations=5000,
    Specs=list(mu=M1$Summary1[1:length(Initial.Values),1]))
```

Table 4. Posterior summaries obtained from fitted object `M10` are reported for the three distributions. These summaries are based on IM algorithm which is an argument in `LaplacesDemon` function.

	Weibull		Normal		Gamma	
	beta	shape	beta	sigma	beta	shape
Mean	3.74	2.30	37.40	17.30	2.10	4.70
SD	0.01	0.06	0.58	0.39	0.05	0.22
LB	3.72	2.19	36.30	16.50	2.00	4.30
Median	3.74	2.30	37.40	17.30	2.10	4.60
UB	3.77	2.41	38.50	18.00	2.20	5.10

4.2. Graphical summary

The graphical presentation can be summarized by caterpillar plots for all the three distributions. However, to save the space caterpillar plot corresponding Weibull and normal is not reported. Caterpillar plots are popular plots in Bayesian inference for summarizing the quantiles of posterior samples. A caterpillar plot is similar to a horizontal box plots making it easier to study more distribution in a single plot. The plot in Figure 3 shows a caterpillar plot of the parameters of the gamma distribution. In this plot quantiles (LB, Median and UB) are plotted as a line for each parameter. The median appears as a black dot. A vertical, gray line is included at zero. Both the parameters are statistically significant as the vertical dotted line does not cross the horizontal lines.

4.3. Model comparison for diameter

Model comparison is summarized in Table 5. On the basis of this table it is suggested that gamma distribution is the best fit as gamma distribution has least value of
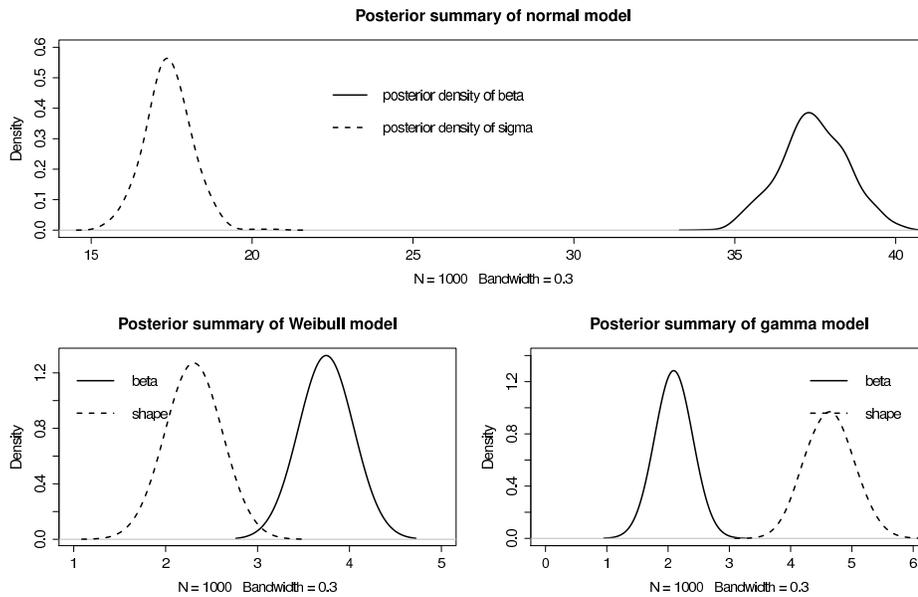
Figure 2. Posterior densities of the parameter of normal model are reported in the top panel whereas Weibull and gamma models are reported in the bottom.
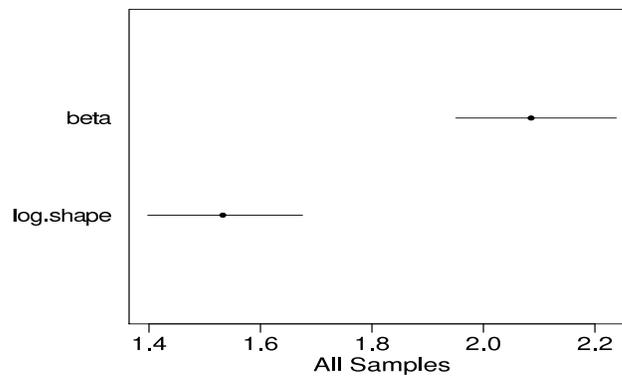

Figure 3. Posterior summarization through caterpillar plot of gamma model.

deviance, BIC and AIC in the case of fitting of diameter.

5. Bayesian regression analysis: Fitting of normal model

In this section `sweetgum` data is used to fit normal regression model for volume given diameter. This data is a part of `FAwR` package. This data is in form of data frame with 39 tree-level observations on 8 variables, namely, `plot`, `tree`, `dbh.in`,

Table 5. Model comparison of the distributions for the fitting of diameter distribution. Gamma is the best choice on the basis of this table as it has least value of the model comparison criterion.

Model	Deviance	BIC	AIC
Normal	2869.50	2881.10	2873.50
Weibull	2834.66	2846.26	2838.66
Gamma	2820.60	2832.20	2824.60

`stump.ht.ft`, `height.ft`, `height.m`, `dbh.cm`, `vol.m3`. Here, fitting is done between volume (`vol.m3`) as response variable and diameter (`dbh.cm`) as input variable.

5.1. Bayesian regression model

Bayesian model consists of the likelihood and prior for the parameters. Fitting with `R` is also discussed in this section.

5.1.1. The model

Let's consider a normal linear regression model, which is often denoted as:

$$y \sim N(\mu, \sigma^2),$$

where $\mu = X\beta$ and σ^2 is the variance. It may be noted that X is model matrix and β is the vector of regressors.

5.1.2. The prior

The prior for each component of β is a normal distribution, whereas prior for σ is half Cauchy with scale 25. These priors are proper but almost flat in the relevant region of parameters. Thus, they are weakly informative priors

$$\beta \sim N(0, 1000),$$
$$\sigma \sim HC(25).$$

5.1.3. The R commands

Creation of data and definition of Bayesian model in **R** for the purpose of fitting are being described now:

```
## Creation of data
library(LaplacesDemon)
library(FAwR)
data(sweetgum)
x<-sweetgum$dbh.cm
X<-cbind(1,x)
y<-sweetgum$vol.m3
J<-2
mon.names<-c("LP","sigma")
parm.names<-as.parm.names(list(beta=rep(0,J),log.sigma=0))
```

```
MyData<-list(J=J,X=X,mon.names=mon.names,parm.names=parm.names,y=y)

## Initial values
Initial.Values<-c(rep(0,J),log(1))

## Model specification
Model <- function(parm, Data)
{
beta<-parm[1:Data$J]
sigma<-exp(parm[Data$J+1])
beta.prior<-sum(dnorm(beta,0,10000,log=T))
sigma.prior<-dhalfcauchy(sigma,25,log=T)
theta<-tcrossprod(Data$X,t(beta))
LL<-sum(dnorm(Data$y,theta,sigma,log=T))
LP<-LL+beta.prior+sigma.prior
Modelout<-list(LP=LP,Dev=-2*LL,Monitor=c(LP,sigma),
    yhat=rnorm(length(theta),theta,sigma),parm=parm)
return(Modelout)
}

## Fiting with LaplaceApproximation
M2<-LaplaceApproximation(Model,Initial.Values,Data=MyData,
    Iterations=10000)
```

The commands for the fitting of Weibull and gamma model are similar to that of normal. The differences are only in the definition of log-likelihoods. The simulated posterior summary of all the three regression models for volume given diameter is reported in Table 6 and Table 7, respectively.

Table 6. Posterior summary of normal, Weibull and gamma regression models for sweetgum data using SIR.

	Mean	SD	LB	Median	UB
		Normal			
beta[1]	-1.83	0.17	-2.14	-1.82	-1.34
beta[2]	0.09	0.00	0.08	0.09	0.10
sigma	0.41	0.05	0.33	0.41	0.57
		Weibull			
	Mean	SD	LB	Median	UB
beta[1]	-1.91	0.12	-2.15	-1.91	-1.68
beta[2]	0.05	0.00	0.05	0.06	0.06
shape	4.66	0.63	3.53	4.65	5.92
		Gamma			
	Mean	SD	LB	Median	UB
beta[1]	-4.63	0.26	-5.11	-4.64	-4.09
beta[2]	0.06	0.00	0.05	0.06	0.06
shape	13.80	3.24	8.05	13.53	20.43

```
## Fitting with LaplacesDemon
Initial.Values<-as.initial.values(M2)
```

```
M20<-LaplacesDemon(Model,Data=MyData,Initial.Values,
    Covar=M2$Covar,Algorithm="IM",Iterations=5000,
    Specs=list(mu=M2$Summary1[1:length(Initial.Values),1]))
```

Table 7. Posterior summary of normal, Weibull and gamma regression models for sweetgum data using IM algorithm.

	Normal				
	Mean	SD	LB	Median	UB
beta[1]	-1.82	0.09	-2.00	-1.82	-1.66
beta[2]	0.09	0.00	0.09	0.09	0.10
sigma	0.40	0.03	0.35	0.40	0.45
	Weibull				
	Mean	SD	LB	Median	UB
beta[1]	-1.91	0.07	-2.05	-1.91	-1.77
beta[2]	0.05	0.00	0.05	0.06	0.06
shape	4.79	0.37	4.11	4.79	5.47
	Gamma				
	Mean	SD	LB	Median	UB
beta[1]	-4.71	0.16	-5.02	-4.71	-4.38
beta[2]	0.06	0.00	0.05	0.06	0.06
shape	14.58	2.03	10.78	14.44	18.97

5.2. Model comparison for volume given diameter

In the case of fitting of volume given diameter, Table 8 suggests that Weibull distribution is the best fit (deviance=16.99) and also gamma distribution (deviance=20.83) is the second best fit.

Table 8. Models comparison for normal, Weibull and gamma regression models.

Model	Deviance	BIC	AIC
Normal	39.18	50.28	45.18
Weibull	16.99	28.09	22.99
Gamma	20.83	31.93	26.83

6. Fitting of regression model for height given diameter

Here again we use the same data `ufc` for illustrative purpose and will show the results of fitting of height (height.m) as responce variable and diameter (dbh.cm) as input variable. Posterior summary of Normal, Weibull and gamma regression models for height given diameter after implementing `LaplaceApproximation` and `LaplacesDemon` functions is reported in Table 9 and Table 10, respectively.
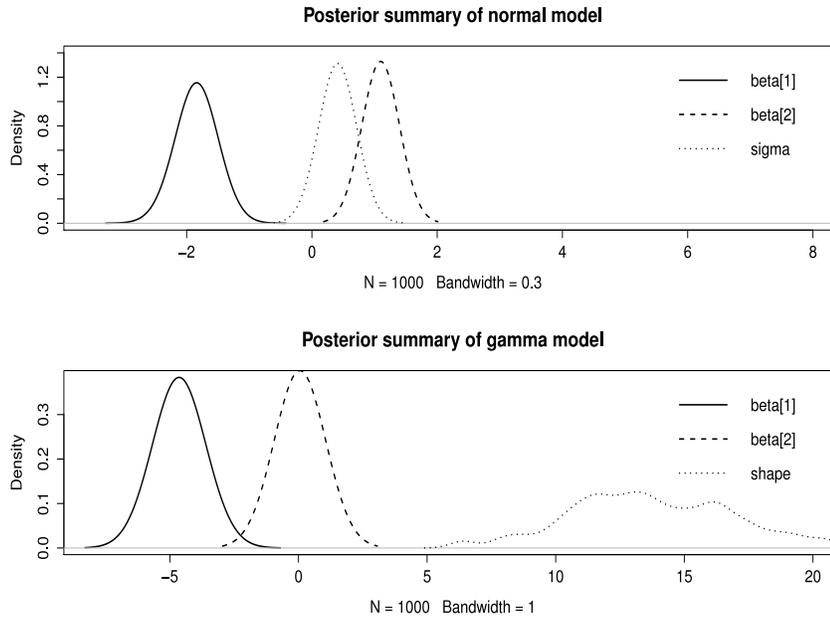
Posterior summary of normal model

Posterior summary of gamma model

Figure 4. Plots of normal and gamma regression models for volume given diameter, which shows the posterior summary of their parameters respectively.

Table 9. Simulated posterior summary of normal, Weibull, and gamma models with SIR.

	Mean	SD	LB	Median	UB
		Normal			
beta[1]	12.11	0.59	10.94	12.11	13.30
beta[2]	0.32	0.01	0.29	0.32	0.35
sigma	4.67	0.19	4.34	4.66	5.04
		Weibull			
	Mean	SD	LB	Median	UB
beta[1]	2.80	0.03	2.70	2.80	2.80
beta[2]	0.01	0.00	0.01	0.01	0.01
shape	5.20	0.22	4.80	5.20	5.70
		Gamma			
	Mean	SD	LB	Median	UB
beta[1]	-0.27	0.09	-0.45	-0.27	-0.86
beta[2]	0.01	0.00	0.01	0.01	0.02
shape	18.00	1.50	16.00	18.00	22.00

6.1. Model comparison for height given diameter

On the basis of goodness of fit measures namely, deviance, BIC and AIC reported in Table 11, it is clear that in the case of fitting regression model for height given diameter, normal distribution is the best fit followed by Weibull and the last choice is gamma.

Table 10. Simulated posterior summary of normal, Weibull, and gamma models with IM algorithm

Normal					
	Mean	SD	LB	Median	UB
beta[1]	12.10	0.36	11.39	12.10	132.80
beta[2]	0.32	0.01	0.31	0.32	0.34
sigma	4.64	0.11	4.43	4.64	4.86
Weibull					
	Mean	SD	LB	Median	UB
beta[1]	2.80	0.02	2.70	2.76	2.80
beta[2]	0.01	0.00	0.01	0.01	0.01
shape	5.30	0.13	5.00	5.22	5.50
Gamma					
	Mean	SD	LB	Median	UB
beta[1]	-0.28	0.05	-0.38	-.28	-0.19
beta[2]	0.01	0.00	0.01	0.01	0.02
shape	19.00	0.86	17.00	19.00	20.00

Table 11. Table for model comparison of the three distributions.

Model	Deviance	BIC	AIC
Normal	1986.08	2003.48	1992.08
Weibull	2000.00	2017.40	2006.00
Gamma	2100.00	2117.40	2106.00

7. Discussion

It has been established that **R** is a very powerful software package for Bayesian modeling of forestry data (R Core Team (2014)). The function `LapalaceApproximation` is the main function for the purpose of optimization in Bayesian scenario whereas `LaplacesDemon` is the function which is meant for implementation of Markov chain Monte Carlo simulation tools. In fitting diameter, gamma model is found to be the best model followed by Weibull and normal. Contrary to this, in regression modeling a normal model was the best choice for regressing height on diameter followed by Weibull and gamma models. However, when volume is regressed on diameter, Weibull was the best model followed by gamma, and normal was the last choice. The criterion for model selection which are used in this paper are Deviance, BIC and AIC. However, as per recommendation of Gelman et al., (2003) deviance is the best criteria for model selection. The code developed in **R** can be used in other areas of regression modeling besides forestry because of their general nature and paradigm. Finally, Bayesian approach is more suitable even if sample size is small and it can be used very effectively in the modeling of forestry data wherein non Guassian model like Weibull and gamma commonly fit. Undoubtedly, **R** enhances the astonishing vigour of Bayesian approach.

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