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STABILITY ANALYSIS OF AN SEIS EPIDEMIC MODEL WITH NONLINEAR INCIDENCE AND TIME DELAY*

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Abstract In this paper, an SEIS epidemic model with nonlinear incidence and time delay is investigated. By analyzing the corresponding characteristic equations, the local stability of each of feasible equilibria of the model is established. By using suitable Lyapunov functional and LaSalle's invariance principle, it is shown that the disease-free equilibrium is globally asymptotically stable if the basic reproduction number is less than unity. If the basic reproduction number is greater than unity, by means of an iteration technique, sufficient conditions are derived for the global stability of the endemic equilibrium. Numerical simulations are carried out to illustrate the theoretical results.

Keywords SEIS epidemic model, nonlinear incidence, time delay, stability.

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1. Introduction

Since the pioneering work of Kermack-McKendrick on compartment modeling [7], studies of epidemic models describing the population dynamics of infectious diseases have become the important research areas of epidemiology. In the previous work, most models assume that the disease incubation is negligible, which implies that once infected, each susceptible individual immediately becomes infectious and later recovers with a permanent or temporary acquired immunity [9]. Based on these assumptions, different types of SIR and SIRS epidemic models have been widely investigated (see, for example, [6, 15–18]). However, many diseases (e.g. tuberculosis, influenza, measles) have a latent or incubation period when a susceptible has become infected but not yet infectious. For example, measles has an 8-13 day latent period and the incubation period for AIDS is anytime from a few months to a few years. Using a compartmental approach, one may assume that a susceptible individual first goes through a latent period (and is said to become exposed or in the class E) after infection, before becoming infectious.

Let $S(t)$ represent the number of individuals who are susceptible to the disease, that is, who are not yet infected at time t; $I(t)$ represent the number of infected individuals who are infectious and are able to spread the disease by contact

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with susceptible individuals, and $E(t)$ represent the number of individuals who are exposed at time t. In [3], Fan et al. considered the following SEIS epidemic model:

$$\begin{aligned} \dot{S}(t) &= A - \mu S(t) - \beta S(t)I(t) + \gamma I(t), \\ \dot{E}(t) &= \beta S(t)I(t) - (\mu + \varepsilon)E(t), \\ \dot{I}(t) &= \varepsilon E(t) - (\mu + \gamma + \alpha)I(t), \end{aligned} \tag{1.1}$$

where the influx of susceptibles comes from two sources, a constant recruitment A and recovered hosts γI. The natural death rate is assumed to be the same constant μ for all hosts, and infectious hosts suffer an extra disease-related death with constant α. $1/\varepsilon$ is the mean latent period and $1/\gamma$ is the mean infectious period. The incidence term is of the bilinear mass-action form βSI. The global stability of a disease-free equilibrium and an endemic equilibrium was investigated in [3] by using Lyapunov function theory and compound matrix theory, respectively. It was proven that the dynamics of (1.1) are completely determined by the basic reproduction number. In recent years, there have been some works on SEIS and SEIR epidemiological models in the literature (see, for example, [14, 21]).

We note that most models in the literature represent the dynamics of disease by systems of ordinary differential equations without time delay. However, inclusion of temporal delays in such models makes them more realistic by allowing the description of the effects of disease latency or immunity (see, for example, [2, 4, 11, 13]). In [11], Li and Ma proposed the following SEIS epidemic model with time delay:

$$\begin{aligned} \dot{S}(t) &= A - \mu S(t) - \beta S(t)I(t) + \gamma I(t), \\ \dot{E}(t) &= \beta S(t)I(t) - \mu E(t) - \beta e^{-\mu\tau} S(t-\tau)I(t-\tau), \\ \dot{I}(t) &= \beta e^{-\mu\tau} S(t-\tau)I(t-\tau) - (\mu + \gamma + \alpha)I(t), \end{aligned} \tag{1.2}$$

where $\tau \geq 0$ represents a time delay describing the latent period of the disease , the term $\beta e^{-\mu\tau} S(t-\tau)I(t-\tau)$ represents the individuals survives in the latent period τ and becoming infective at time t. The dynamics of system (1.2) was investigated in [11].

Incidence rate plays a very important role in the research of epidemiological models. In classical epidemic models, bilinear incidence rate βSI is frequently used (see, for example, [3, 11, 12]). Bilinear incidence rate is based on the law of mass action. This contact law is more appropriate for communicable diseases such as influenza, but not for sexually transmitted diseases. After a study of the cholera epidemic spread in Bari in 1973, Capasso and Serio [1] introduced a saturated incidence rate $g(I)S$ into epidemic models, where $g(I) = \beta I/(1 + \alpha I)$, here βI measures the infection force of the disease, and $1/(1 + \alpha I)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the susceptible individuals. In [20], Yuan and Li pointed out that the infectious force ought to depend on the densities of both infective individuals and susceptible individuals, and proposed the following nonlinear infectious incidence force function :

$$g(I, S) = g\left(\frac{I}{S}\right) = \frac{k(I/S)^l}{1 + \alpha(I/S)^h},$$

where the parameters k, l and h are positive constants and α is a nonnegative constant. This incidence rate seems more biologically reasonable than the bilinear incidence rate.

Motivated by the works of Li and Ma [11] and Yuan and Li [20], in this paper, we are concerned with the effect of time delay representing latent period and nonlinear incidence on the dynamic of an SEIS epidemic model. To this end, we study the following delayed differential system:

$$\begin{aligned}
\dot{S}(t) &= A - \mu S(t) - \frac{\beta S(t) I(t)}{S(t) + I(t)} + \gamma I(t), \\
\dot{E}(t) &= \frac{\beta S(t) I(t)}{S(t) + I(t)} - \mu E(t) - \frac{\beta e^{-\mu\tau} S(t-\tau) I(t-\tau)}{S(t-\tau) + I(t-\tau)}, \\
\dot{I}(t) &= \frac{\beta e^{-\mu\tau} S(t-\tau) I(t-\tau)}{S(t-\tau) + I(t-\tau)} - (\mu + \gamma + \alpha) I(t),
\end{aligned} \tag{1.3}$$

where the parameters $\mu, \alpha, \gamma, \beta$ and A are the same as that defined in model (1.1) and τ is the same as that defined in model (1.2).

The initial conditions for system (1.3) take the form

$$\begin{aligned}
&S(\theta) = \phi_1(\theta), \quad E(\theta) = \phi_2(\theta), \quad I(\theta) = \phi_3(\theta), \\
&\phi_i(\theta) \geq 0, \quad \theta \in [-\tau, 0], \quad \phi_i(0) > 0 \ (i = 1, 2, 3),
\end{aligned} \tag{1.4}$$

where $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], \mathbb{R}^3_{+0})$, here $\mathbb{R}^3_{+0} = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$.

For continuity of the initial conditions, we require

$$E(0) = \int_{-\tau}^{0} \beta e^{\mu\theta} \frac{\phi_1(\theta)\phi_3(\theta)}{\phi_1(\theta) + \phi_3(\theta)} d\theta. \tag{1.5}$$

It is easy to show that all solutions of system (1.3) with initial conditions (1.4) and (1.5) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The organization of this paper is as follows. In the next section, by analyzing the corresponding characteristic equations, we study the local asymptotic stability of the disease-free equilibrium and the endemic equilibrium of model (1.3). In Sec. 3, we discuss the global stability of the disease-free equilibrium and the endemic equilibrium by means of suitable Lyapunov functionals, LaSalle's invariance principle and iteration technique, respectively. In Sec. 4, numerical simulations are carried out to illustrate the theoretical results. A brief concluding remark is given in Sec. 5.

2. Local stability

In this section, we discuss the local stability of equilibria of system (1.3) by analyzing the corresponding characteristic equations, respectively.

Denote

$$\mathscr{R}_0 = \frac{\beta e^{-\mu\tau}}{\mu + \gamma + \alpha}.$$

$\mathscr{R}_0$ is called the basic reproduction number of system (1.3), which represents the average number of secondary infections produced by one infected individual during the mean course of infection in a completely susceptible population. It is easy to

show that system (1.3) always has a disease-free equilibrium $P^0(A/\mu, 0, 0)$. Further, if $\mathscr{R}_0 > 1$, system (1.3) admits a unique endemic equilibrium $P^*(S^*, E^*, I^*)$, where

$$S^* = \frac{A}{\mu + [e^{\mu\tau}(\mu+\gamma+\alpha) - \gamma](\mathscr{R}_0 - 1)},$$
$$E^* = \frac{(e^{\mu\tau}-1)(\mu+\gamma+\alpha)}{\mu}(\mathscr{R}_0 - 1)S^*, \quad I^* = (\mathscr{R}_0 - 1)S^*.$$

The characteristic equation of system (1.3) at the disease-free equilibrium P^0 is of the form

$$(\lambda+\mu)^2(\lambda+\mu+\gamma+\alpha-\beta e^{-\mu\tau}e^{-\lambda\tau}) = 0. \tag{2.1}$$

Clearly, Eq. (2.1) always has two negative real roots $\lambda_1 = \lambda_2 = -\mu$. All other roots of (2.1) are given by the following equation

$$\lambda+\mu+\gamma+\alpha-\beta e^{-\mu\tau}e^{-\lambda\tau} = 0. \tag{2.2}$$

Let $f(\lambda) = \lambda+\mu+\gamma+\alpha-\beta e^{-\mu\tau}e^{-\lambda\tau}$. We note that $\mathscr{R}_0 > 1$ implies $\beta e^{-\mu\tau} > \mu+\gamma+\alpha$. Hence, for λ real,

$$f(0) = \mu+\gamma+\alpha-\beta e^{-\mu\tau} < 0, \quad \lim_{\lambda\to+\infty} f(\lambda) = +\infty.$$

Hence, $f(\lambda) = 0$ has at least one positive real root. Therefore, P^0 is unstable. If $\mathscr{R}_0 < 1$, we now claim that roots of $f(\lambda) = 0$ have only negative real parts. Suppose that $\mathrm{Re}\lambda \geq 0$. Then it follows from (2.2) that

$$\mathrm{Re}\lambda = -(\mu+\gamma+\alpha) + \beta e^{-\mu\tau}e^{-\tau\mathrm{Re}\lambda}\cos(\tau\mathrm{Im}\lambda) \leq -(\mu+\gamma+\alpha) + \beta e^{-\mu\tau} < 0,$$

which leads to a contradiction. Hence, we have $\mathrm{Re}\lambda < 0$. Therefore, if $\mathscr{R}_0 < 1$, the disease-free equilibrium $P^0(A/\mu, 0, 0)$ is locally asymptotically stable.

The characteristic equation of system (1.3) at the endemic equilibrium $P^*(S^*, E^*, I^*)$ takes the form

$$(\lambda+\mu)\left[\lambda^2 + p_1(\tau)\lambda + p_0(\tau) + (q_1(\tau)\lambda + q_0(\tau))e^{-\lambda\tau}\right] = 0, \tag{2.3}$$

where

$$p_0(\tau) = (\mu+\gamma+\alpha)\left(\mu + \frac{\beta I^{*2}}{(S^*+I^*)^2}\right),$$
$$p_1(\tau) = 2\mu+\gamma+\alpha+\frac{\beta I^{*2}}{(S^*+I^*)^2},$$
$$q_0(\tau) = -\frac{\gamma\beta e^{-\mu\tau}I^{*2}}{(S^*+I^*)^2} - \frac{\mu\beta e^{-\mu\tau}S^{*2}}{(S^*+I^*)^2},$$
$$q_1(\tau) = -\frac{\beta e^{-\mu\tau}S^{*2}}{(S^*+I^*)^2}.$$

Obviously, Eq. (2.3) always has a negative real root $\lambda = -\mu$. Other roots of Eq. (2.3) are determined by the following equation

$$\lambda^2 + p_1(\tau)\lambda + p_0(\tau) + (q_1(\tau)\lambda + q_0(\tau))e^{-\lambda\tau} = 0. \tag{2.4}$$

When $\tau = 0$, Eq. (2.4) becomes

$$\lambda^2 + (p_1(0)+q_1(0))\lambda + p_0(0) + q_0(0) = 0. \tag{2.5}$$

It is easy to show that

$$\begin{aligned} p_1(0)+q_1(0) &= \left\{\mu+\frac{\beta I^*}{S^*+I^*}\right\}_{\tau=0}>0,\\ p_0(0)+q_0(0) &= \left\{\frac{\beta I^*}{S^*+I^*}\left(\mu+\frac{\alpha I^*}{S^*+I^*}\right)\right\}_{\tau=0}>0. \end{aligned} \tag{2.6}$$

Hence, if $\mathscr{R}_0>1$, the endemic equilibrium P^* is locally asymptotically stable when $\tau=0$.

If $\mathrm{i}\omega(\omega>0)$ is a solution of Eq. (2.4), by calculation, we have:

$$\omega^4+\left(p_1^2(\tau)-q_1^2(\tau)-2p_0(\tau)\right)\omega^2+p_0^2(\tau)-q_0^2(\tau)=0. \tag{2.7}$$

By direct calculations, one can show that

$$\begin{aligned} p_0^2(\tau)-q_0^2(\tau) &= \frac{\beta I^*}{(S^*+I^*)^2}[\mu^{-\mu\tau}S^*+(\mu+\alpha)I^*+\gamma(1-e^{-\mu\tau})I^*]\\ &\quad\times\left[\frac{\beta e^{-\mu\tau}}{(S^*+I^*)^2}(\gamma I^{*2}+\mu S^{*2})+(\mu+\gamma+\alpha)\left(\mu+\frac{\beta I^{*2}}{(S^*+I^*)^2}\right)\right]>0,\\ p_1^2(\tau)-q_1^2(\tau)-2p_0(\tau) &= \frac{\beta e^{-\mu\tau}S^*I^*}{(S^*+I^*)^2}\left(\mu+\gamma+\alpha+\frac{\beta^{-\mu\tau}S^{*2}}{(S^*+I^*)^2}\right)\\ &\quad+\left(\mu+\frac{\beta I^{*2}}{(S^*+I^*)^2}\right)^2>0. \end{aligned}$$

Hence, if $\mathscr{R}_0>1$, Eq. (2.7) has no positive roots. Noting that the equilibrium P^* is locally asymptotically stable when $\tau=0$, by the general theory on characteristic equations of delay differential equations from Kuang [8](Theorem 3.4.1), we see that if $\mathscr{R}_0>1$, the equilibrium P^* is locally asymptotically stable.

Based on the discussions above, we have the following result.

Theorem 2.1. *For system* (1.3)*, if* $\mathscr{R}_0<1$*, the disease-free equilibrium* $P^0(A/\mu,0,0)$ *is locally asymptotically stable; if* $\mathscr{R}_0>1$*,* P^0 *is unstable and the endemic equilibrium* $P^*(S^*,E^*,I^*)$ *exists and is locally asymptotically stable.*

3. Global stability

In this section, we are concerned with the global stability of the endemic equilibrium P^* and the disease-free equilibrium P^0 of system (1.3).

We first consider the following equation with time delay

$$\dot{u}(t)=\frac{a_1\beta e^{-\mu\tau}u(t-\tau)}{a_1+u(t-\tau)}-au(t) \tag{3.1}$$

with initial condition

$$u(s)=\phi(s)\geq 0,\quad s\in[-\tau,0),\quad \phi(0)>0,$$

here a,a_1,β and μ are positive constants, $\tau\geq 0$.

By using a similar argument as in the proof of Lemma 4.1 in [19], one can show the following Lemma.

Lemma 3.1. *For system* (3.1)*, we have*

(i) If $\beta e^{-\mu\tau} > a$*, then the unique positive equilibrium* $u^* = a_1(\beta e^{-\mu\tau} - a)/a$ *is globally asymptotically stable.*

(ii) If $\beta e^{-\mu\tau} < a$*, then the trivial equilibrium* $u^0 = 0$ *is globally asymptotically stable.*

We now investigate the global asymptotic stability of the endemic equilibrium P^* of system (1.3). The technique of proof is to use an iteration scheme.

Theorem 3.1. *Let* $\mathscr{R}_0 > 1$*. The endemic equilibrium* $P^*(S^*, E^*, I^*)$ *of system* (1.3) *is globally asymptotically stable provided that*

$$\text{(H1)}\quad 0 < A_1 - A_2 < \frac{4A^2 A_1 A_2(\gamma A_2 - \mu A_1)}{\mu A_1^2 + \gamma A_2^2 - A_1 A_2(\mu + \gamma + \beta)},$$

where

$$A_1 = \mu + \gamma + \alpha, \quad A_2 = \beta e^{-\mu\tau} - (\mu + \gamma + \alpha).$$

Proof. Let $(S(t), E(t), I(t))$ be any positive solution of system (1.3) with initial conditions (1.4) and (1.5).

Since the first and the third equations are independent of the second equation in (1.3), we need only to consider the following subsystem:

$$\begin{aligned} \dot{S}(t) &= A - \mu S(t) - \frac{\beta S(t) I(t)}{S(t) + I(t)} + \gamma I(t), \\ \dot{I}(t) &= \frac{\beta e^{-\mu\tau} S(t-\tau) I(t-\tau)}{S(t-\tau) + I(t-\tau)} - (\mu + \gamma + \alpha) I(t). \end{aligned} \tag{3.2}$$

Let

$$\begin{aligned} U_1 &= \limsup_{t\to+\infty} S(t), \quad V_1 = \liminf_{t\to+\infty} S(t), \\ U_2 &= \limsup_{t\to+\infty} I(t), \quad V_2 = \liminf_{t\to+\infty} I(t). \end{aligned}$$

Now we prove that $U_1 = V_1 = S^*$, $U_2 = V_2 = I^*$.

Letting $N(t) = S(t) + E(t) + I(t)$, we derive from (1.3) that

$$\dot{N}(t) = A - \mu N(t) - \alpha I(t).$$

By comparison it follows that

$$\limsup_{t\to+\infty} N(t) \le \frac{A}{\mu},$$

which yields

$$\limsup_{t\to+\infty} S(t) \le \frac{A}{\mu} := M_1^S.$$

Hence, for $\varepsilon > 0$ sufficiently small there exists a $T_1 > 0$ such that if $t > T_1$, $S(t) \le M_1^S + \varepsilon$.

We derive from the second equation of system (3.2) that, for $t > T_1 + \tau$,

$$\dot{I}(t) \le \frac{\beta e^{-\mu\tau}\left(M_1^S + \varepsilon\right) I(t-\tau)}{M_1^S + \varepsilon + I(t-\tau)} - (\mu + \gamma + \alpha) I(t).$$

Consider the following auxiliary equation

$$\dot{u}(t) = \frac{\beta e^{-\mu\tau}\left(M_1^S + \varepsilon\right) u(t-\tau)}{M_1^S + \varepsilon + u(t-\tau)} - (\mu+\gamma+\alpha)u(t). \tag{3.3}$$

Since $\mathscr{R}_0 > 1$, by Lemma 3.1 it follows from (3.3) that

$$\lim_{t\to+\infty} u(t) = \frac{[\beta e^{-\mu\tau} - (\mu+\gamma+\alpha)](M_1^S+\varepsilon)}{\mu+\gamma+\alpha}.$$

By comparison, we obtain that

$$\limsup_{t\to+\infty} I(t) \le \frac{[\beta e^{-\mu\tau} - (\mu+\gamma+\alpha)](M_1^S+\varepsilon)}{\mu+\gamma+\alpha}.$$

Since this inequality holds true for arbitrary $\varepsilon > 0$, it follows that $U_2 \le M_1^I$, where

$$M_1^I = \frac{[\beta e^{-\mu\tau} - (\mu+\gamma+\alpha)]M_1^S}{\mu+\gamma+\alpha}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 \ge T_1 + \tau$ such that if $t > T_2$, $I(t) \le M_1^I + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the first equation of system (3.2) that, for $t > T_2$,

$$\begin{aligned}\dot{S}(t) &\ge A - \mu S(t) - \frac{\beta S(t)(M_1^I+\varepsilon)}{S(t)+M_1^I+\varepsilon} \\ &= \frac{1}{S(t)+M_1^I+\varepsilon}\left\{-\mu S^2(t) + [A-(\mu+\beta)(M_1^I+\varepsilon)]S(t) + A(M_1^I+\varepsilon)\right\}.\end{aligned}$$

A comparison argument shows that

$$\liminf_{t\to+\infty} S(t) \ge \frac{A-(\mu+\beta)(M_1^I+\varepsilon) + \sqrt{[A-(\mu+\beta)(M_1^I+\varepsilon)]^2 + 4\mu A(M_1^I+\varepsilon)}}{2\mu}.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_1 \ge N_1^S$, where

$$N_1^S = \frac{A-(\mu+\beta)M_1^I + \sqrt{[A-(\mu+\beta)M_1^I]^2 + 4\mu A M_1^I}}{2\mu}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_3 \ge T_2$ such that if $t > T_3$, $S(t) \ge N_1^S - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the second equation of system (3.2) that, for $t > T_3 + \tau$,

$$\dot{I}(t) \ge \frac{\beta e^{-\mu\tau}(N_1^S-\varepsilon)I(t-\tau)}{N_1^S - \varepsilon + I(t-\tau)} - (\mu+\gamma+\alpha)I(t).$$

By Lemma 3.1 and by comparison it follows that

$$\liminf_{t\to+\infty} I(t) \ge \frac{[\beta e^{-\mu\tau} - (\mu+\gamma+\alpha)](N_1^S-\varepsilon)}{\mu+\gamma+\alpha}.$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_2 \geq N_1^I$, where

$$N_1^I = \frac{[\beta e^{-\mu\tau} - (\mu + \gamma + \alpha)]N_1^S}{\mu + \gamma + \alpha}.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_4 \geq T_3 + \tau$ such that if $t > T_4$, $I(t) \geq N_1^I - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the first equation of system (3.2) that, for $t > T_4$,

$$\begin{aligned}\dot{S}(t) &\leq A - \mu S(t) - \frac{\beta S(t)(N_1^I - \varepsilon)}{S(t) + N_1^I - \varepsilon} + \gamma(M_1^I + \varepsilon) \\ &= \frac{1}{S(t) + N_1^I - \varepsilon}\left\{-\mu S^2(t) + [A - (\mu + \beta)(N_1^I - \varepsilon) + \gamma(M_1^I + \varepsilon)]S(t)\right. \\ &\quad \left. + [A + \gamma(M_1^I + \varepsilon)](N_1^I - \varepsilon)\right\}.\end{aligned}$$

A comparison argument yields

$$\begin{aligned}&\limsup_{t\to+\infty} S(t) \\ \leq &\frac{1}{2\mu}\Big\{A - (\mu + \beta)(N_1^I - \varepsilon) + \gamma(M_1^I + \varepsilon) \\ &+ \sqrt{[A - (\mu + \beta)(N_1^I - \varepsilon) + \gamma(M_1^I + \varepsilon)]^2 + 4\mu\big[A + \gamma(M_1^I + \varepsilon)\big](N_1^I - \varepsilon)}\Big\}.\end{aligned}$$

Since this is true for arbitrary $\varepsilon > 0$, it follows that $U_1 \leq M_2^S$, where

$$M_2^S = \frac{A - (\mu + \beta)N_1^I + \gamma M_1^I + \sqrt{[A - (\mu + \beta)N_1^I + \gamma M_1^I]^2 + 4\mu\big(A + \gamma M_1^I\big)N_1^I}}{2\mu}.$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_5 \geq T_4$ such that if $t > T_5$, $S(t) \leq M_2^S + \varepsilon$. It therefore follows from the second equation of system (3.2) that, for $t > T_5 + \tau$,

$$\dot{I}(t) \leq \frac{\beta e^{-\mu\tau}(M_2^S + \varepsilon)I(t - \tau)}{M_2^S + \varepsilon + I(t - \tau)} - (\mu + \gamma + \alpha)I(t). \tag{3.4}$$

By Lemma 3.1 and a comparison argument we derive from (3.4) that

$$\limsup_{t\to+\infty} I(t) \leq \frac{[\beta e^{-\mu\tau} - (\mu + \gamma + \alpha)](M_2^S + \varepsilon)}{\mu + \gamma + \alpha}.$$

Since this inequality is true for arbitrary $\varepsilon > 0$, it follows that $U_2 \leq M_2^I$, where

$$M_2^I = \frac{[\beta e^{-\mu\tau} - (\mu + \gamma + \alpha)]M_2^S}{\mu + \gamma + \alpha}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there exists a $T_6 \geq T_5 + \tau$ such that if $t > T_6$, $I(t) \leq M_2^I + \varepsilon$.

Again, for $\varepsilon > 0$ sufficiently small, we derive from the first equation of system (3.2) that, for $t > T_6$,

$$\begin{aligned}\dot{S}(t) &\geq A - \mu S(t) - \frac{\beta S(t)(M_2^I + \varepsilon)}{S(t) + M_2^I + \varepsilon} + \gamma(N_1^I - \varepsilon) \\ &= \frac{1}{S(t) + M_2^I + \varepsilon} \left\{ -\mu S^2(t) + [A - (\mu + \beta)(M_2^I + \varepsilon) + \gamma(N_1^I - \varepsilon)]S(t) \right. \\ &\quad \left. + [A + \gamma(N_1^I - \varepsilon)](M_2^I + \varepsilon) \right\}.\end{aligned}$$

A comparison argument shows that

$$\begin{aligned}&\liminf_{t \to +\infty} S(t) \\ \geq &\frac{1}{2\mu} \Big\{ A - (\mu + \beta)(M_2^I + \varepsilon) + \gamma(N_1^I - \varepsilon) \\ &+ \sqrt{[A - (\mu + \beta)(M_2^I + \varepsilon) + \gamma(N_1^I - \varepsilon)]^2 + 4\mu \big[A + \gamma(N_1^I - \varepsilon)\big](M_2^I + \varepsilon)} \Big\}.\end{aligned}$$

Since this is true for arbitrary $\varepsilon > 0$, we derive that $V_1 \geq N_2^S$, where

$$N_2^S = \frac{A - (\mu + \beta)M_2^I + \gamma N_1^I + \sqrt{[A - (\mu + \beta)M_2^I + \gamma N_1^I]^2 + 4\mu \big(A + \gamma N_1^I\big) M_2^I}}{2\mu}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_7 \geq T_6$ such that if $t > T_7$, $S(t) \geq N_2^S - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the second equation of system (3.2) that, for $t > T_7 + \tau$,

$$\dot{I}(t) \geq \frac{\beta e^{-\mu\tau}(N_2^S - \varepsilon) I(t - \tau)}{N_2^S - \varepsilon + I(t - \tau)} - (\mu + \gamma + \alpha) I(t). \tag{3.5}$$

By Lemma 3.1 and a comparison argument, it follows from (3.5) that

$$\liminf_{t \to +\infty} I(t) \geq \frac{[\beta e^{-\mu\tau} - (\mu + \gamma + \alpha)](N_2^S - \varepsilon)}{\mu + \gamma + \alpha}.$$

Since this inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_2 \geq N_2^I$, where

$$N_2^I = \frac{[\beta e^{-\mu\tau} - (\mu + \gamma + \alpha)] N_2^S}{\mu + \gamma + \alpha}.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there exists a $T_8 \geq T_7 + \tau$ such that if $t > T_8$, $I(t) \geq N_2^I - \varepsilon$.

Continuing this process, we obtain four sequences $M_n^S, M_n^I, N_n^S, N_n^I$, $(n = 1, 2, \cdots)$

such that, for $n \geq 2$,

$$\begin{aligned}
M_n^S &= \frac{A-(\mu+\beta)N_{n-1}^I+\gamma M_{n-1}^I+\sqrt{[A-(\mu+\beta)N_{n-1}^I+\gamma M_{n-1}^I]^2+4\mu\big(A+\gamma M_{n-1}^I\big)N_{n-1}^I}}{2\mu},\\
M_n^I &= \frac{[\beta e^{-\mu\tau}-(\mu+\gamma+\alpha)]M_n^S}{\mu+\gamma+\alpha},\\
N_n^S &= \frac{A-(\mu+\beta)M_n^I+\gamma N_{n-1}^I+\sqrt{[A-(\mu+\beta)M_n^I+\gamma N_{n-1}^I]^2+4\mu\big(A+\gamma N_{n-1}^I\big)M_n^I}}{2\mu},\\
N_n^I &= \frac{[\beta e^{-\mu\tau}-(\mu+\gamma+\alpha)]N_n^S}{\mu+\gamma+\alpha}.
\end{aligned} \tag{3.6}$$

It is readily seen that

$$N_n^S \leq V_1 \leq U_1 \leq M_n^S, \quad N_n^I \leq V_2 \leq U_2 \leq M_n^I. \tag{3.7}$$

It is easy to show that the sequences M_n^S, M_n^I are nonincreasing, and the sequences N_n^S, N_n^I are nondecreasing. Hence, the limit of each sequence in $M_n^S, M_n^I, N_n^S, N_n^I$ exists. Denote

$$\begin{aligned}
\overline{S} &= \lim_{n\to+\infty} M_n^S, \quad \underline{S} = \lim_{n\to+\infty} N_n^S,\\
\overline{I} &= \lim_{n\to+\infty} M_n^I, \quad \underline{I} = \lim_{n\to+\infty} N_n^I.
\end{aligned} \tag{3.8}$$

We therefore derive from (3.6) and (3.8) that

$$(\mu A_1^2-\gamma A_1A_2)\overline{S}-[(\mu+\beta)A_1A_2-\gamma A_2^2]\overline{S}\,\underline{S} = AA_1^2\overline{S}+AA_1A_2\underline{S}, \tag{3.9}$$

$$(\mu A_1^2-\gamma A_1A_2)\underline{S}-[(\mu+\beta)A_1A_2-\gamma A_2^2]\overline{S}\,\underline{S} = AA_1^2\underline{S}+AA_1A_2\overline{S}. \tag{3.10}$$

(3.9) minus (3.10),

$$(\mu A_1-\gamma A_2)(\overline{S}^2-\underline{S}^2) = A(A_1-A_2)(\overline{S}-\underline{S}). \tag{3.11}$$

Assume that $\overline{S} \neq \underline{S}$. Then we obtain from (3.11) that

$$\overline{S}+\underline{S} = \frac{A(A_1-A_2)}{\mu A_1-\gamma A_2}. \tag{3.12}$$

(3.9) plus (3.10),

$$A_1(\mu A_1-\gamma A_2)(\overline{S}+\underline{S})^2-2[\mu A_1^2+\gamma A_2^2-A_1A_2(\mu+\gamma+\beta)]\overline{S}\,\underline{S} = AA_1(A_1+A_2)(\overline{S}+\underline{S}). \tag{3.13}$$

On substituting (3.12) into (3.13), it follows that

$$\overline{S}\,\underline{S} = \frac{A^2A_1A_2(A_2-A_1)}{(\mu A_1-\gamma A_2)[\mu A_1^2+\gamma A_2^2-A_1A_2(\mu+\gamma+\beta)]}. \tag{3.14}$$

Note that $\overline{S}>0, \underline{S}>0$. Let (H1) hold. It follows from (3.12) and (3.14) that

$$(\overline{S}+\underline{S})^2-4\overline{S}\,\underline{S} = \frac{A^2(A_1-A_2)}{(\mu A_1-\gamma A_2)^2}\left[A_1-A_2+\frac{4A^2A_1A_2(\mu A_1-\gamma A_2)}{\mu A_1^2+\gamma A_2^2-A_1A_2(\mu+\gamma+\beta)}\right]. \tag{3.15}$$

Hence, we have $(\overline{S}+\underline{S})^2-4\overline{S}\underline{S}<0$. This is a contradiction. Accordingly, we have $\overline{S}=\underline{S}$. This, together with (3.6), yields that $\overline{I}=\underline{I}$. Therefore, we have

$$\lim_{t\to+\infty} S(t)=S^*, \quad \lim_{t\to+\infty} I(t)=I^*. \tag{3.16}$$

We derive from (1.3) and (1.5) that

$$E(t)=\int_{t-\tau}^{t} \beta e^{-\mu(t-u)}\frac{S(u)I(u)}{S(u)+I(u)}du, \tag{3.17}$$

which together with (3.16) yields $\lim_{t\to+\infty} E(t)=E^*$. Noting that if $\mathscr{R}_0>1$, by Theorem 2.1, the endemic equilibrium P^* is locally asymptotically stable, we conclude that if $\mathscr{R}_0>1$ and (H1) hold, P^* is globally asymptotically stable. The proof is complete. □

In the following, by using suitable Lyapunov functional, we prove our result on the global stability of the disease-free equilibrium $P^0(A/\mu,0,0)$.

Theorem 3.2. *If $\mathscr{R}_0<1$, then the disease-free equilibrium $P^0(A/\mu,0,0)$ of system (1.3) is globally asymptotically stable.*

Proof. Let $(S(t),E(t),I(t))$ be any positive solution of system (1.3) with initial conditions (1.4) and (1.5).

Define

$$V_1(t)=I(t)+\beta e^{-\mu\tau}\int_{t-\tau}^{t}\frac{S(u)I(u)}{S(u)+I(u)}du. \tag{3.18}$$

Calculating the derivative of $V_1(t)$ along positive solutions of system (1.3), we obtain that

$$\frac{d}{dt}V_1(t)=\frac{\beta e^{-\mu\tau}S(t)I(t)}{S(t)+I(t)}-(\mu+\gamma+\alpha)I(t)\le[\beta e^{-\mu\tau}-(\mu+\gamma+\alpha)]I(t). \tag{3.19}$$

Hence, if $\mathscr{R}_0<1$, then $V_1'(t)\le 0$. By Theorem 5.3.1 in [5], solutions limit to $\mathcal{M}$, the largest invariant subset of $\{V_1'(t)=0\}$. Clearly, we see from (3.19) that $V_1'(t)=0$ if and only if $I=0$. Noting that $\mathcal{M}$ is invariant, for each element in $\mathcal{M}$, we have $I(t)=0$. It therefore follows from the first and the second equations of system (1.3) that

$$\dot{S}(t)=A-\mu S(t), \quad \dot{E}(t)=-\mu E(t),$$

which yields $S(t)=A/\mu, E(t)=0$. Hence, $V_1'(t)=0$ if and only if $(S,E,I)=(A/\mu,0,0)$. Accordingly, the global asymptotic stability of P^0 follows from LaSalle's invariance principle. This completes the proof. □

4. Numerical Examples

In this section, we give two examples to illustrate the main theoretical results in Sec. 3.

Example 4.1. In system (1.3), let $A=2,\alpha=0.5,\beta=0.85,\gamma=0.03,\mu=0.1,\tau=2$. It is easy to show that the basic reproduction number $\mathscr{R}_0\approx 1.2210>1$, and system (1.3) has a unique endemic equilibrium $P^*(11.2754,1.1798,1.6457)$. Further, by calculation we have

$$A_1-A_2\approx 0.5641, \quad \frac{4A^2A_1A_2(\gamma A_2-\mu A_1)}{\mu A_1^2+\gamma A_2^2-A_1A_2(\mu+\gamma+\beta)}\approx 45.5481.$$

Hence, (H1) holds. By Theorem 3.1, we see that P^* is globally asymptotically stable. Numerical simulation supports the above result (see, Fig. 1).

Example 4.2. In system (1.4), let $A = 4, \alpha = 1, \beta = 0.2, \gamma = 0.5, \mu = 1, \tau = 1$. By calculation, we get the basic reproduction number $\mathscr{R}_0 \approx 0.2496 < 1$. In this case, system (1.4) has a disease-free equilibrium $P^0(4, 0, 0)$. By Theorem 3.2, we see that P^0 is globally asymptotically stable. Numerical simulation illustrates this fact (see, Fig. 2).

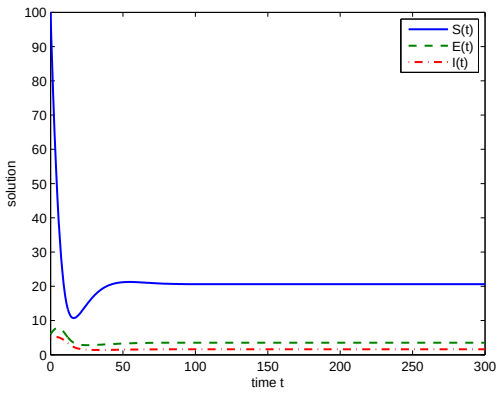

Figure 1. The temporal solution found by numerical integration of system (1.3) with $A = 2, \alpha = 0.5, \beta = 0.85, \gamma = 0.03, \mu = 0.1, \tau = 2$ and $(\phi_1, \phi_2, \phi_3) = (100, 425(1 - e^{-0.2})/13, 4)$.

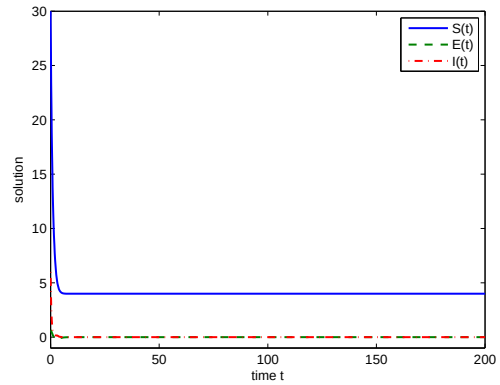

Figure 2. The temporal solution found by numerical integration of system (1.3) with $A = 4, \alpha = 1, \beta = 2, \gamma = 0.5, \mu = 1, \tau = 1$ and $(\phi_1, \phi_2, \phi_3) = (30, 1 - e^{-1}, 6)$.

5. Concluding remark

In this paper, we have discussed an SEIS epidemic model with nonlinear incidence and time delay. The basic reproduction number $\mathscr{R}_0$ was calculated. We investigat-

ed the global asymptotic stability of the disease-free equilibrium and the endemic equilibrium of system (1.3), respectively.

When the basic reproduction number is greater than unity, by using the iteration scheme, we have established sufficient conditions for the global stability of the endemic equilibrium of system (1.3). By Theorem 3.1, we see that if the latent period τ of the disease is small enough and the transmission rate β is large enough such that $\mathscr{R}_0 > 1$, the disease becomes endemic. On the other hand, we see that if the basic reproduction number is less than unity, the disease-free equilibrium is globally asymptotically stable. Biologically, these indicate that when the latent period of the disease is long enough and the transmission rate is sufficiently small such that $\mathscr{R}_0 < 1$, the disease will fade out.

We would like to point out here that the additional assumption $0 < A_1 - A_2 < [4A^2 A_1 A_2\ (\gamma A_2 - \mu A_1)]/[\mu A_1^2 + \gamma A_2^2 - A_1 A_2(\mu + \gamma + \beta)]$ was imposed on the global stability of the endemic equilibrium P^*. Noting that P^* is locally asymptotically stable if $\mathscr{R}_0 > 1$, we conjecture that P^* is also globally asymptotically stable in this case. We leave this for our future work.

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