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SOME GENERALIZED GRONWALL-LIKE RETARDED INEQUALITIES IN TWO INDEPENDENT VARIABLES ON TIME SCALES

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Abstract In this paper, we establish some new Gronwall-like inequalities in two independent variables which can be used as tools in the theory of integral equations with delay on time scales.

Keywords Gronwall inequalities, time scales, retarded integral inequalities, two independent variables.

MSC(2000) 26D15, 26D10.

1. Introduction

Integral inequalities play an important role in the qualitive analysis of differential and integral equations. Gronwall [5] proved the famous integral inequalities named after his name in 1919 and Bellman [1] was the first one who generalized the Gronwall inequality in 1934. From then on, the well-known Gronwall inequality has been extended as a useful tool in various contexts and among them many retarded inequalities have been discovered [3, 4, 6–9, 11]. Very recently, some scholars obtained some new integral inequalities with two independent variables. In the paper Ma & Yang [9], the authors established some new retarded Volterra integral inequalities with two independent variables.

Meanwhile, the study of the theory of dynamic equations on time scales is a new area of mathematics that received a lot of attentions. Very recently, in the paper Xu & Sun [10] the authors investigated some integral inequalities in two independent variables on time scales. For there is not many results of integral inequalities in two variables on time scales, in this paper, we intend to establish some new retarded Gronwall-type integral inequalities with two variables on time scales, which generalize several results that have been found.

First, we give some lemmas which will be used in the main results.

Lemma 1.1. *[6] Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$. Then for any $K > 0$,*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}.$$

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From Theorem 2.77 of Bohner & Peterson [2], it is easy to obtain the following Lemma.

Lemma 1.2. *Suppose* $u,\ b \in C_{rd}(\mathbb{T},\ \mathbb{R}),\ a \in \mathcal{R}^{+}$. *Then*

$$u^{\triangle}(t) \leq a(t)u(t) + b(t),\ t \in \mathbb{T},$$

implies

$$u(t) \leq u(t_0)e_a(t,t_0) + \int_{t_0}^{t} e_a(t,\sigma(s))b(s)\triangle s.$$

Where we define $u^{\triangle}(t)$ *to be the number with the property that given any* $\epsilon > 0$, *there is a neighborhood* U *of* t *such that*

$$\mid [u(\sigma(t)) - u(s)] - u^{\triangle}(t)[\sigma(t) - s] \mid \leq \varepsilon \mid \sigma(t) - s \mid,\ \forall s \in U.$$

2. Main Results

We suppose that the reader is familiar with the basic concepts on time scales for dynamic equations, which are omitted here.

Theorem 2.1. *Suppose that* $u(x,y),\ p(x,y),\ \phi(x,y)$ *are nonnegative functions defined on* $[x_0 - r,\infty)_{\mathbb{T}} \times [y_0 - r,\infty)_{\mathbb{T}}$ *which are right-dense continuous,* $\alpha(x),\ \beta(y)$, $a(x,y,t,s),\ b(x,y,t,s)$ *are nonnegative right-dense continuous functions and* $a(x,y,t,s), b(x,y,t,s)$ *are nondecreasing in the variables* $x,\ y$. $x_0 \leq \alpha(x) \leq x$, $y_0 \leq \beta(y) \leq y$, $\alpha^{\triangle}(x) > 0$, $\beta^{\triangle}(y) > 0$, $m \geq n \geq 0$, *with* $m \neq 0$, $r \geq 0$, *if* $u(x,y)$ *satisfies*

$$\begin{cases} u^m(x,y) \leq p(x,y) + \displaystyle\int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s)u^n(t-r,s-r)\triangle s\triangle t \\ \qquad\qquad + \displaystyle\int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s)u^n(t-r,s-r)\triangle s\triangle t, \quad x \geq x_0, \quad y \geq y_0, \\ u(x,y) \leq \phi(x,y),\ \phi(x,y) \leq p^{\frac{1}{m}}(x,y),\ x \in [x_0 - r, x_0] \text{ or } y \in [y_0 - r, y_0], \end{cases} \tag{2.1}$$

then for $x \geq x_0,\ y \geq y_0$, *any* $K > 0$, *we have*

$$\begin{aligned} u^m(x,y) \leq & p(x,y) + \int_{x_0}^{x} e_{-1}(q(l,y),q(x,y)) \\ & \times \frac{\partial}{\partial_l}[\int_{x_0}^{\alpha(l)} \int_{y_0}^{\beta(y)} a(l,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)\triangle s\triangle t \\ & + \frac{m-n}{m}K^{\frac{n}{m}} \int_{x_0}^{\alpha(l)} \int_{y_0}^{\beta(y)} a(l,y,t,s)\triangle s\triangle t \\ & + \int_{x_0}^{l} \int_{y_0}^{y} b(l,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)\triangle s\triangle t \\ & + \frac{m-n}{m}K^{\frac{n}{m}} \int_{x_0}^{l} \int_{y_0}^{y} b(l,y,t,s)\triangle s\triangle t]\triangle l, \end{aligned} \tag{2.2}$$

where $q(x,y) = \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \frac{n}{m}K^{\frac{n-m}{m}}a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x} \int_{y_0}^{y} \frac{n}{m}K^{\frac{n-m}{m}}b(x,y,t,s)\triangle s\triangle t$.

Proof. Define

$$z(x,y) := \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s) u^n(t-r,s-r) \triangle s \triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s) u^n(t-r,s-r) \triangle s \triangle t,$$

for $x \geq x_0,\ y \geq y_0$, and $z(x,y) = 0$, for $x,\ y \in R^2 \backslash E$, where $E = \{(x,y) | x \geq x_0,\ y \geq y_0\}$. Then

$$u^m(x,y) \leq p(x,y) + z(x,y),$$

for $x \geq x_0,\ y \geq y_0$ and z is nondecreasing in each of the variables.

From lemma 1.1, we obtain

$$\begin{aligned} u^n(x,y) \leq & [p(x,y)+z(x,y)]^{\frac{n}{m}} \\ \leq & \frac{n}{m} K^{\frac{n-m}{m}} [p(x,y)+z(x,y)] + \frac{m-n}{m} K^{\frac{n}{m}}, \\ & x \geq x_0,\ y \geq y_0, \end{aligned}$$

and

$$\begin{aligned} u^n(x,y) & \leq \phi^n(x,y) \leq p^{\frac{n}{m}}(x,y) = (p(x,y)+z(x,y))^{\frac{n}{m}} \\ & \leq \frac{n}{m} K^{\frac{n-m}{m}} [p(x,y)+z(x,y)] + \frac{m-n}{m} K^{\frac{n}{m}}, \\ & \quad x \in [x_0 - r, x_0] \text{ or } y \in [y_0 - r, y_0]. \end{aligned}$$

For $x \geq x_0,\ y \geq y_0$, we have

$$\begin{aligned} z_x^{\triangle} = & \alpha^{\triangle}(x) \int_{y_0}^{\beta(y)} a(\sigma(x), y, \alpha(x), s) u^n(\alpha(x)-r, s-r) \triangle s \\ & + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \partial_x a(x,y,t,s) u^n(t-r,s-r) \triangle s \triangle t \\ & + \int_{y_0}^{y} b(\sigma(x), y, x, s) u^n(x-r, s-r) \triangle s \\ & + \int_{x_0}^{x} \int_{y_0}^{y} \partial_x b(x,y,t,s) u^n(t-r,s-r) \triangle s \triangle t \\ \leq & \alpha^{\triangle}(x) \int_{y_0}^{\beta(y)} a(\sigma(x), y, \alpha(x), s) [\frac{n}{m} K^{\frac{n-m}{m}} p(\alpha(x)-r, s-r) \\ & + \frac{n}{m} K^{\frac{n-m}{m}} z(\alpha(x)-r, s-r) + \frac{m-n}{m} K^{\frac{n}{m}}] \triangle s \\ & + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \partial_x a(x,y,t,s) [\frac{n}{m} K^{\frac{n-m}{m}} p(t-r,s-r) \\ & + \frac{n}{m} K^{\frac{n-m}{m}} z(t-r,s-r) + \frac{m-n}{m} K^{\frac{n}{m}}] \triangle s \triangle t \\ & + \int_{y_0}^{y} b(\sigma(x), y, x, s) [\frac{n}{m} K^{\frac{n-m}{m}} p(x-r, s-r) \end{aligned}$$

$$
\begin{aligned}
&+\frac{n}{m}K^{\frac{n-m}{m}}z(x-r,s-r)+\frac{m-n}{m}K^{\frac{n}{m}}]\triangle s\\
&+\int_{x_0}^{x}\int_{y_0}^{y}\partial_x b(x,y,t,s)[\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)\\
&+\frac{n}{m}K^{\frac{n-m}{m}}z(t-r,s-r)+\frac{m-n}{m}K^{\frac{n}{m}}]\triangle s\triangle t,
\end{aligned}
$$

then

$$
\begin{aligned}
&z_x^{\triangle}-z(\sigma(x),y)[\alpha^{\triangle}(x)\int_{y_0}^{\beta(y)}a(\sigma(x),y,\alpha(x),s)\frac{n}{m}K^{\frac{n-m}{m}}\triangle s\\
&+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\partial_x a(x,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}\triangle s\triangle t+\int_{y_0}^{y}b(\sigma(x),y,x,s)\frac{n}{m}K^{\frac{n-m}{m}}\triangle s\\
&+\int_{x_0}^{x}\int_{y_0}^{y}\partial_x b(x,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}\triangle s\triangle t]\\
\leq&\alpha^{\triangle}(x)\int_{y_0}^{\beta(y)}a(\sigma(x),y,\alpha(x),s)[\frac{n}{m}K^{\frac{n-m}{m}}p(\alpha(x)-r,s-r)+\frac{m-n}{m}K^{\frac{n}{m}}]\triangle s\\
&+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\partial_x a(x,y,t,s)[\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)+\frac{m-n}{m}K^{\frac{n}{m}}]\triangle s\triangle t\\
&+\int_{y_0}^{y}b(\sigma(x),y,x,s)[\frac{n}{m}K^{\frac{n-m}{m}}p(x-r,s-r)+\frac{m-n}{m}K^{\frac{n}{m}}]\triangle s\\
&+\int_{x_0}^{x}\int_{y_0}^{y}\partial_x b(x,y,t,s)[\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)+\frac{m-n}{m}K^{\frac{n}{m}}]\triangle s\triangle t,
\end{aligned}
$$

then

$$
\begin{aligned}
&z_x^{\triangle}-z(\sigma(x),y)\frac{\partial}{\partial_x}[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\frac{n}{m}K^{\frac{n-m}{m}}a(x,y,t,s)\triangle s\triangle t\\
&+\int_{x_0}^{x}\int_{y_0}^{y}\frac{n}{m}K^{\frac{n-m}{m}}b(x,y,t,s)\triangle s\triangle t]\\
\leq&\frac{\partial}{\partial_x}[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}a(x,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)\triangle s\triangle t\\
&+\frac{m-n}{m}K^{\frac{n}{m}}\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}a(x,y,t,s)\triangle s\triangle t\\
&+\int_{x_0}^{x}\int_{y_0}^{y}b(x,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)\triangle s\triangle t\\
&+\frac{m-n}{m}K^{\frac{n}{m}}\int_{x_0}^{x}\int_{y_0}^{y}b(x,y,t,s)\triangle s\triangle t].
\end{aligned}
$$

Let

$$
\begin{aligned}
q(x,y):=&\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\frac{n}{m}K^{\frac{n-m}{m}}a(x,y,t,s)\triangle s\triangle t\\
&+\int_{x_0}^{x}\int_{y_0}^{y}\frac{n}{m}K^{\frac{n-m}{m}}b(x,y,t,s)\triangle s\triangle t,
\end{aligned}
$$

multiplying both sides by $e_{-1}(q(x,y),x_0)$, one obtains that

$$\begin{aligned}
&[z(x,y)e_{-1}(q(x,y),x_0)]_x^{\triangle}\\
\leq&\frac{\partial}{\partial_x}[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}a(x,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)\triangle s\triangle t\\
&+\frac{m-n}{m}K^{\frac{n}{m}}\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}a(x,y,t,s)\triangle s\triangle t\\
&+\int_{x_0}^{x}\int_{y_0}^{y}b(x,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)\triangle s\triangle t\\
&+\frac{m-n}{m}K^{\frac{n}{m}}\int_{x_0}^{x}\int_{y_0}^{y}b(x,y,t,s)\triangle s\triangle t]e_{-1}(q(x,y),x_0).
\end{aligned}$$

Integrating the both sides of above inequality from x_0 to x, we have

$$\begin{aligned}
&z(x,y)e_{-1}(q(x,y),x_0)\\
\leq&\int_{x_0}^{x}e_{-1}(q(l,y),x_0)\frac{\partial}{\partial_l}[\int_{x_0}^{\alpha(l)}\int_{y_0}^{\beta(y)}a(l,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)\triangle s\triangle t\\
&+\frac{m-n}{m}K^{\frac{n}{m}}\int_{x_0}^{\alpha(l)}\int_{y_0}^{\beta(y)}a(l,y,t,s)\triangle s\triangle t\\
&+\int_{x_0}^{l}\int_{y_0}^{y}b(l,y,t,s)\frac{n}{m}K^{\frac{n-m}{m}}p(t-r,s-r)\triangle s\triangle t\\
&+\frac{m-n}{m}K^{\frac{n}{m}}\int_{x_0}^{l}\int_{y_0}^{y}b(l,y,t,s)\triangle s\triangle t]\triangle l,
\end{aligned}$$

then we get the inequality (2.2). The proof is complete. □

Remark 2.1. When $m=n,\ \mathbb{T}=R,\ r=0$, the result is the Theorem 2.1 of paper Zhang & Meng [12].

For $p(x,y)\equiv p$, we can get the following corallary.

Corollary 2.1. *The conditions are the same as Theorem 2.1, with $p(x,y)\equiv p>0$, if $u(x,y)$ satisfies*

$$\begin{cases}
u^m(x,y)\leq p+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}a(x,y,t,s)u^n(t-r,s-r)\triangle s\triangle t\\
\qquad\qquad+\int_{x_0}^{x}\int_{y_0}^{y}b(x,y,t,s)u^n(t-r,s-r)\triangle s\triangle t,\ x\geq x_0,\ y\geq y_0,\\
u(x,y)\leq\phi(x,y),\ \phi(x,y)\leq p^{\frac{1}{m}},\quad x\in[x_0-r,x_0]\ or\ y\in[y_0-r,y_0],
\end{cases}\tag{2.3}$$

then for $x\geq x_0,\ y\geq y_0$, any $K>0$, we have

$$u^m(x,y)\leq(p+\frac{m-n}{n}k)e_{-1}(0,q(x,y))-\frac{m-n}{n}k,\tag{2.4}$$

where

$$q(x,y)=\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\frac{n}{m}K^{\frac{n-m}{m}}a(x,y,t,s)\triangle s\triangle t+\int_{x_0}^{x}\int_{y_0}^{y}\frac{n}{m}K^{\frac{n-m}{m}}b(x,y,t,s)\triangle s\triangle t.$$

Proof. For $x \geq x_0,\ y \geq y_0$, similar to the Theorem 2.1, we get

$$\begin{aligned}
z(x,y) \leq & \int_{x_0}^{x} e_{-1}(q(l,y),q(x,y)) \frac{\partial}{\partial l}\Big(\frac{n}{m} K^{\frac{n-m}{m}} p \int_{x_0}^{\alpha(l)} \int_{y_0}^{\beta(y)} a(l,y,t,s) \triangle s \triangle t \\
& + \frac{m-n}{m} K^{\frac{n}{m}} \int_{x_0}^{\alpha(l)} \int_{y_0}^{\beta(y)} a(l,y,t,s) \triangle s \triangle t \\
& + \frac{n}{m} K^{\frac{n-m}{m}} p \int_{x_0}^{l} \int_{y_0}^{y} b(l,y,t,s) \triangle s \triangle t \\
& + \frac{m-n}{m} K^{\frac{n}{m}} \int_{x_0}^{l} \int_{y_0}^{y} b(l,y,t,s) \triangle s \triangle t \Big) \triangle l \\
= & \int_{x_0}^{x} e_{-1}(q(l,y),q(x,y)) \frac{\partial}{\partial l}\Big((\frac{n}{m} K^{\frac{n-m}{m}} p + \frac{m-n}{m} K^{\frac{n}{m}}) \\
& [\int_{x_0}^{\alpha(l)} \int_{y_0}^{\beta(y)} a(l,y,t,s) \triangle s \triangle t + \int_{x_0}^{l} \int_{y_0}^{y} b(l,y,t,s) \triangle s \triangle t] \Big) \triangle l \\
= & (p + \frac{m-n}{n} K) \int_{x_0}^{x} e_{-1}(q(l,y),q(x,y)) \triangle q(l,y) \\
= & (p + \frac{m-n}{n} K)(e_{-1}(q(x_0,y),q(x,y)) - 1).
\end{aligned}$$

Then

$$z(x,y) \leq (p + \frac{m-n}{n} K)(e_{-1}(0,q(x,y)) - 1),$$

we get

$$u^m(x,y) \leq p + z(x,y) = (p + \frac{m-n}{n} K) e_{-1}(0,q(x,y)) - \frac{m-n}{n} K.$$

The proof is complete. □

Theorem 2.2. *Let $p,\ a,\ b,\ \alpha,\ \beta$ be the same as Theorem 2.1, p is nondecreasing in each variables and a, b have continual partial derivative about $x,\ y$, w is nonnegative and nondecreasing with $w(x_0) = x_0$, let $\int_{x_0}^{\infty} \frac{\triangle t}{w(t)} = \infty$ and $F(\gamma) = \int_{x_0+1}^{\gamma} \frac{\triangle t}{w(t)}$. If $u(x,y)$ satisfies*

$$\begin{cases} u(x,y) \leq p(x,y) + \displaystyle\int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(x)} a(x,y,t,s) w(u(t-r,s-r)) \triangle s \triangle t \\ \qquad + \displaystyle\int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s) w(u(t-r,s-r)) \triangle s \triangle t,\ x \geq x_0,\ y \geq y_0, \\ u(x,y) \leq \phi(x,y),\ \phi(x,y) \leq p(x_0,y_0),\ x \in [x_0 - r, x_0] \text{ or } y \in [y_0 - r, y_0], \end{cases} \tag{2.5}$$

then for $x \geq x_0 + r,\ y \geq y_0 + r,$ we have

$$u(x,y) \leq F^{-1}\Big(F(p(x,y)) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s) \triangle s \triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s) \triangle s \triangle t\Big),$$

and for $x_0 \leq x \leq x_0 + r$ or $y_0 \leq y \leq y_0 + r,$ we have

$$\begin{aligned} u(x,y) \leq & p(x,y) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s) w(\phi(t-r,s-r)) \triangle s \triangle t \\ & + \int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s) w(\phi(t-r,s-r)) \triangle s \triangle t. \end{aligned}$$

Proof. Let

$$z(x,y) := \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s)w(u(t-r,s-r))\triangle s\triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s)w(u(t-r,s-r))\triangle s\triangle t.$$

For $x_0 \le x \le x_0 + r$ or $y_0 \le y \le y_0 + r$, we have

$$u(x,y) \le p(x,y) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s)w(\phi(t-r,s-r))\triangle s\triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s)w(\phi(t-r,s-r))\triangle s\triangle t.$$

For $x \ge x_0 + r$, $y \ge y_0 + r$, we have $u(x,y) \le p(x,y) + z(x,y)$.

Let $T_1,\ T_2$ be arbitrary real numbers and $T_1 > x_0 + r,\ T_2 > y_0 + r$.

For $x_0 + r \le x \le T_1,\ y_0 + r \le y \le T_2$, we have

$$\begin{aligned} z_x^{\triangle} =& \alpha^{\triangle}(x) \int_{y_0}^{\beta(y)} a(\sigma(x),y,\alpha(x),s)w(u(\alpha(x)-r,s-r))\triangle s \\ &+ \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \partial_x a(x,y,t,s)w(u(t-r,s-r))\triangle s\triangle t \\ &+ \int_{y_0}^{y} b(\sigma(x),y,x,s)w(u(x-r,s-r))\triangle s \\ &+ \int_{x_0}^{x} \int_{y_0}^{y} \partial_x b(x,y,t,s)w(u(t-r,s-r))\triangle s\triangle t, \end{aligned}$$

and

$$\begin{aligned} z_{xy}^{\triangle\triangle} =& \alpha^{\triangle}(x)\beta^{\triangle}(y)a(\sigma(x),\sigma(y),\alpha(x),\beta(y))w(u(\alpha(x)-r,\beta(y)-r)) \\ &+ \alpha^{\triangle}(x) \int_{y_0}^{\beta(y)} \partial_y a(\sigma(x),y,\alpha(x),s)w(u(\alpha(x)-r,s-r)\triangle s \\ &+ \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \partial_x\partial_y a(x,y,t,s)w(u(t-r,s-r))\triangle s\triangle t \\ &+ \beta^{\triangle}(y) \int_{x_0}^{\alpha(x)} \partial_x a(x,\sigma(y),t,\beta(y))w(u(t-r,\beta(y)-r))\triangle t \\ &+ b(\sigma(x),\sigma(y),x,y)w(u(x-r,y-r)) \\ &+ \int_{y_0}^{y} \partial_y b(\sigma(x),y,x,s)w(u(x-r,s-r))\triangle s \\ &+ \int_{x_0}^{x} \int_{y_0}^{y} \partial_x\partial_y b(x,y,t,s)w(u(t-r,s-r))\triangle s\triangle t \\ &+ \int_{x_0}^{x} \partial_x b(x,\sigma(y),t,y)w(u(t-r,y-r))\triangle t. \end{aligned}$$

If $T_1 \ge x \ge t \ge x_0 + r,\ T_2 \ge y \ge s \ge y_0 + r$, we can easily get

$$u(t-r,s-r) \le p(T_1-r,T_2-r) + z(\sigma(x)-r,\sigma(y)-r),$$

and if $x_0 \leq t < x_0 + r$ *or* $y_0 \leq s < y_0 + r$, then

$$u(t-r, s-r) \leq p(x_0, y_0) \leq p(T_1-r, T_2-r) \leq p(T_1-r, T_2-r) + z(\sigma(x)-r, \sigma(y)-r).$$

We have

$$\begin{aligned} z_{xy}^{\triangle\triangle} \leq & w(p(T_1 - r, T_2 - r) + z(\sigma(x) - r, \sigma(y) - r))\times \\ & \frac{\partial^2}{\partial_x \partial_y}\Big(\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t\Big). \end{aligned}$$

In other words

$$\begin{aligned} & \frac{z_{xy}^{\triangle\triangle}}{w(p(T_1 - r, T_2 - r) + z(\sigma(x) - r, \sigma(y) - r))} \\ \leq & \frac{\partial^2}{\partial_x \partial_y}\Big(\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t\Big). \end{aligned}$$

Further we can get

$$\begin{aligned} & \frac{\partial}{\partial_y}\Big(\frac{z_x^{\triangle}(x,y)}{w(p(T_1 - r, T_2 - r) + z(x,y))}\Big) \\ \leq & \frac{z_{xy}^{\triangle\triangle}}{w(p(T_1 - r, T_2 - r) + z(\sigma(x) - r, \sigma(y) - r))} \\ \leq & \frac{\partial^2}{\partial_x \partial_y}\Big(\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t\Big). \end{aligned}$$

Integrating both sides from y_0 to y, we obtain

$$\begin{aligned} & \frac{z_x^{\triangle}(x,y)}{w(p(T_1 - r, T_2 - r) + z(x,y))} \\ \leq & \frac{\partial}{\partial_x}\Big(\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t\Big). \end{aligned}$$

Integrating both sides from x_0 to x, we obtain

$$\begin{aligned} & F(p(T_1, T_2) + z(x,y)) \\ \leq & F(p(T_1, T_2)) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t. \end{aligned}$$

So

$$\begin{aligned} & p(T_1, T_2) + z(x,y) \\ \leq & F^{-1}\Big(F(p(T_1, T_2)) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t\Big). \end{aligned}$$

Let $x = T_1,\ y = T_2$, we have

$$\begin{aligned} p(T_1, T_2) + z(T_1, T_2) \leq & F^{-1}\Big(F(p(T_1, T_2)) + \int_{x_0}^{\alpha(T_1)}\int_{y_0}^{\beta(T_2)} a(T_1, T_2, t, s)\triangle s\triangle t \\ & + \int_{x_0}^{T_1}\int_{y_0}^{T_2} b(T_1, T_2, t, s)\triangle s\triangle t\Big). \end{aligned}$$

Due to $T_1,\ T_2$ are arbitrary real numbers, we get

$$u(x,y) \le F^{-1}\Big(F(p(x,y)) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t\Big).$$

The proof is complete. $\square$

Theorem 2.3. *Let $p,\ a,\ b,\ \alpha,\ \beta$ be the same as Theorem 2.1, $p(x,y)$ is nondecreasing in each variables $x,\ y$, and $a,\ b$ have continual partial derivative, τ is defined on $[\alpha,\infty)_{\mathbb{T}}$, where $\alpha = \min\{\tau(x_0),\tau(y_0)\}$, $\tau(t) \le t$, w, h are nonnegative and nondecreasing with $w(x_0) = x_0$, let $\int_{x_0}^{\infty} \frac{\triangle t}{q(t)} = \infty$ and $F(\gamma) = \int_{x_0+1}^{\gamma} \frac{\triangle t}{q(t)}$, where $q(t) = w(h^{-1}(t))$. If $u(x,y)$ satisfies*

$$\begin{cases} h(u(x,y)) \le p(x,y) + \displaystyle\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)w(u(\tau(t),\tau(s)))\triangle s\triangle t \\ \qquad + \displaystyle\int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)w(u(\tau(t),\tau(s)))\triangle s\triangle t,\ x \ge x_0\ ,y \ge y_0, \\ u(x,y) \le \phi(x,y),\ \phi(x,y) \le h^{-1}(p(x,y)),\ x \in [\tau(x_0),x_0]\ \text{or}\ y \in [\tau(y_0),y_0], \end{cases} \tag{2.6}$$

then for $x_0 \le x \le \tau^{-1}(x_0)$ or $y_0 \le y \le \tau^{-1}(y_0)$, we have

$$\begin{aligned} u(x,y) \le & h^{-1}\Big(p(x,y) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)w(\phi(\tau(t),\tau(s)))\triangle s\triangle t \\ & + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)w(\phi(\tau(t),\tau(s)))\triangle s\triangle t\Big), \end{aligned}$$

and for $x \ge \tau^{-1}(x_0)$, $y \ge \tau^{-1}(y_0)$, we have

$$\begin{aligned} u(x,y) \le & h^{-1}\Big(F^{-1}(F(p(x,y)) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t \\ & + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t)\Big). \end{aligned}$$

Proof. Let

$$\begin{aligned} z(x,y) := & \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)w(u(\tau(t),\tau(s)))\triangle s\triangle t \\ & + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)w(u(\tau(t),\tau(s)))\triangle s\triangle t. \end{aligned}$$

For $x_0 \le x \le \tau^{-1}(x_0)$ or $y_0 \le y \le \tau^{-1}(y_0)$, we can easily get

$$\begin{aligned} u(x,y) \le & h^{-1}\Big(p(x,y) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)w(\phi(\tau(t),\tau(s)))\triangle s\triangle t \\ & + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)w(\phi(\tau(t),\tau(s)))\triangle s\triangle t\Big). \end{aligned}$$

For $x \ge \tau^{-1}(x_0)$, $y \ge \tau^{-1}(y_0)$, we have $u(x,y) \le h^{-1}(p(x,y) + z(x,y))$.

Let $T_1,\ T_2$ be arbitrary real numbers and $T_1 > \tau^{-1}(x_0),\ T_2 > \tau^{-1}(y_0)$.

For $\tau^{-1}(x_0) \leq x \leq T_1,\ \tau^{-1}(y_0) \leq y \leq T_2$, we have

$$\begin{aligned}
z_x^{\triangle} =&\alpha^{\triangle}(x) \int_{y_0}^{\beta(y)} a(\sigma(x), y, \alpha(x), s) w(u(\tau(\alpha(x)), \tau(s))) \triangle s \\
&+ \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \partial_x a(x, y, t, s) w(u(\tau(t), \tau(s))) \triangle s \triangle t \\
&+ \int_{y_0}^{y} b(\sigma(x), y, x, s) w(u(\tau(x), \tau(s))) \triangle s \\
&+ \int_{x_0}^{x} \int_{y_0}^{y} \partial_x b(x, y, t, s) w(u(\tau(t), \tau(s))) \triangle s \triangle t.
\end{aligned}$$

If $T_1 \geq x \geq t \geq \tau^{-1}(x_0),\ T_2 \geq y \geq s \geq \tau^{-1}(y_0)$, we can easily get $u(\tau(t), \tau(s)) \leq h^{-1}(p(\tau(T_1), \tau(T_2)) + z(x, y))$, but if $x_0 \leq t \leq \tau^{-1}(x_0)$ or $y_0 \leq s \leq \tau^{-1}(y_0)$, then $u(\tau(t), \tau(s)) \leq h^{-1}(p(\tau(t), \tau(s))) \leq h^{-1}(p(\tau(T_1), \tau(T_2))) \leq h^{-1}(p(\tau(T_1), \tau(T_2)) + z(x, y))$, we have

$$\begin{aligned}
z_{xy}^{\triangle\triangle} =&\alpha^{\triangle}(x) \beta^{\triangle}(y) a(\sigma(x), \sigma(y), \alpha(x), \beta(y)) w(u(\tau(\alpha(x)), \tau(\beta(y)))) \\
&+ \alpha^{\triangle}(x) \int_{y_0}^{\beta(y)} \partial_y a(\sigma(x), y, \alpha(x), s) w(u(\tau(\alpha(x)), \tau(s)) \triangle s \\
&+ \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \partial_x \partial_y a(x, y, t, s) w(u(\tau(t), \tau(s))) \triangle s \triangle t \\
&+ \beta^{\triangle}(y) \int_{x_0}^{\alpha(x)} \partial_x a(x, \sigma(y), t, \beta(y)) w(u(\tau(t), \tau(\beta(y)))) \triangle t \\
&+ b(\sigma(x), \sigma(y), x, y) w(u(\tau(x), \tau(y))) \\
&+ \int_{y_0}^{y} \partial_y b(\sigma(x), y, x, s) w(u(\tau(x), \tau(s))) \triangle s \\
&+ \int_{x_0}^{x} \int_{y_0}^{y} \partial_x \partial_y b(x, y, t, s) w(u(\tau(t), \tau(s))) \triangle s \triangle t \\
&+ \int_{x_0}^{x} \partial_x b(x, \sigma(y), t, y) w(u(\tau(t), \tau(y))) \triangle t \\
\leq& w(h^{-1}(p(\tau(T_1), \tau(T_2)) + z(x, y))) \times \\
&\frac{\partial^2}{\partial_x \partial_y} \Big(\int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x, y, t, s) \triangle s \triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x, y, t, s) \triangle s \triangle t \Big).
\end{aligned}$$

Let $w(h^{-1}(t)) = q(t)$ and we get

$$\begin{aligned}
&\frac{z_{xy}^{\triangle\triangle}}{q(p(\tau(T_1), \tau(T_2)) + z(x, y))} \\
\leq& \frac{\partial^2}{\partial_x \partial_y} \Big(\int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x, y, t, s) \triangle s \triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x, y, t, s) \triangle s \triangle t \Big).
\end{aligned}$$

For

$$\frac{\partial}{\partial_y} \left(\frac{z_x^{\triangle}(x, y)}{q(p(\tau(T_1), \tau(T_2)) + z(x, y))} \right) \leq \frac{z_{xy}^{\triangle\triangle}}{q(p(\tau(T_1), \tau(T_2)) + z(x, y))}$$

$$\leq \frac{\partial^2}{\partial_x\partial_y}\Big(\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t\Big),$$

we integrate both sides from y_0 to y, then

$$\begin{aligned}&\frac{z_x^{\triangle}(x,y)}{q(p(\tau(T_1),\tau(T_2))+z(x,y))}\\ \leq&\frac{\partial}{\partial_x}\Big(\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t\Big).\end{aligned}$$

Integrating both sides from x_0 to x, we obtain

$$\begin{aligned}&F(p(T_1,T_2)+z(x,y))\\ \leq&F(p(T_1,T_2)) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t.\end{aligned}$$

Inversing the both sides of above inequality, we have

$$\begin{aligned}&p(T_1,T_2)+z(x,y)\\ \leq&F^{-1}\Big(F(p(T_1,T_2)) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t\Big).\end{aligned}$$

Let $x=T_1,\ y=T_2$, we have

$$\begin{aligned}p(T_1,T_2)+z(T_1,T_2) \leq&F^{-1}\Big(F(p(T_1,T_2)) + \int_{x_0}^{\alpha(T_1)}\int_{y_0}^{\beta(T_2)} a(T_1,T_2,t,s)\triangle s\triangle t\\ &+ \int_{x_0}^{T_1}\int_{y_0}^{T_2} b(T_1,T_2,t,s)\triangle s\triangle t\Big).\end{aligned}$$

For $T_1,\ T_2$ are arbitrary real numbers, we get

$$\begin{aligned}u(x,y) \leq&h^{-1}\Big(F^{-1}(F(p(x,y)) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(x,y,t,s)\triangle s\triangle t\\ &+ \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,t,s)\triangle s\triangle t)\Big).\end{aligned}$$

The proof is complete. □

Remark 2.2. When $\mathbb{T}=R,\ x_0=y_0=0,\ h(t)=t,\ \tau(t)=t$, the result is the Theorem 2.3 of Zhang & Meng [12].

For the special cases, we can get the following results.

Theorem 2.4. *The conditions are the same as Theorem 2.3. If* $q(t)\leq t$, *if* $u(x,y)$ *satisfies* (2.6). *Then for* $x\geq x_0,\ y\geq y_0$, *we get*

$$u(x,y)\leq h^{-1}[p(x,y)e_{k(\cdot,y)}(x,x_0)],$$

where

$$\begin{aligned}k(x,y)=&\alpha^{\triangle}(x)\int_{y_0}^{\beta(y)} a(\sigma(x),y,\alpha(x),s)\triangle s + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \partial_x a(x,y,t,s)\triangle s\triangle t\\ &+\int_{y_0}^{y} b(\sigma(x),y,x,s)\triangle s + \int_{x_0}^{x}\int_{y_0}^{y} \partial_x b(x,y,t,s)\triangle s\triangle t.\end{aligned}$$

Proof. Let

$$h(z(x,y)) := p(x,y) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s) w(u(\tau(t),\tau(s))) \triangle s \triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s) w(u(\tau(t),\tau(s))) \triangle s \triangle t.$$

When $\tau(x) \geq x_0,\ \tau(y) \geq y_0$, we have

$$u(\tau(x),\tau(y)) \leq z(\tau(x),\tau(y)) \leq z(x,y).$$

When $\tau(x_0) \leq \tau(x) \leq x_0,\ or\ \tau(y_0) \leq \tau(y) \leq y_0$, we have

$$u(\tau(x),\tau(y)) \leq \phi(\tau(x),\tau(y)) \leq h^{-1}(p(\tau(x),\tau(y))) \leq z(\tau(x),\tau(y)) \leq z(x,y).$$

Then we get

$$h(z(x,y)) \leq p(x,y) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s) w(z(t,s)) \triangle s \triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s) w(z(t,s)) \triangle s \triangle t.$$

Note that $p(x,y)$ is nonnegative and nondecreasing. Let $\varepsilon > 0$ be given, we obtain

$$\frac{h(z(x,y))}{p(x,y)+\varepsilon} \leq 1 + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s) \frac{w(z(t,s))}{p(t,s)+\varepsilon} \triangle s \triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s) \frac{w(z(t,s))}{p(t,s)+\varepsilon} \triangle s \triangle t.$$

Let

$$g(x,y) := 1 + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} a(x,y,t,s) \frac{w(z(t,s))}{p(t,s)+\varepsilon} \triangle s \triangle t + \int_{x_0}^{x} \int_{y_0}^{y} b(x,y,t,s) \frac{w(z(t,s))}{p(t,s)+\varepsilon} \triangle s \triangle t,$$

we have

$$z(x,y) \leq h^{-1}[(p(x,y)+\varepsilon) g(x,y)].$$

By the above inequality and $q(t) \leq t$, we get

$$\begin{aligned} g_x^{\triangle} =& \alpha^{\triangle}(x) \int_{y_0}^{\beta(y)} a(\sigma(x),y,\alpha(x),s) \frac{w(z(\alpha(x),s))}{p(\alpha(x),s)+\varepsilon} \triangle s \\ &+ \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \partial_x a(x,y,t,s) \frac{w(z(t,s))}{p(t,s)+\varepsilon} \triangle s \triangle t \\ &+ \int_{y_0}^{y} b(\sigma(x),y,x,s) \frac{w(z(x,s))}{p(x,s)+\varepsilon} \triangle s \\ &+ \int_{x_0}^{x} \int_{y_0}^{y} \partial_x b(x,y,t,s) \frac{w(z(t,s))}{p(t,s)+\varepsilon} \triangle s \triangle t \\ \leq& g(x,y) [\alpha^{\triangle}(x) \int_{y_0}^{\beta(y)} a(\sigma(x),y,\alpha(x),s) \triangle s + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \partial_x a(x,y,t,s) \triangle s \triangle t \\ &+ \int_{y_0}^{y} b(\sigma(x),y,x,s) \triangle s + \int_{x_0}^{x} \int_{y_0}^{y} \partial_x b(x,y,t,s) \triangle s \triangle t]. \end{aligned}$$

Let

$$k(x,y) :=\alpha^{\triangle}(x)\int_{y_0}^{\beta(y)} a(\sigma(x),y,\alpha(x),s)\triangle s+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \partial_x a(x,y,t,s)\triangle s\triangle t$$
$$+\int_{y_0}^{y} b(\sigma(x),y,x,s)\triangle s+\int_{x_0}^{x}\int_{y_0}^{y} \partial_x b(x,y,t,s)\triangle s\triangle t.$$

Then we have

$$g^{\triangle}(x,y)\le g(x,y)k(x,y).$$

From lemma 1.2 we get

$$g(x,y)\le e_{k(\cdot,y)}(x,x_0).$$

Then

$$u(x,y)\le h^{-1}[(p(x,y)+\varepsilon)e_{k(\cdot,y)}(x,x_0)],$$

letting $\varepsilon\to 0$, we immediately obtain the required inequality. The proof is complete. □

Theorem 2.5. *The conditions are the same as Theorem 2.3. If* $p(x,y)=p_1(x)+p_2(y)$, *and* $u(x,y)$ *satisfies*

$$\begin{cases} h(u(x,y))\le p_1(x)+p_2(y)+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(x)} a(t,s)w(u(\tau(t),\tau(s)))\triangle s\triangle t \\ \qquad +\int_{x_0}^{x}\int_{y_0}^{y} b(t,s)w(u(\tau(t),\tau(s)))\triangle s\triangle t\ , \quad x\ge x_0\ , y\ge y_0, \\ u(x,y)\le\phi(x,y),\ \ \phi(x,y)\le h^{-1}(p(x,y)),\ \ x\in[\tau(x_0),x_0]\ \text{or}\ y\in[\tau(y_0),y_0], \end{cases} \tag{2.7}$$

then for $x\ge x_0$, $y\ge y_0$, *we get*

$$u(x,y)\le h^{-1}\Big(F^{-1}(F(p_1(x_0)+p_2(y))+F(p_1(x)+p_2(y_0))-F(p_1(x_0)+p_2(y_0))$$
$$+\int_{x_0}^{x}\int_{y_0}^{y}[\alpha^{\triangle}(t)\beta^{\triangle}(s)a(\alpha(t),\beta(s))+b(t,s)]\triangle s\triangle t)\Big).$$

Proof. From the proof of Theorem 2.4, we can get

$$h(z(x,y))\le p_1(x)+p_2(y)+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(t,s)w(z(t,s))\triangle s\triangle t$$
$$+\int_{x_0}^{x}\int_{y_0}^{y} b(t,s)w(z(t,s))\triangle s\triangle t.$$

Let

$$g(x,y) :=p_1(x)+p_2(y)+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} a(t,s)w(z(t,s))\triangle s\triangle t$$
$$+\int_{x_0}^{x}\int_{y_0}^{y} b(t,s)w(z(t,s))\triangle s\triangle t.$$

Then we have

$$g(x_0,y)=p_1(x_0)+p_2(y),\ \ g(x,y_0)=p_1(x)+p_2(y_0),$$

and

$$z(x,y) \leq h^{-1}(g(x,y)).$$

Then

$$g_x^{\triangle} = p_1^{\triangle}(x) + \alpha^{\triangle}(x)\int_{y_0}^{\beta(y)} a(\alpha(x),s)w(z(\alpha(x),s))\triangle s + \int_{y_0}^{y} b(x,s)w(z(x,s))\triangle s,$$

and

$$\begin{aligned} g_{xy}^{\triangle\triangle} &= \alpha^{\triangle}(x)\beta^{\triangle}(y)a(\alpha(x),\beta(y))w(z(\alpha(x),\beta(y))) + b(x,y)w(z(x,y)) \\ &\leq w(z(x,y))[\alpha^{\triangle}(x)\beta^{\triangle}(y)a(\alpha(x),\beta(y)) + b(x,y)] \\ &\leq q(g(x,\sigma(y)))[\alpha^{\triangle}(x)\beta^{\triangle}(y)a(\alpha(x),\beta(y)) + b(x,y)]. \end{aligned}$$

For

$$\frac{\partial}{\partial_y}(\frac{g_x^{\triangle}(x,y)}{q(g(x,y))}) \leq \frac{g_{xy}^{\triangle\triangle}}{q(g(x,\sigma(y)))},$$

we have

$$\frac{\partial}{\partial_y}(\frac{g_x^{\triangle}(x,y)}{q(g(x,y))}) \leq \alpha^{\triangle}(x)\beta^{\triangle}(y)a(\alpha(x),\beta(y)) + b(x,y).$$

Integrating both sides from y_0 to y, we get

$$\frac{g_x^{\triangle}(x,y)}{q(g(x,y))} \leq \frac{p_1^{\triangle}(x)}{q(p_1(x)+p_2(y_0))} + \int_{y_0}^{y}[\alpha^{\triangle}(x)\beta^{\triangle}(s)a(\alpha(x),\beta(s)) + b(x,s)]\triangle s,$$

then integrating both sides from x_0 to x, we get

$$\begin{aligned} F(g(x,y)) \leq& F(p_1(x_0)+p_2(y)) + \int_{x_0}^{x}\frac{p_1^{\triangle}(t)}{q(p_1(t)+p_2(y_0))}\triangle t \\ &+ \int_{x_0}^{x}\int_{y_0}^{y}[\alpha^{\triangle}(t)\beta^{\triangle}(s)a(\alpha(t),\beta(s)) + b(t,s)]\triangle s\triangle t \\ =& F(p_1(x_0)+p_2(y)) + F(p_1(x)+p_2(y_0)) - F(p_1(x_0)+p_2(y_0)) \\ &+ \int_{x_0}^{x}\int_{y_0}^{y}[\alpha^{\triangle}(t)\beta^{\triangle}(s)a(\alpha(t),\beta(s)) + b(t,s)]\triangle s\triangle t. \end{aligned}$$

Inversing the both sides of above inequality, we have

$$\begin{aligned} g(x,y) \leq& F^{-1}\Big(F(p_1(x_0)+p_2(y)) + F(p_1(x)+p_2(y_0)) - F(p_1(x_0)+p_2(y_0)) \\ &+ \int_{x_0}^{x}\int_{y_0}^{y}[\alpha^{\triangle}(t)\beta^{\triangle}(s)a(\alpha(t),\beta(s)) + b(t,s)]\triangle s\triangle t\Big). \end{aligned}$$

Finally, we get the required inequality

$$\begin{aligned} u(x,y) \leq& h^{-1}\Big(F^{-1}(F(p_1(x_0)+p_2(y)) + F(p_1(x)+p_2(y_0)) - F(p_1(x_0)+p_2(y_0)) \\ &+ \int_{x_0}^{x}\int_{y_0}^{y}[\alpha^{\triangle}(t)\beta^{\triangle}(s)a(\alpha(t),\beta(s)) + b(t,s)]\triangle s\triangle t)\Big). \end{aligned}$$

The proof is complete. $\square$

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