

GLOBAL STABILITY OF A DELAYED RATIO-DEPENDENT PREDATOR-PREY MODEL WITH GOMPERTZ GROWTH FOR PREY*

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Abstract A delayed ratio-dependent predator-prey model with Gompertz growth for prey is investigated. The local stability of a predator-extinction equilibrium and a coexistence equilibrium is discussed. Furthermore, the existence of Hopf bifurcation at the coexistence equilibrium is established. By constructing a Lyapunov functional, sufficient conditions are obtained for the global stability of the coexistence equilibrium.

Keywords Predator-prey model, ratio-dependence, Gompertz growth, stability, bifurcation.

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1. Introduction

Predator-prey systems are very important in analyzing the population dynamics between different species. In many classical predator-prey models, the per capita growth function is chosen as the logistic growth $r[1 - x(t)/K]$, where $x(t)$ is the density of population at time t , r is the intrinsic growth rate and K is the carrying capacity. In [11], Laird discovered that the experimental data for a variety of primary and transplanted tumours of the mouse, rat and rabbit were well fit to the Gompertz curve, which was first proposed by Benjamin Gompertz [7]. In [12], Laird et al. showed that the Gompertz model, due to its ability to exhibit exponential retardation, could describe the normal growth of an organism such as the gumea pig over an incredible 10000-fold range of the growth. Since that time the Gompertz equation is often used to describe the population dynamics and the inner growth of tumour, i.e.

$$\frac{dx(t)}{dt} = rx(t) \ln \frac{K}{x(t)}. \quad (1.1)$$

Note that the per capita rate of predation only depends on the prey numbers in a large body of existing predator-prey theories. In reality, as the number of

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predators changes slowly relative to prey change, there is often competition among the predators, and the per capita rate of predation should therefore depend on the numbers of both prey and predator, most probably and likely on their ratio. Therefore, a more suitable general predator-prey model should be based on the "ratio-dependent" theory. These hypotheses are strongly supported by numerous field and laboratory experiment and observation (see, for example, [1–3, 8]).

On the other hand, many models in the literature represent the population dynamics by systems of ordinary differential equations without time delay. However, inclusion of temporal delays in such models makes them more realistic. It is generally recognized that some kind of time delays are tend to be destabilizing in the sense that longer delays may destroy the stability of coexistence equilibrium [10]. Time delay due to gestation is among them, because generally a duration of τ time units elapses when an individual prey is killed and the moment when the corresponding increase in the predator population is realized. The effect of this kind of delay on the asymptotic behavior of populations has been studied by a number of papers (see, for example, [5, 6, 13]).

Motivated by the works of Laird [12], Arditi [3] and Freedman [5], in this paper we study a delayed ratio-dependent predator-prey model with Gompertz growth for prey:

$$\begin{aligned} \dot{x}(t) &= r_1 x(t) \ln \frac{K}{x(t)} - \frac{a_1 x(t)y(t)}{my(t) + x(t)}, \\ \dot{y}(t) &= y(t) \left[\frac{a_2 x(t - \tau)}{my(t - \tau) + x(t - \tau)} - r_2 \right], \end{aligned} \quad (1.2)$$

with initial conditions

$$x(\theta) = \phi_1(\theta) \geq 0, y(\theta) = \phi_2(\theta) \geq 0, \phi_1(0) > 0, \phi_2(0) > 0, \quad \theta \in [-\tau, 0]. \quad (1.3)$$

In system (1.2), $x(t)$ and $y(t)$ represent the densities of the prey and predator at time t , respectively. The parameters a_1, a_2, m, r_1, r_2 and K are positive constants in which a_1 is the capturing rate of predators, a_2/a_1 is the conversion rate of the predator by consuming prey, m is the half saturation constant, r_1 is the intrinsic growth rate, r_2 denotes the death rate of the predator, K is the carrying capacity, respectively. $\tau > 0$ is a constant delay due to the gestation of predators.

The organization of this paper is as follows. In the next section, by analyzing the corresponding characteristic equations, the local stability of a predator-extinction equilibrium and a coexistence equilibrium of system (1.2) is discussed. The existence of Hopf bifurcations at the coexistence equilibrium is established. Section 3 provides sufficient conditions for the predator-extinction equilibrium and coexistence equilibrium of system (1.2) to be globally asymptotically stable. In Section 4, an example is given to illustrate the main results in this paper.

2. Local stability of equilibria

In this section, we investigate the local stability of a predator-extinction equilibrium and a coexistence equilibrium by analyzing the corresponding characteristic equations, respectively.

System (1.2) always has a predator-extinction equilibrium $E_1(K, 0)$. Let

$$\mathcal{R}_0 = \frac{a_2}{r_2}.$$

It is easy to show that if $\mathcal{R}_0 > 1$, then system (1.2) has a unique coexistence equilibrium $E^*(x^*, y^*)$, where

$$x^* = K e^{\frac{a_1(r_2 - a_2)}{ma_2 r_1}}, \quad y^* = \frac{K(a_2 - r_2)}{mr_2} e^{\frac{a_1(r_2 - a_2)}{ma_2 r_1}}.$$

The characteristic equation of system (1.2) at the predator-extinction equilibrium $E_1(K, 0)$ is of the form

$$(\lambda + r_1)(\lambda - a_2 + r_2) = 0. \quad (2.1)$$

Clearly, Eq. (2.1) always has one negative root $\lambda_1 = -r_1$ and another root $\lambda_2 = a_2 - r_2$. Hence, if $\mathcal{R}_0 < 1$, the equilibrium E_1 is locally asymptotically stable; if $\mathcal{R}_0 > 1$, E_1 is unstable.

To study the local stability of the coexistence equilibrium, we consider the linearization of system (1.2) at $E^*(x^*, y^*)$

$$\begin{aligned} \dot{\hat{x}}(t) &= r_1 \left[\frac{x^* \ln \frac{K}{x^*}}{my^* + x^*} - 1 \right] \hat{x}(t) - \frac{a_1 x^{*2}}{(my^* + x^*)^2} \hat{y}(t), \\ \dot{\hat{y}}(t) &= \frac{a_2 m y^{*2}}{(my^* + x^*)^2} \hat{x}(t - \tau) - \frac{a_2 m x^* y^*}{(my^* + x^*)^2} \hat{y}(t - \tau). \end{aligned}$$

The characteristic equation of system (1.2) at the coexistence equilibrium $E^*(x^*, y^*)$ is of the form

$$\lambda^2 + P_1 \lambda + (Q_1 \lambda + Q_0) e^{-\lambda \tau} = 0, \quad (2.2)$$

where

$$P_1 = r_1 - \frac{a_1 r_2^2 y^*}{a_2^2 x^*}, \quad Q_0 = \frac{mr_1 r_2^2 y^*}{a_2 x^*}, \quad Q_1 = \frac{mr_2^2 y^*}{a_2 x^*}.$$

When $\tau = 0$, (2.2) becomes

$$\lambda^2 + (P_1 + Q_1) \lambda + Q_0 = 0,$$

where

$$P_1 + Q_1 = r_1 + \frac{r_2(ma_2 - a_1)(a_2 - r_2)}{ma_2^2}.$$

Therefore, if the following condition holds,

$$(H1) \quad P_1 + Q_1 > 0,$$

the coexistence equilibrium $E^*(x^*, y^*)$ is locally asymptotically stable for $\tau = 0$.

When $\tau > 0$, if $i\omega$ ($\omega > 0$) is a solution of (2.2), separating real and imaginary parts, we obtain that

$$Q_1 \omega \sin \omega \tau + Q_0 \cos \omega \tau = \omega^2, \quad Q_1 \omega \cos \omega \tau - Q_0 \sin \omega \tau = -P_1 \omega. \quad (2.3)$$

Squaring and adding the two equations of (2.3), it follows that

$$\omega^4 + (P_1^2 - Q_1^2) \omega^2 - Q_0^2 = 0. \quad (2.4)$$

It is easy to see that Eq. (2.4) has a unique positive root $\omega(\tau)$. From Eq. (2.3), we can get the corresponding $\tau_k > 0$,

$$\tau_k = \frac{1}{\omega} \arccos \left(\frac{(Q_0 - P_1 Q_1) \omega^2}{Q_1^2 \omega^2 + Q_0^2} \right) + \frac{2k\pi}{\omega}, \quad k = 0, 1, \dots, \quad (2.5)$$

such that Eq. (2.2) has a pair of purely imaginary roots.

Now we turn to determine the direction in which the pair of imaginary roots $\lambda = \pm i\omega(\tau_k)$ cross the imaginary axis as τ increases.

Differentiating (2.2) with respect to τ , we obtain

$$\frac{d\lambda}{d\tau} = \frac{\lambda(Q_1\lambda + Q_0)e^{-\lambda\tau}}{2\lambda + P_1 + (Q_1 - Q_1\lambda\tau - Q_0\tau)e^{-\lambda\tau}}. \quad (2.6)$$

It is convenient to consider $(d\lambda/d\tau)^{-1}$. Hence, from (2.6) we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + P_1)e^{\lambda\tau} + Q_1 - Q_1\lambda\tau - Q_0\tau}{\lambda(Q_1\lambda + Q_0)}, \quad (2.7)$$

where, due to (2.2)

$$e^{\lambda\tau} = -\frac{Q_1\lambda + Q_0}{\lambda^2 + P_1\lambda}. \quad (2.8)$$

Therefore, substituting (2.8) in (2.7), we have that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{2\lambda + P_1}{\lambda(\lambda^2 + P_1\lambda)} + \frac{Q_1}{\lambda(Q_1\lambda + Q_0)} - \frac{\tau}{\lambda}.$$

Simple computation at $\lambda = i\omega(\tau_0)$ yields

$$\begin{aligned} \text{sign} \left\{ \frac{d(\text{Re}\lambda)}{d\tau} \right\}_{\lambda=i\omega(\tau_0)} &= \text{sign} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\lambda=i\omega(\tau_0)} \\ &= \text{sign} \left\{ \frac{Q_1^2\omega^4 + 2Q_0^2\omega^2 + P_1^2Q_0^2}{\omega^2(P_1^2 + \omega^2)(Q_1^2\omega^2 + Q_0^2)} \right\} \\ &= 1. \end{aligned}$$

This root of equation (2.2) crosses the imaginary axis from the left to the right as τ continuously varies from a number less than τ_k to one greater than τ_k by Rouché's theorem [4]. Therefore, the transversality condition holds and the conditions for Hopf bifurcation [9] are satisfied at $\tau = \tau_k$.

In conclusion, we have the following result.

Theorem 2.1. *Let $\mathcal{R}_0 > 1$ and (H1) hold.*

- (i) *If $\tau \in [0, \tau_0)$, the coexistence equilibrium $E^*(x^*, y^*)$ of system (1.2) is locally asymptotically stable;*
- (ii) *If $\tau > \tau_0$, the equilibrium $E^*(x^*, y^*)$ is unstable and system (1.2) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$.*

3. Global asymptotic stability

In this section, we are concerned with the global asymptotic stability of the predator-extinction equilibrium E_1 and coexistence equilibrium E^* , respectively. In order to do this, we need the following result.

Lemma 3.1. *Positive solutions of system (1.2) with initial conditions (1.3) are ultimately bounded if $\mathcal{R}_0 > 1$.*

Proof. Let $(x(t), y(t))$ be any positive solution of system (1.2) with conditions (1.3). From the first equation of system (1.2), we obtain that

$$\dot{x}(t) \leq r_1 x(t) \ln \frac{K}{x(t)} \leq r_1 x(t) \left(\frac{K}{x(t)} - 1 \right) = r_1 (K - x(t)).$$

A standard comparison argument shows that

$$\limsup_{t \rightarrow +\infty} x(t) \leq K. \quad (3.1)$$

Hence, there exists a $T_1 > 0$, such that if $t > T_1$, $x(t) \leq M_1$ for some $M_1 > K$.

In addition, from the second equation of system (1.2), we derive that

$$\dot{y}(t) \leq (a_2 - r_2)y(t),$$

for $t > \tau$, which implies that

$$y(t - \tau) \geq y(t)e^{-(a_2 - r_2)\tau}.$$

Therefore, for $t > T_1 + \tau$, we have that

$$\begin{aligned} \dot{y}(t) &\leq y(t) \left[\frac{a_2 M_1}{m e^{-(a_2 - r_2)\tau} y(t) + M_1} - r_2 \right] \\ &= y(t) \frac{M_1(a_2 - r_2) - m r_2 e^{-(a_2 - r_2)\tau} y(t)}{m e^{-(a_2 - r_2)\tau} y(t) + M_1}. \end{aligned}$$

By a comparison argument, we have that

$$\limsup_{t \rightarrow +\infty} y(t) \leq \frac{M_1(a_2 - r_2)}{m r_2} e^{(a_2 - r_2)\tau} \triangleq M^*. \quad (3.2)$$

Therefore, there exists a $T_2 \geq T_1$, such that if $t \geq T_2$, $y(t) \leq M_2$ for some $M_2 > M^*$. \square

Theorem 3.1. *Suppose that $\mathcal{R}_0 < 1$. Then the predator-extinction equilibrium $E_1(K, 0)$ of system (1.2) is globally asymptotically stable provided that*

$$(H2) \quad \ln K + 1 - \frac{a_1}{m r_1} > 0.$$

Proof. Since $\mathcal{R}_0 < 1$, we derive from (3.2) that

$$\lim_{t \rightarrow +\infty} y(t) = 0. \quad (3.3)$$

From the first equation of system (1.2), we have that

$$\begin{aligned} \dot{x}(t) &\geq r_1 x(t) (\ln K - \ln x(t)) - \frac{a_1}{m} x(t) \\ &\geq x(t) \left[r_1 \ln K + r_1 - \frac{a_1}{m} - r_1 x(t) \right]. \end{aligned}$$

A comparison argument shows that

$$\liminf_{t \rightarrow +\infty} x(t) \geq \ln K + 1 - \frac{a_1}{m r_1}. \quad (3.4)$$

Define

$$v(t) = \frac{y(t)}{x(t)}.$$

If (H2) holds, we derive from (3.3) and (3.4) that

$$\lim_{t \rightarrow +\infty} v(t) = 0. \quad (3.5)$$

Rewrite the first equation of system (1.2) as

$$\dot{x}(t) = r_1 x(t) \ln \frac{K}{x(t)} - \frac{a_1 x(t) v(t)}{m v(t) + 1}.$$

By (3.5), for arbitrary $\varepsilon > 0$, there is a $T_3 > 0$, such that if $T > T_3$, $v(t) \leq \varepsilon$. Therefore, we get that

$$\dot{x}(t) \geq x(t) \left(r_1 \ln \frac{K}{x(t)} - a_1 \varepsilon \right).$$

By a comparison argument, we derive that

$$x(t) \geq \frac{K}{e^{\frac{a_1 \varepsilon}{r_1}}}.$$

It is easy to show that

$$\liminf_{t \rightarrow +\infty} x(t) \geq K.$$

Using (3.1), we derive that

$$\lim_{t \rightarrow +\infty} x(t) = K.$$

□

Next we will investigate the global stability of the coexistence equilibrium of system (1.2). The strategy is to construct an appropriate Lyapunov functional.

Define

$$P(u) = \frac{u}{m + u},$$

then system (1.2) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= x(t) \left\{ r_1 \ln \frac{x^*}{x(t)} + a_1 \left[\frac{y^*}{x^*} P\left(\frac{x^*}{y^*}\right) - \frac{y(t)}{x(t)} P\left(\frac{x(t)}{y(t)}\right) \right] \right\}, \\ \dot{y}(t) &= a_2 y(t) \left[P\left(\frac{x(t-\tau)}{y(t-\tau)}\right) - P\left(\frac{x^*}{y^*}\right) \right]. \end{aligned} \quad (3.6)$$

Define

$$u(t) = \frac{x(t)}{y(t)}, \quad u^* = \frac{x^*}{y^*},$$

then system (3.6) becomes

$$\begin{aligned} \dot{x}(t) &= x(t) \left[r_1 \ln \frac{x^*}{x(t)} + a_1 \left(\frac{P(u^*)}{u^*} - \frac{P(u)}{u} \right) \right], \\ \dot{u}(t) &= u(t) \left[r_1 \ln \frac{x^*}{x(t)} + a_1 \left(\frac{P(u^*)}{u^*} - \frac{P(u)}{u} \right) - a_2 (P(u(t-\tau)) - P(u^*)) \right]. \end{aligned} \quad (3.7)$$

Define $v(t) = (v_1(t), v_2(t))$ by

$$\begin{aligned} v_1(t) &= x(t) - x^*, & v_2(t) &= u(t) - u^*, \\ F(v_2) &= P(u) - P(u^*) = \frac{mv_2}{(m + u^*)(m + u)}. \end{aligned} \quad (3.8)$$

It is easy to verify that

$$\begin{aligned} v_2 F(v_2) &> 0, & v_2 &\neq 0; \\ F'(v_2)u &= \frac{mu}{(m + u)^2} < 1; \\ \frac{P(u^*)}{u^*} - \frac{P(u)}{u} &= \frac{1}{m} F(v_2). \end{aligned} \quad (3.9)$$

Therefore, from (3.7) and (3.8), we finally obtain

$$\begin{aligned} \dot{v}_1(t) &= (v_1(t) + x^*) \left[r_1 \ln \frac{x^*}{v_1(t) + x^*} + \frac{a_1}{m} F(v_2(t)) \right], \\ \dot{v}_2(t) &= (v_2(t) + u^*) \left[r_1 \ln \frac{x^*}{v_1(t) + x^*} + \frac{a_1}{m} F(v_2(t)) - a_2 F(v_2(t - \tau)) \right]. \end{aligned} \quad (3.10)$$

It is clearly that the global stability of the equilibrium E^* of (1.2) is equivalent to that of the trivial solution of (3.10). Now we formulate the corresponding result as follows.

Theorem 3.2. *Assume that $\mathcal{R}_0 > 1$. Then the coexistence equilibrium $E^*(x^*, y^*)$ of system (1.2) is globally asymptotically stable provided that*

(H3) $A_i > 0, i = 1, 2$,

where

$$\begin{aligned} A_1 &= \frac{r_1}{M_1} - \frac{a_1}{m} - \tau \frac{a_2 r_1}{M_1}, \\ A_2 &= 2a_2 - \frac{3a_1}{m} - \frac{r_1}{M_1} - \tau a_2 \left(\frac{r_1}{M_1} + \frac{a_1}{m} + 2a_2 \right). \end{aligned}$$

Proof. Define

$$V_1(t) = v_1(t) - x^* \ln \frac{v_1(t) + x^*}{x^*} + \int_{u^*}^{u(t)} \frac{P(\nu) - P(u^*)}{\nu} d\nu. \quad (3.11)$$

Calculating the derivative of $V_1(t)$ along solutions of system (3.10), it follows that

$$\begin{aligned} \frac{dV_1(t)}{dt} &= \frac{v_1(t)}{v_1(t) + x^*} \dot{v}_1(t) + \frac{F(v_2(t))}{u(t)} \dot{v}_2(t) \\ &= v_1(t) \left[r_1 \ln \frac{x^*}{v_1(t) + x^*} + \frac{a_1}{m} F(v_2(t)) \right] \\ &\quad + F(v_2(t)) \left[r_1 \ln \frac{x^*}{v_1(t) + x^*} + \frac{a_1}{m} F(v_2(t)) - a_2 F(v_2(t - \tau)) \right] \\ &= r_1 v_1(t) \ln \frac{x^*}{v_1(t) + x^*} + \frac{a_1}{m} v_1(t) F(v_2(t)) + r_1 \ln \frac{x^*}{v_1(t) + x^*} F(v_2(t)) \\ &\quad + \left(\frac{a_1}{m} - a_2 \right) F^2(v_2(t)) + a_2 F(v_2(t)) \int_{t-\tau}^t F'(v_2(s)) v_2'(s) ds \end{aligned}$$

$$\begin{aligned}
&= r_1 v_1(t) \ln \frac{x^*}{v_1(t) + x^*} + \frac{a_1}{m} v_1(t) F(v_2(t)) + r_1 \ln \frac{x^*}{v_1(t) + x^*} F(v_2(t)) \\
&\quad + \left(\frac{a_1}{m} - a_2 \right) F^2(v_2(t)) + a_2 F(v_2(t)) \int_{t-\tau}^t F'(v_2(s)) u(s) r_1 \ln \frac{x^*}{v_1(s) + x^*} \\
&\quad + F'(v_2(s)) u(s) \left[\frac{a_1}{m} F(v_2(s)) - a_2 F(v_2(s-\tau)) \right] ds.
\end{aligned}$$

Using the inequality $\ln x \leq x - 1$, $a^2 + b^2 \geq 2ab$ and the Cauchy-Schwarz inequality, by Lemma 3.1, we derive from (3.9) that for $t > T^*$,

$$\begin{aligned}
\frac{dV_1(t)}{dt} &\leq r_1 v_1(t) \left(\frac{x^*}{v_1(t) + x^*} - 1 \right) + \frac{a_1}{m} v_1(t) F(v_2(t)) - r_1 \frac{v_1(t)}{v_1(t) + x^*} F(v_2(t)) \\
&\quad + \left(\frac{a_1}{m} - a_2 \right) F^2(v_2(t)) + a_2 |F(v_2(t))| \int_{t-\tau}^t F'(v_2(s)) u(s) r_1 \left| -\frac{v_1(s)}{v_1(s) + x^*} \right| \\
&\quad + F'(v_2(s)) u(s) \left[\frac{a_1}{m} |F(v_2(s))| + a_2 |F(v_2(s-\tau))| \right] ds \\
&\leq -\frac{r_1}{M_1} v_1^2(t) + \frac{a_1}{2m} [v_1^2(t) + F^2(v_2(t))] + \frac{r_1}{2M_1} [v_1^2(t) + F^2(v_2(t))] \\
&\quad + \left(\frac{a_1}{m} - a_2 \right) F^2(v_2(t)) + a_2 |F(v_2(t))| \int_{t-\tau}^t F'(v_2(s)) u(s) \frac{r_1}{M_1} |v_1(s)| \\
&\quad + F'(v_2(s)) u(s) \left[\frac{a_1}{m} |F(v_2(s))| + a_2 |F(v_2(s-\tau))| \right] ds \\
&\leq -\frac{1}{2} \left[\frac{r_1}{M_1} - \frac{a_1}{m} \right] v_1^2(t) + \left[\frac{3a_1}{2m} + \frac{r_1}{2M_1} - a_2 \right] F^2(v_2(t)) \\
&\quad + a_2 |F(v_2(t))| \int_{t-\tau}^t \left[\frac{r_1}{M_1} |v_1(s)| + \frac{a_1}{m} |F(v_2(s))| + a_2 |F(v_2(s-\tau))| \right] ds \\
&\leq -\frac{1}{2} \left[\frac{r_1}{M_1} - \frac{a_1}{m} \right] v_1^2(t) + \left[\frac{3a_1}{2m} + \frac{r_1}{2M_1} - a_2 \right] F^2(v_2(t)) \\
&\quad + \frac{1}{2} a_2 \tau \left[\frac{r_1}{M_1} + \frac{a_1}{m} + a_2 \right] F^2(v_2(t)) \\
&\quad + \frac{1}{2} a_2 \int_{t-\tau}^t \left[\frac{r_1}{M_1} v_1^2(s) + \frac{a_1}{m} F^2(v_2(s)) + a_2 F^2(v_2(s-\tau)) \right] ds.
\end{aligned} \tag{3.12}$$

Now define a Lyapunov functional $V(t)$ as

$$\begin{aligned}
V(t) &= V_1(t) + \frac{1}{2} a_2 \int_{t-\tau}^t \int_{\nu}^t \left[\frac{r_1}{M_1} v_1^2(s) + \frac{a_1}{m} F^2(v_2(s)) + a_2 F^2(v_2(s-\tau)) \right] ds d\nu \\
&\quad + \frac{1}{2} \tau a_2^2 \int_{t-\tau}^t F^2(v_2(s)) ds.
\end{aligned} \tag{3.13}$$

From (3.11)-(3.13), it is easy to verify that

$$\frac{dV(t)}{dt} \leq -\frac{1}{2} [A_1 v_1^2(t) + A_2 F^2(v_2(t))].$$

Define

$$\omega_1(|v(t)|) = V_1(t), \quad \omega_2(|v(t)|) = A_1 v_1^2(t) + A_2 F^2(v_2(t)),$$

where $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are continuous definite functions such that $\omega_i(0) = 0$, and $\lim_{s \rightarrow +\infty} \omega(s) = +\infty, i = 1, 2$. Applying the corollary 5.2 on Page 30 in [10], we conclude that the equilibrium $E^*(x^*, y^*)$ of system (1.2) is asymptotically stable with respect to positive solutions. \square

4. Numerical simulations

In this section, we give an example to show the feasibility of Theorems 2.1 and 3.2. Numerical simulations are carried out to illustrate the local stability of the coexistence equilibrium and the existence of periodic solutions near the equilibrium.

In system (1.2), let $r_1 = 2, r_2 = 1, a_1 = 1, a_2 = 4, K = 3, m = 12$. Then system (1.2) has a predator-extinction equilibrium $E_1(3, 0)$ and a coexistence equilibrium $E^*(3e^{-1/32}, 0.75e^{-1/32})$. It is easy to verify that the assumption in (H1) holds. From (2.4) and (2.5), we get $\omega = 0.7552, \tau_0 = 2.0766$. Numerical simulations show that the coexistence equilibrium E^* is asymptotically stable for $\tau \in [0, 2.0766)$ (see, for example, Fig.1) and unstable for $\tau > 2.0766$ (see, for example, Fig.2). By using Theorem 1, we know that system (1.2) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$. Theorem 2 provides that the coexistence equilibrium $E^*(3e^{-1/32}, 0.75e^{-1/32})$ is globally stable if $\tau < 0.1885$. Therefore, the global stability of E^* will impose restrictions on the length of time delay τ . In other words, time delay destabilizes E^* for system (1.2).

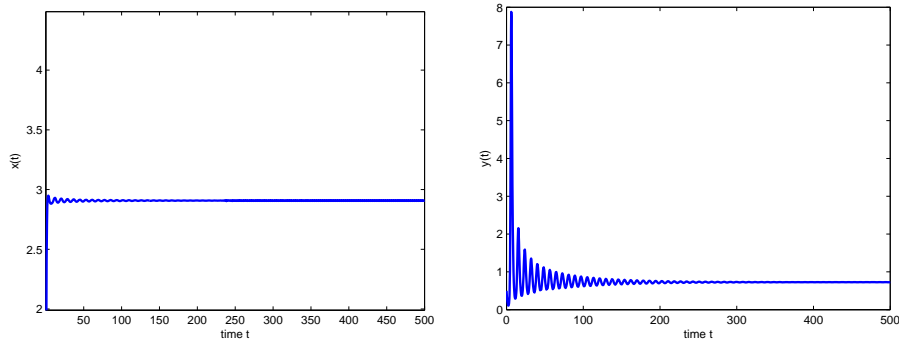


Figure 1. The equilibrium E^* is locally asymptotically stable with $\tau = 2$.

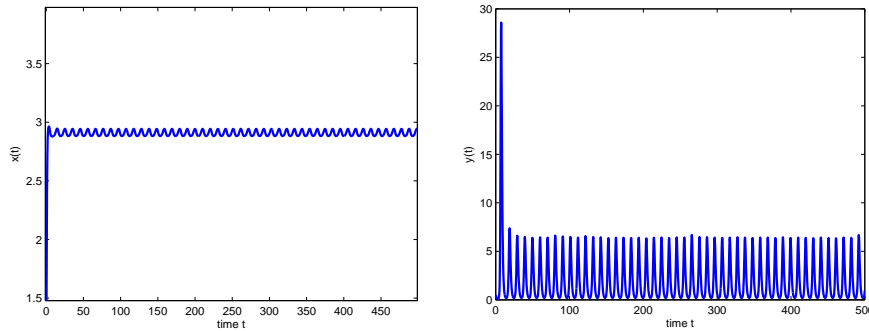


Figure 2. System (1.2) admits a periodic solution near the equilibrium E^* with $\tau = 2.5$.

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