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ON A CLASS OF SINGULAR P-LAPLACIAN SEMIPOSITONE PROBLEMS WITH SIGN-CHANGING WEIGHT

S.H. Rasouli[1,†] and Z. Firouzjahi[1]

Abstract We study existence of positive weak solution for a class of p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda g(x)[f(u) - \frac{1}{u^\alpha}], & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter and $\alpha \in (0,1)$, Ω is a bounded domain in R^N for $(N > 1)$ with smooth boundary, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator for $(p > 2)$, $g(x)$ is C^1 sign-changing function such that maybe negative near the boundary and be positive in the interior and f is C^1 nondecreasing function $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$. We discuss the existence of positive weak solution when f and g satisfy certain additional conditions. We use the method of sub-supersolution to establish our result.

Keywords Positive solution, singular problem, sub-supersolution.

MSC(2000) 35J55, 35J65.

1. Introduction

We consider the singular following problem

$$\begin{cases} -\Delta_p u = \lambda g(x)[f(u) - \frac{1}{u^\alpha}], & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter and $\alpha \in (0,1)$, Ω is a bounded domain in R^N for $(N > 1)$ with smooth boundary and $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator for $(p > 2)$, $g(x)$ is C^1 sign-changing function that maybe negative near the boundary and be positive in the interior and f is C^1 nondecreasing function.

Problems involving the p-Laplacian arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [1]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

Let $\tilde{f}(y) = f(y) - \frac{1}{y^\alpha}$. Then $\lim_{y\to 0} \tilde{f}(y) = -\infty$, and hence we refer to (1.1) as an infinite semipositone problem (see [4–6, 8]). See [6] where the authors discussed the problem (1.1) when $a \equiv 1$ and $p = 2$. In [5], the authors extended the study of [6], to the case when $p > 1$. Here we focus on further extending the study in [5] to the problem (1.1). In fact, we study the existence of positive solution to the

†the corresponding author. Email address:s.h.rasouli@mit.ac.ir (S.H. Rasouli)
1Department of Mathematics, Faculty of Basic Sciences, Babol University of Technology, Babol, Iran

problem (1.1) with sign-changing weight function $g(x)$. Due to the weight function, the extensions are challenging and nontrivial. Our approach is based on the method of sub-super solutions (see [2, 3, 7]).

To precisely state our existence result, we consider the eigenvalue problem

$$\begin{cases} -\Delta_p\phi = \lambda|\phi|^{p-2}\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Let ϕ be the eigenfunction corresponding to the first eigenvalue λ_1 of (1.2) such that $\phi(x) > 0$ in Ω, and $\|\phi\|_\infty = 1$. Let $m, \sigma, \delta > 0$ be such that

$$\sigma \le \phi \le 1, \qquad x \in \Omega - \overline{\Omega_\delta}, \tag{1.3}$$

$$(1 - \frac{\alpha p}{p-1+\alpha})|\nabla\phi|^p \ge m, \qquad x \in \overline{\Omega_\delta}, \tag{1.4}$$

where $\overline{\Omega_\delta} := \{x \in \Omega | d(x, \partial\Omega) \le \delta\}$. This is possible since $|\nabla\phi| \ne 0$ on $\partial\Omega$ while $\phi = 0$ on $\partial\Omega$. We will also consider the unique solution $e \in W_0^{1,p}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta_p e = 1, & x \in \Omega, \\ e = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is known that $e > 0$ in Ω and $\frac{\partial e}{\partial n} < 0$ on $\partial\Omega$.

Here we assume that the weight function $g(x)$ takes negative values in $\overline{\Omega_\delta}$, but require $g(x)$ be strictly positive in $\Omega - \overline{\Omega_\delta}$. To be precise we assume that there exist positive constants a, b such that $g(x) \ge -a$, on $\overline{\Omega_\delta}$ and $g(x) \ge b$ on $\Omega - \overline{\Omega_\delta}$.

2. Existence result

In this section, we shall establish our existence result via the method of sub-super solution. A function ψ is said to be a subsolution of (1.1), if it is in $W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ such that $\psi = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla\psi|^{p-2}\, \nabla\psi \,.\nabla w\, dx \le \int_\Omega \lambda g(x)[f(\psi) - \frac{1}{\psi^\alpha}]\, w\, dx, \quad \forall w \in W,$$

where $W = \{w \in C_0^\infty(\Omega) \,|\, w \ge 0, x \in \Omega\}$ (see [8]).

A function z is said supersolution of (1.1), if it is in $W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ such that $z = o$ on $\partial\Omega$, and

$$\int_\Omega |\nabla z|^{p-2}\, \nabla z \,.\nabla w\, dx \ge \int_\Omega \lambda g(x)[f(z) - \frac{1}{z^\alpha}]\, w\, dx, \; \forall w \in W.$$

Then the following result holds :

Lemma 2.1. *(See [3]). If there exist a subsolution ψ and supersolution z such that $\psi \le z$ in Ω then (1.1) has a weak solution u such that $\psi \le u \le z$.*

We make the following assumptions :

(H1) $f : (0, \infty) \to (0, \infty)$ *is C^1 nondecreasing function.*

(H2) $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$.

(H3) *Suppose that there exists $\epsilon > 0$ such that*

(i) $f(\frac{\epsilon^{\frac{1}{p-1}}(p-1+\alpha)\sigma}{p}) > (\frac{p}{\epsilon^{\frac{1}{p-1}}\sigma(p-1+\alpha)})^{\alpha}$,

(ii) $\frac{\epsilon^{\alpha+p-1}\lambda_1(p-1+\alpha)^{\alpha}}{ap^{\alpha}} > \frac{m\epsilon}{af(\epsilon^{\frac{1}{p-1}})}$,

(iii) $\frac{\epsilon\lambda_1}{Nb} > \frac{m\epsilon}{af(\epsilon^{\frac{1}{p-1}})}$,

where

$$N = f(\frac{\epsilon^{\frac{1}{p-1}}(p-1+\alpha)\sigma}{p}) - (\frac{p}{\epsilon^{\frac{1}{p-1}}\sigma(p-1+\alpha)})^{\alpha}.$$

We are now ready to give our existence result.

Theorem 2.1. *Let $(H1)-(H3)$ hold. Then there exists a positive weak solution of (1.1) for every $\lambda \in [\lambda_*(\epsilon),\ \lambda^*(\epsilon)]$, where*

$$\lambda^* = \frac{m\epsilon}{af(\epsilon^{\frac{1}{p-1}})} \qquad \textit{and} \qquad \lambda_* = \max\left\{\frac{\epsilon^{\frac{\alpha+p-1}{p-1}}\lambda_1(p-1+\alpha)^{\alpha}}{ap^{\alpha}},\ \frac{\epsilon\lambda_1}{Nb}\right\}.$$

Remark 2.1. Note that $(H3)$ implies $\lambda_* < \lambda^*$.

Example 2.1. Let $B > 0$ and $f(x) = e^{\frac{Bx^2}{B+x^2}}$. Then, $f(x) > 0$ for $x > 0$ and f is nondecreasing and

$$\lim_{x\to\infty}\frac{f(x)}{x^{p-1}} = \lim_{x\to\infty}\frac{e^{\frac{Bx^2}{B+x2}}}{x^{p-1}} = 0.$$

We can choose $\epsilon > 0$ such that f satisfy $(H3)$.

Proof of the Theorem 2.1. First we construct a positive subsolution of (1.1). For this, we let $\psi = \frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\alpha}}$. Let $w \in W$. Since $\nabla\psi = \epsilon^{\frac{1}{p-1}}\phi^{\frac{1-\alpha}{p-1+\alpha}}\nabla\phi$, then a calculation shows that

$$\begin{aligned}
&\int_\Omega |\nabla\psi|^{p-2}\,\nabla\psi\,.\nabla w\,dx\\
=&\epsilon\int_\Omega \phi^{\frac{(p-1)(1-\alpha)}{p-1+\alpha}}\,|\nabla\phi|^{p-2}\,\nabla\phi\,.\nabla w\,dx\\
=&\epsilon\int_\Omega |\nabla\phi|^{p-2}\,\nabla\phi\Big[\nabla(\phi^{1-\frac{\alpha p}{p-1+\alpha}}w) - \nabla(\phi^{1-\frac{\alpha p}{p-1+\alpha}})w\Big]dx\\
=&\epsilon\int_\Omega |\nabla\phi|^{p-2}\,\nabla\phi\,\nabla(\phi^{1-\frac{\alpha p}{p-1+\alpha}}w)dx - \epsilon\int_\Omega |\nabla\phi|^{p-2}\,\nabla\phi\,.\nabla(\phi^{1-\frac{\alpha p}{p-1+\alpha}})\,wdx\\
=&\epsilon\int_\Omega \lambda_1\phi^{-\frac{\alpha p}{p-1+\alpha}}\,\phi^p\,wdx - \epsilon\int_\Omega (1-\frac{\alpha p}{p-1+\alpha})\,\phi^{-\frac{\alpha p}{p-1+\alpha}}\,|\nabla\phi|^p\,wdx\\
=&\epsilon\int_\Omega [\lambda_1\phi^{p-\frac{\alpha p}{p-1+\alpha}} - (1-\frac{\alpha p}{p-1+\alpha})\phi^{-\frac{\alpha p}{p-1+\alpha}}\,|\nabla\phi|^p]\,wdx.
\end{aligned}$$

First we consider the case when $x \in \overline{\Omega_\delta}$. We have $(1-\frac{\alpha p}{p-1+\alpha})|\nabla\phi|^p \geq m$ and

$g(x) \geq -a$. Hence since $\lambda \leq \lambda^* = \frac{m\epsilon}{af(\epsilon^{\frac{1}{p-1}})}$, we have

$$\begin{aligned} -\epsilon(1-\frac{\alpha p}{p-1+\alpha})\phi^{-\frac{\alpha p}{p-1+\alpha}}|\nabla\phi|^p &\leq -m\epsilon\phi^{-\frac{\alpha p}{p-1+\alpha}} \\ &\leq -m\epsilon \\ &\leq -\lambda af(\epsilon^{\frac{1}{p-1}}) \\ &\leq -\lambda af(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\alpha}}), \end{aligned} \tag{2.1}$$

and since $\lambda \geq \lambda_* = \frac{\epsilon^{\frac{\alpha+p-1}{p-1}}\lambda_1(p-1+\alpha)^\alpha}{ap^\alpha}$, we have

$$\begin{aligned} \epsilon\phi^{-\frac{\alpha p}{p-1+\alpha}}\lambda_1\phi^p &\leq \epsilon\phi^{-\frac{\alpha p}{p-1+\alpha}}\lambda_1\phi^p \\ &\leq \frac{\lambda ap^\alpha}{\epsilon^{\frac{\alpha}{p-1}}(p-1+\alpha)^\alpha} \\ &\leq \frac{\lambda a}{(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\alpha}})^\alpha}. \end{aligned} \tag{2.2}$$

By combining (2.1) and (2.2) we see that

$$\begin{aligned} &\epsilon[\phi^{p-\frac{\alpha p}{p-1+\alpha}}\lambda_1-(1-\frac{\alpha p}{p-1+\alpha})\phi^{-\frac{\alpha p}{p-1+\alpha}}|\nabla\psi|^p] \\ \leq&\lambda g(x)[f(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\alpha}})-\frac{1}{(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\alpha}})^\alpha}]. \end{aligned}$$

On the other hand, on $\Omega-\overline{\Omega_\delta}$, we have $g(x) \geq b$ and $\sigma \leq \phi^{\frac{p}{p-1+\alpha}} \leq 1$. Thus for $\lambda \geq \lambda_* = \frac{\epsilon\lambda_1}{Nb}$, we have

$$\begin{aligned} &\epsilon\phi^{-\frac{\alpha p}{p-1+\alpha}}\lambda_1\phi^p-(1-\frac{\alpha p}{p-1+\alpha})\phi^{-\frac{\alpha p}{p-1+\alpha}}|\nabla\psi|^p) \\ \leq&\epsilon\lambda_1\phi^{p-\frac{\alpha p}{p-1+\alpha}} \\ \leq&\epsilon\lambda_1 \\ \leq&\lambda bN \\ \leq&\lambda b[f(\frac{p-1+\alpha}{p})\sigma\epsilon^{\frac{1}{p-1}}-\frac{1}{\left((\frac{p-1+\alpha}{p})\sigma\epsilon^{\frac{1}{p-1}}\right)^\alpha}] \\ \leq&\lambda g(x)[f(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\alpha}})-\frac{1}{\left((\frac{p-1+\alpha}{p})\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\alpha}}\right)^\alpha}] \\ =&\lambda g(x)[f(\psi)-\frac{1}{\psi^\alpha}]. \end{aligned}$$

Hence

$$\begin{aligned}
&\int_\Omega |\nabla\psi|^{p-2}\,\nabla\psi\,.\nabla w\,dx\\
=&\epsilon\int_\Omega [\lambda_1(\phi^{p-\frac{\alpha p}{p-1+\alpha}} - (1-\frac{\alpha p}{p-1+\alpha})\,\phi^{-\frac{\alpha p}{p-1+\alpha}}\,|\nabla\phi|^p]\,w\,dx\\
\leq&\int_\Omega \lambda g(x)[f(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\alpha}}) - \frac{1}{(\frac{p-1+\alpha}{p}\epsilon^{\frac{1}{p-1}}\phi^{\frac{p}{p-1+\alpha}})^\alpha}]\,w\,dx\\
=&\int_\Omega \lambda g(x)[f(\psi) - \frac{1}{\psi^\alpha}]\,w\,dx.
\end{aligned}$$

i.e. ψ is a subsolution of (1) for $\lambda \in [\lambda_*, \lambda^*]$.

Now we will construct a supersolution of (1.1). For this, we let $z := ce$ and $w \in W$. Since $\nabla z = c\,\nabla e$ then a calculation shows that

$$\begin{aligned}
\int_\Omega |\nabla z|^{p-2}\,\nabla z\,.\nabla\,w\,dx &= c^{p-1}\int_\Omega |\nabla e|^{p-2}\,\nabla e\,.\nabla w\,dx\\
&= c^{p-1}\int_\Omega w\,dx.
\end{aligned}$$

By $(H2)$ we can choose c large enough so that

$$(c\|e\|_\infty)^{p-1}(\lambda\|g(x)\|_\infty\|e\|_\infty)^{-1} \geq f(c\|e\|_\infty).$$

Hence

$$\begin{aligned}
c^{p-1} &\geq \lambda\|g(x)\|_\infty f(c\|e\|_\infty)\\
&\geq \lambda g(x) f(ce)\\
&\geq \lambda g(x)[f(ce) - \frac{1}{(ce)^\alpha}]\\
&= \lambda g(x)[f(z) - \frac{1}{z^\alpha}].
\end{aligned}$$

Thus we have

$$\begin{aligned}
\int_\Omega |\nabla z|^{p-2}\,\nabla z\,.\nabla w\,dx &= c^{p-1}\int_\Omega w dx\\
&\geq \int_\Omega \lambda g(x)[f(ce) - \frac{1}{(ce)^\alpha}]\,w\,dx\\
&= \int_\Omega \lambda g(x)[f(z) - \frac{1}{z^\alpha}]w\,dx.
\end{aligned}$$

i.e., z is a supersolution of (1.1) with $z \geq \psi$ for c large (note $|\nabla e| \neq 0$; on $\partial\Omega$). Thus, there exist a positive weak solution u of (1.1) such that $\psi \leq u \leq z$. This completes the proof of Theorem 2.1. □

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