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A NOTE ON MONOTONE ITERATION METHOD FOR TRAVELING WAVES OF REACTION-DIFFUSION SYSTEMS WITH TIME DELAY

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Abstract For a monotone reaction-diffusion system with or without time delay, a standard approach to show the existence of a mono-stable traveling wave solution is the monotone iteration that requires the construction of a pair of upper and lower solution. In this note we will show that the monotone iteration approach can be improved by just constructing an upper solution. This improvement gives more freedom for the construction of an upper solution.

Keywords Reaction-diffusion system, time delay, traveling waves, monotone iteration.

MSC(2000) 34K, 35K.

1. Introduction

The reaction-diffusion systems with or without time delay have been served as models for many problems in biology, epidemiology, ecology, engineering and physics [1,3,4,6]. One of important solutions, the traveling wave solution that is of interest both in application and mathematics itself, has been extensively studied, and variety of techniques have been developed to investigate the existence of wave solutions [2,5].

In the case that a system is monotonic, the monotone iteration now becomes a standard approach in literature to show the existence of a mono-stable traveling wave solution (a traveling wave connecting an unstable equilibrium and a stable equilibrium). That is, to show the existence of a traveling wave solution by constructing a pair of upper and lower solution. In this note we will show that, by a truncation approach similar to the idea used in [5], the monotone iteration approach can be improved by just constructing an upper solution. The advantage of this improvement is twofold: First, requiring only an upper solution not only simplifies the procedure but will release a restriction on the upper solution. To be specific, recall that in the monotone iteration approach we need not only to construct an upper solution and a lower solution, says $U(\xi)$ and $u(\xi)$, respectively, but also has the constraint on the upper and lower solution as

$$U(\xi) \geq u(\xi). \tag{1.1}$$

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Hence, the need of only an upper solution without the constraint (1.1) gives more freedom for the construction of an upper solution. The second advantage can be seen in the following Corollary 1.1.

In this paper, we will consider the following class of time delayed reaction-diffusion systems

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + F\left(u(x,t), \int_{-\sigma}^{0} d\eta(\theta)u(x,t+\theta)\right), \tag{1.2}$$

where $u(x,t) \in \mathbb{R}^n$, $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function, $D = \mathrm{dig}(d_1, \cdots, d_n)$ is a nonnegative and nonzero diagonal matrix, $\eta : [-\sigma, 0] \to \mathbb{R}^{n\times n}$ is of bounded variation with $\sigma > 0$. We suppose that

H1 There is a strictly positive vector $E^* \in \mathbb{R}^n$ such that

$$F(0,0) = F(E^*, \eta^* E^*) = 0, \text{ where } \eta^* = \int_{-\sigma}^{0} d\eta(\theta).$$

Thus 0 and E^* are two equilibrium points of $(2,1)$. We look for a traveling wave front of (1.2) connecting the equilibrium points 0 and E^*, i.e., a solution of the form $u(x,s) = U(x\cdot\nu + cs)$ satisfying the boundary condition

$$U(-\infty) = 0, \qquad U(\infty) = E^*, \tag{1.3}$$

where $\nu \in \mathbb{R}^n$ is a unit vector and $c \in \mathbb{R}$ is a wave speed. A straightforward substitution yields that $U(t)$ with $t = x\cdot k + cs$ satisfying the system

$$c\dot{U}(t) = D\ddot{U}(t) + F\Big(U(t), \int_{-\sigma}^{0} d\eta(\theta)U(t+c\theta)\Big). \tag{1.4}$$

Let $\mathcal{R}$ be a rectangle region:

$$\mathcal{R} = \big\{0 \le u \le E^*\big\}.$$

(Here for $u = (u_1, \cdots, u_n), v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, $u \ll (\le) v$ if $u_i < (\le) v_i$ for $i = 1, \cdots, n$.)

We further suppose that (1.2) is a monotone system. That is, the functions $\eta(\theta) = [\eta_{ij}(\theta)]$ and $F(u,v) = (F_1(u,v), \cdots, F_n(u,v))$ satisfy the following conditions:

H2 $\eta_{ij}(\theta)$ is non-decreasing for all $i, j = 1, \cdots, n$.

H3 For $u, v \in \mathcal{R}$,

$$\begin{aligned} \frac{\partial F_i(u,v)}{\partial u_j} &\ge 0, \quad i,j = 1,\cdots,n, i \ne j, \\ \frac{\partial F_i(u,v)}{\partial v_j} &\ge 0, \quad i,j = 1,\cdots. \end{aligned} \tag{1.5}$$

In addition to **H1 - H3**, we suppose

H4 There is a positive vector $p \in \mathbb{R}^n$ and a positive number s_0 such that $F(sp, s\eta^* p) \ge 0$ for all $0 < s \le s_0$.

In a rough sense Assumption **H4** is equivalent to say that the equilibrium point 0 is unstable with respect to the following delay differential equations, which is called the reaction equation corresponding to (1.2):

$$\frac{\partial u(t)}{\partial t} = F\left(u(t), \int_{-\sigma}^{0} d\eta(\theta)u(t+\theta)\right),$$

The main purpose of this chapter to prove that following Theorem.

Theorem 1.1. *Suppose that Assumptions* **H1** - **H4** *are satisfied. If, in addition, suppose that, for a fixed* $c > 0$,

(a1) *There is a monotone increasing function* $U \in C(\mathbb{R}; \mathcal{R})$ *such that*

$$U(t_0) << E^* \quad \text{for some } t_0 \quad \text{and} \quad U(-\infty) = 0 << U(t), \qquad t > -\infty.$$

(a2) *For* $i = 1, \cdots, n$, *there is a* $\delta < t_i \leq \infty$ *such that* $\dot{U}_i(t)$ *and* $\ddot{U}_i(t)$ *are continuous on* $(\delta, t_i]$, *in addition,*

$$\begin{aligned} &c\dot{U}_i(t) \geq d_i \ddot{U}_i(t) + F_i\big(U_i(t), \int_{-\sigma}^{0} d\eta(\theta)U(t+c\theta)\big), \quad t \in (-\infty, t_i). \\ &U_i(t) = E_i^*, \qquad\qquad t \geq t_i \\ &i = 1, 2, \cdots, n. \end{aligned} \tag{1.6}$$

(a3) *For* $V_0 \in \mathcal{R}$, $F(V_0, \eta^* V_0) = 0$ *if and only if* V_0 *or* $V_0 = E^*$.

Then the system (1.2) has a traveling wave solution of wave speed c connecting 0 *and* E^*.

To prove Theorem 1.1, we need to establish a few lemmas, which will be introduced in the next section. A complete proof of Theorem 1.1 will be given in Section 3. With the aid of Theorem 1.1, we have the following

Corollary 1.1. *Suppose that Assumptions* **H1** - **H4** *and (a3) in Theorem 1.1 are satisfied. If for some* c_*, *(1.2) has a monotone increasing traveling wave solution connecting* 0 *and* E^*, *then for any* $c > c_*$, *(1.2) has a monotone increasing traveling wave solution connecting* 0 *and* E_*.

Proof. If for some $c^* \geq 0$, (1.2) has an monotone increasing traveling wave solution $U^*(t)$ connecting 0 and E^*, then we have

$$c^* \dot{U}^*(t) = D\ddot{U}^*(t) + F\Big(U^*(t), \int_{-\sigma}^{0} d\eta(\theta)U^*(t+c^*\theta)\Big). \tag{1.7}$$

From (1.7) and the fact that $\dot{U}^*(t) \geq 0$ it follows that, for $c > c^*$,

$$c\dot{U}^*(t) \geq c^* \dot{U}^*(t) = D\ddot{U}^*(t) + F\Big(U^*(t), \int_{-\sigma}^{0} d\eta(\theta)U^*(t+c^*\theta)\Big). \tag{1.8}$$

Moreover, $c > c^*$ implies that $U^*(t+c\theta) \leq U^*(t+c^*\theta)$ for all $\theta \leq 0$ and $t \in \mathbb{R}$. Hence, we deduce by Assumption **H2** and **H3** that

$$F\Big(U^*(t), \int_{-\sigma}^{0} d\eta(\theta)U^*(t+c^*\theta)\Big) \geq F\Big(U^*(t), \int_{-\sigma}^{0} d\eta(\theta)U^*(t+c\theta)\Big).$$

The above inequality and (1.7) therefore yield that

$$c\dot{U}^*(t) \geq D\ddot{U}^*(t) + F\Big(U^*(t), \int_{-\sigma}^{0} d\eta(\theta)U^*(t+c\theta)\Big). \tag{1.9}$$

That is, $U^*(t)$ is an upper solution of (1.4). Thus, Corollary 1.1 is a direct consequence of Theorem 1.1. □

Remark 1.1. The implication of Corollary 1.1 is that if the system (1.2) has a positive traveling wave solution connecting 0 and E^* with some wave speed $\bar{c} > 0$, then the number c_m defined by

$$\begin{aligned} c_m = \inf\{c > 0 :\ &\text{(1.2) has a traveling wave solution} \\ &\text{of wave speed } c \text{ connecting } 0 \text{ and } E^*\} \end{aligned}$$

gives the minimum wave speed of traveling waves of (1.2).

2. Preliminaries

Lemma 2.1. *Fix any constants a and δ with $a > 0$. Consider the second order non-homogeneous linear equation*

$$\begin{aligned} &\ddot{u} - c\dot{u} - ku = -f(t), \quad t \geq \delta, \\ &u(\delta) = a, \end{aligned} \tag{2.1}$$

where $f : [\delta, \infty) \to \mathbb{R}$ is piecewise continuous and $|f(t)| \leq mt$ as $t \to \infty$ for some constant m. Then $u(t)$ is a solution of (2.1) and $\dot{u}(t)$ is continuous with $u(t) = O(t)$ (order of t) as $t \to \infty$ if and only if

$$\begin{aligned} u(t) =& a\Big[1 - \frac{1}{\lambda_2 - \lambda_1}\int_{\delta}^{\infty} e^{\lambda_2(\delta-s)} f(s)ds\Big] e^{\lambda_1(t-\delta)} \\ &+ \frac{1}{\lambda_2 - \lambda_1}\Big[\int_{\delta}^{t} e^{\lambda_1(t-s)} f(s)ds + \int_{t}^{\infty} e^{\lambda_2(t-s)} f(s)ds\Big] \\ =& ae^{\lambda_1(t-\delta)} + \frac{1}{\lambda_2 - \lambda_1}\Big[\int_{\delta}^{t} K_1(t,s)f(s)ds + \int_{t}^{\infty} K_2(t,s)f(s)ds\Big], \end{aligned} \tag{2.2}$$

where

$$\lambda_1 = \frac{c - \sqrt{c^2 + 4k}}{2} < 0,$$

$$\lambda_2 = \frac{c + \sqrt{c^2 + 4k}}{2} > 0,$$

and

$$\begin{aligned} K_1(t,s) &= e^{\lambda_1(t-s)} - e^{\lambda_1(t-\delta)} e^{\lambda_2(\delta-s)}, \\ K_2(t,s) &= e^{\lambda_2(t-s)} - e^{\lambda_2(\delta-s)} e^{\lambda_1(t-\delta)}. \end{aligned} \tag{2.3}$$

Proof. With the use of variational-of-parameter formula we obtain

$$\begin{aligned} u(t) =& c_1 e^{\lambda_1(t-\delta)} + c_2 e^{\lambda_2(t-\delta)} \\ &+ \frac{1}{\lambda_2 - \lambda_1}\left[\int_{\delta}^{t} e^{\lambda_1(t-s)} f(s)ds - \int_{\delta}^{t} e^{\lambda_2(t-s)} f(s)ds\right]. \end{aligned} \tag{2.4}$$

Multiplying both sides of (2.4) by $e^{\lambda_2(t-\delta)}$, we obtain

$$\begin{aligned} e^{-\lambda_2(t-\delta)}u(t) \quad &= c_1 e^{-\lambda_2(t-\delta)+\lambda_1(t-\delta)} + c_2 \\ &+\frac{1}{\lambda_2-\lambda_1} e^{-\lambda_2(t-\delta)} \int_\delta^t e^{\lambda_1(t-s)} f(s)ds \\ &-\frac{1}{\lambda_2-\lambda_1} e^{-\lambda_2(t-\delta)} \int_\delta^t e^{\lambda_2(t-s)} f(s)ds. \end{aligned} \tag{2.5}$$

Since $\lambda_1 < 0$, $\lambda_2 > 0$, and $\lambda_2 > |\lambda_1|$, then when $t \to \infty$, we have

$$\begin{aligned} &e^{-\lambda_2(t-\delta)}u(t) \to 0, \\ &c_1 e^{-\lambda_2(t-\delta)+\lambda_1(t-\delta)} \to 0, \\ &\frac{1}{\lambda_2-\lambda_1} e^{-\lambda_2(t-\delta)} \int_\delta^t e^{\lambda_1(t-s)} f(s)ds \to 0. \end{aligned} \tag{2.6}$$

Therefore, (2.5) and (2.6) yield that

$$c_2 = \frac{1}{\lambda_2-\lambda_1} \int_\delta^\infty e^{\lambda_2(\delta-s)} f(s)ds.$$

Upon a substitution of the above equality into (2.4) yields that

$$\begin{aligned} u(t) =&c_1 e^{\lambda_1(t-\delta)} + \frac{1}{\lambda_2-\lambda_1} \int_\delta^\infty e^{-\lambda_2(t-s)} f(s)ds \\ &+ \frac{1}{\lambda_2-\lambda_1} \int_\delta^t e^{\lambda_1(t-s)} f(s)ds - \frac{1}{\lambda_2-\lambda_1} e^{-\lambda_2(t-\delta)} \int_\delta^t e^{\lambda_2(t-s)} f(s)ds \\ =&c_1 e^{\lambda_1(t-\delta)} + \frac{1}{\lambda_2-\lambda_1} \int_\delta^t e^{\lambda_1(t-s)} f(s)ds + \frac{1}{\lambda_2-\lambda_1} \int_t^\infty e^{\lambda_2(t-s)} f(s)ds. \end{aligned} \tag{2.7}$$

The initial condition $u(\delta) = a$ and (2.7) therefore imply that

$$a = u(\delta) = c_1 + \frac{1}{\lambda_2-\lambda_1} e^{\lambda_2\delta} \int_\delta^\infty e^{-\lambda_2 s} f(s)ds.$$

Hence

$$c_1 = a - \frac{1}{\lambda_2-\lambda_1} \int_\delta^\infty e^{\lambda_2(\delta-s)} f(s)ds. \tag{2.8}$$

Thus we have

$$\begin{aligned} u(t) \quad &= a e^{\lambda_1(t-\delta)} - \frac{e^{\lambda_1(t-\delta)}}{\lambda_2-\lambda_1} \int_\delta^\infty e^{\lambda_2(\delta-s)} f(s)ds \\ &+\tfrac{1}{\lambda_2-\lambda_1}\Big[\textstyle\int_\delta^t e^{\lambda_1(t-s)} f(s)ds + \int_t^\infty e^{\lambda_2(t-s)} f(s)ds \Big] \\ &= a e^{\lambda_1(t-\delta)} + \tfrac{1}{\lambda_2-\lambda_1}\Big[\textstyle\int_\delta^t \Big(e^{\lambda_1(t-s)} - e^{\lambda_1(t-\delta)} e^{\lambda_2(\delta-s)} \Big) f(s)ds \\ &+ \textstyle\int_t^\infty \Big(e^{\lambda_2(t-s)} - e^{\lambda_2(\delta-s)} e^{\lambda_1(t-\delta)} \Big) f(s)ds \Big] \\ &= a e^{\lambda_1(t-\delta)} + \tfrac{1}{\lambda_2-\lambda_1}\Big[\textstyle\int_\delta^t K_1(t,s) f(s)ds + \int_t^\infty K_2(t,s) f(s)ds \Big]. \end{aligned} \tag{2.9}$$

Moreover, it is easy to verify that the solution $u(t)$ given in (2.9) has the order of t as $t \to \infty$. $\square$

Now for each piecewise continuous function $f : [\delta, \infty) \to \mathbb{R}$ that has the order of t as $t \to \infty$, the equation (2.1) has a unique solution $u(t)$ which has an order of t as $t \to \infty$. Let this unique solution be denoted by $u_f(t) : t \geq \delta$. Then we have

Lemma 2.2. *The following hold:*

(a) *u_f is monotone increasing with respect to f. i.e., if $f(t) \geq g(t)$ for all $t \geq \delta$, then $u_f(t) \geq u_g(t)$ for all $t \geq \delta$.*

(b) *If $f(t)$ is an increasing function on t and $f(t) \geq ka$ for all $t \geq \delta$, then*

$$\dot{u}_f(t) \geq 0 \ , \quad t \geq \delta.$$

Hence $u_f(t)$ is nondecreasing function of t and $u_f(t) \geq a$ for $t \geq \delta$.

Proof. Recall that $\lambda_1 < 0 < \lambda_2$. If $t > s > \delta$, then $t - s < t - \delta$, $\lambda_2(\delta - s) < 0$. Hence

$$\lambda_1(t-s) > \lambda_1(t-\delta) > \lambda_1(t-\delta) + \lambda_2(\delta - s).$$

It follows that for $t \geq s \geq \delta$,

$$K_1(t,s) = e^{\lambda_1(t-s)} - e^{\lambda_1(t-\delta)+\lambda_2(\delta-s)} > 0, \quad \text{for} \quad t > s > \delta.$$

By a similar argument one is able to verify that

$$K_2(t,s) = e^{\lambda_2(t-s)} - e^{\lambda_2(\delta-s)}e^{\lambda_1(t-\delta)} > 0, \quad \text{for} \quad \delta < t < s.$$

Thus, by the formula (2.2) we easily conclude that $u_f(t)$ is monotone increasing with respect to f. So that (a) holds.

Next, suppose that $f(t) \geq ka$ is nondecreasing. Differentiating the first equality of (2.2) with respect to t we obtain

$$\begin{aligned} \dot{u}_f(t) =& \lambda_1\Big[a - \frac{1}{\lambda_2 - \lambda_1}\int_\delta^\infty e^{\lambda_2(\delta-s)}f(s)ds\Big]e^{\lambda_1(t-s)} \\ &+ \frac{\lambda_1}{\lambda_2-\lambda_1}\int_\delta^t e^{\lambda_1(t-s)}f(s)ds + \frac{\lambda_2}{\lambda_2-\lambda_1}\int_t^\infty e^{\lambda_2(t-s)}f(s)ds. \end{aligned} \tag{2.10}$$

If, in addition to $f(t) \geq ka$, suppose that $f(t)$ is increasing function, then, the fact that $\lambda_1 < 0$, $\lambda_2 > 0$, and (2.10) imply that

$$\begin{aligned} \dot{u}_f(t) \geq& \lambda_1\Big[a - \frac{ka}{\lambda_2 - \lambda_1}\int_\delta^\infty e^{\lambda_2(\delta-s)}ds\Big]e^{\lambda_1(t-s)} \\ &+ \left[\frac{\lambda_1}{\lambda_2-\lambda_1}\int_\delta^t e^{\lambda_1(t-s)}ds + \frac{\lambda_2}{\lambda_2-\lambda_1}\int_t^\infty e^{\lambda_2(t-s)}ds\right] f(t). \end{aligned} \tag{2.11}$$

Notice that

$$\begin{aligned} &\int_\delta^\infty e^{\lambda_2(\delta-s)}ds = \frac{1}{\lambda_2}, \\ &\lambda_1\int_\delta^t e^{\lambda_1(t-s)}ds = e^{\lambda_1(t-\delta)} - 1, \\ &\lambda_2\int_t^\infty e^{\lambda_2(t-s)}ds = 1. \end{aligned} \tag{2.12}$$

(2.11) and (2.12) yield that

$$\begin{aligned} \dot{u}_f(t) &\geq \lambda_1 a e^{\lambda_1(t-s\delta)} + \Big[-\frac{\lambda_1}{\lambda_2(\lambda_2-\lambda_1)} + \frac{1}{\lambda_2-\lambda_1}\Big] e^{\lambda_1(t-\delta)}(ka) \\ &= \lambda_1 a e^{\lambda_1(t-\delta)} + \frac{ka}{\lambda_2} e^{\lambda_1(t-\delta)} \\ &= \Big[\lambda_1 + \delta\frac{k}{\lambda_2}\Big] a e^{\lambda_1(t-\delta)}. \end{aligned} \tag{2.13}$$

Recall that λ_1 and λ_2 are solution of $\lambda^2 - c\lambda - k = 0$. so we have $\lambda_1\lambda_2 = -k$. So that

$$\lambda_1 + \frac{k}{\lambda_2} = 0. \tag{2.14}$$

From (2.13) and (2.14) we deduce that

$$\dot{u}_f(t) \geq 0 \quad \text{for all} \ \ t \geq \delta.$$

This completes the proof of Part (b). □

Now we turn to consider the system (1.4) with the boundary condition (1.3). Fix real numbers δ and $c > 0$, and let $U_0 \in \mathbb{R}^n$ be a strictly positive vector. We begin with studying the existence of a solution for the following truncated problem:

$$\begin{aligned} &c\dot{U}(t) = D\ddot{U}(t) + F\Big(U(t), \int_{-\sigma}^{0} d\eta(\theta)U(t+c\theta)\Big), \quad t \geq \delta, \\ &U(\theta) = U_0, \qquad\qquad\qquad\qquad\qquad\qquad \theta \in [\delta - c\sigma, \delta], \end{aligned} \tag{2.15}$$

where $D = \operatorname{diag}(d_1, \cdots, d_n)$ is a diagonal matrix with $d_i > 0$ for $i = 1, \ldots, n$ (see **Remark 3.1** for the case that not all $d_i > 0$). Let $U = (U_1, \cdots, U_n)$, $F = (F_1, \cdots, F_n)$, $U_0 = (a_1, \cdots, a_n)$, and let $k > 0$ be a constant. We then can rewrite (2.15) an equivalent system as

$$\begin{aligned} &\ddot{U}_i(t) - c_i U_i(t) - kU_i(t) = -\left[kU_i(t) + \frac{1}{d_i}F_i\Big(U(t), \int_{-\sigma}^{0} d\eta(\theta)U(t+c\theta)\Big)\right], t \geq \delta, \\ &U_i(\theta) = a_i, \qquad \theta \in [\delta - c\sigma, \delta], \\ &i = 1, \cdots, n, \end{aligned} \tag{2.16}$$

where $c_i = c/d_i$. The smoothness of $F(u, v)$ implies that we can pick $k > 0$ sufficiently large such that

$$k + \frac{1}{d_i}\frac{\partial F_i(u,v)}{\partial u_i} > 0, \qquad (u,v) \in \mathcal{R} \times \mathcal{R}, \qquad i = 1, \cdots, n. \tag{2.17}$$

Let

$$\begin{aligned} \lambda_1^i &= \frac{c_i - \sqrt{c_i^2 + 4k}}{2} < 0, \\ \lambda_2^i &= \frac{c_i + \sqrt{c_i^2 + 4k}}{2} > 0, \\ K_1^i(t,s) &= \quad e^{\lambda_1^i(t-s)} - e^{\lambda_1^i(t-\delta)}e^{\lambda_2^i(\delta-s)}, \\ K_2^i(t,s) &= \quad e^{\lambda_2^i(t-s)} - e^{\lambda_2^i(\delta-s)}e^{\lambda_1^i(t-\delta)}. \end{aligned} \tag{2.18}$$

We define an operator $T = T^{U_0} : C([\delta - c\sigma, \infty), \mathbb{R}^n) \to C([\delta - c\sigma, \infty), \mathbb{R}^n)$ by

$$T(f) = [T_1(f), \cdots, T_n(f)]$$

with

$$\begin{aligned}
&T_i(f)(\theta) = a_i, \qquad \theta \in [\delta - c\sigma, \delta], \\
&T_i(f)(t) = a_i e^{\lambda_1^i (t-\delta)} + \frac{1}{\lambda_2^i - \lambda_1^i} \int_\delta^t K_1^i(t,s) f_i(s) ds \\
&\qquad\qquad + \frac{1}{\lambda_2^i - \lambda_1^i} \int_t^\infty K_2^i(t,s) f_i(s) ds, \qquad t \geq \delta, \\
&i = 1, 2, \ldots, n.
\end{aligned} \tag{2.19}$$

From the definition of the operator T and Lemma 2.2 it follows that T is a monotone operator.

Let

$$X_\delta = C([\delta - c\sigma, \infty); \mathcal{R})$$

and define a function $F^k = [F_1^k, \cdots, F_n^k] : X_\delta \to C(\delta, \mathbb{R}^n)$ by

$$F_i^k(U)(t) = kU_i(t) + \frac{1}{d_i} F_i\Big(U(t), \int_{-\sigma}^0 d\eta(\theta) U(t + c\theta)\Big), \quad t \geq \delta. \tag{2.20}$$

As a consequence of Lemma 2.1 and the inequality (2.17) we have

Lemma 2.3. *Let the operators T and F^k be defined as above. Then*

(A) *$U \in X_\delta$ is a solution of the system (2.15) or (2.16) if and only if*

$$U(t) = T(F^k(U))(t), \quad t \geq \delta - c\sigma. \tag{2.21}$$

(B) *The operator $T\big(F^k(\cdot)\big) : X_\delta \to C([\delta - c\sigma, \infty), \mathbb{R}^n)$ is monotone increasing with respect to $U \in X_\delta$. That is,*

$$T\big(F^K(U)\big)(t) \geq T\big(F^K(V)\big)(t), \quad \text{for all } \ t \geq \delta - c\sigma$$

if $U, V \in X_\delta$ and $U(t) \geq V(t)$, for $t \geq \delta - c\sigma$.

Since the operator $T\big(F^K(\cdot)\big)$ is a monotone operator, it is natural to show the existence of a solution to (2.21) by a monotone iteration approach.

Definition 2.1. A function $U \in X_\delta$ is an upper (lower) solution of (2.21) if

$$U(t) \geq (\leq)\ T(F^k(U))(t), \qquad t \geq \delta - c\sigma.$$

Lemma 2.4. *Suppose $c > 0$ in (2.21). Let $U = (U_1, \cdots, U_n) \in X_\delta = C([\delta - c\sigma, \infty); \mathcal{R})$. If $U(t)$ is nondecreasing and there are constants $t_1, \cdots, t_n$, such that $\delta < t_i \leq \infty$ for $i = 1, \cdots, n$, $\dot{U}_i(t)$ and $\ddot{U}_i(t)$ are continuous on (δ, t_i), and the left limit of $u_i(t)$ at t_i exists, in addition,*

$$\begin{aligned}
&U(\theta) \geq U_0, \quad \theta \in [\delta - c\sigma, \delta], \\
&c\dot{U}_i(t) \geq d_i \ddot{U}_i(t) + F_i\big(U_i(t), \int_{-\sigma}^0 d\eta(\theta) U(t + c\theta)\big), \quad t \in (\delta, t_i), \\
&U_i(t) = E_i^*, \quad t \geq t_i, \\
&i = 1, 2, \cdots, n,
\end{aligned} \tag{2.22}$$

then $U(t)$ is an upper solution of (2.21).

Proof. Since $U_i(t)$ is non-decreasing, $\dot{U}_i(t_i) \geq 0$, here $\dot{U}_i(t_i)$ is the left derivative. Define $\bar{U} = (\bar{U}_1, \cdots, \bar{U}_n)$ by

$$\bar{U}_i(t) = \begin{cases} U_i(t), & t \leq t_i, \\ E_i^* + \dot{U}_i(t_i)(t - t_i), & t > t_i. \end{cases} \tag{2.23}$$

Then $U(t) \leq \bar{U}(t)$, $t \geq \delta - c\sigma$ and $\bar{U}(t)$ is order of $|t|$ as $t \to \infty$. For $i = 1, \cdots, n$, let

$$\ddot{\bar{U}}_i(t) - c_i \dot{\bar{U}}_i(t) - k\bar{U}_i(t) = -h_i(t), \quad t \geq \delta. \tag{2.24}$$

Then (2.23)- (2.24) yield that

$$\begin{aligned} h_i(t) \quad &\geq kU_i(t) + \frac{1}{d_i} F_i\big(U(t), \int_{-\sigma}^{0} d\eta_i(\theta) U(t + c\theta)\big) \\ &= F_i^k(U)(t), \qquad t \in (\delta, t_i]. \end{aligned} \tag{2.25}$$

Recall that $F_i(E^*, \eta^* E^*) = 0$ and $U(t) \leq E^*$. The monotonicity of $F^k(U)$ implies that

$$kE_i^* = kE_i^* + \frac{1}{d_i} F_i\big(E^*, \int_{-\sigma}^{0} d\eta(\theta) E^*\big) \geq F_i^k\big(U\big)(t). \tag{2.26}$$

Hence for $t > t_i$,

$$\begin{aligned} h_i(t) \quad &= c_i \dot{U}_i(t_i) + k\bar{U}_i(t) \\ &\geq kE_i^* \\ &= kE_i^* + \frac{1}{d_i} F_i(E^*, \eta^* E^*) \\ &\geq kU_i(t) + \frac{1}{d_i} F_i\big(U(t), \int_{-\sigma}^{0} d\eta(\theta) U(t + c\theta)\big) \\ &= F_i^k(U)(t), \qquad t > t_i. \end{aligned} \tag{2.27}$$

From (2.25) and (2.27) it follows that

$$h_i(t) \geq F_i^k(U)(t), \quad t \geq \delta.$$

Hence, for $t \in (\delta, t_i)$,

$$U_i(t) \quad = \bar{U}_i(t) = T_i(h_i)(t) \geq T_i(F^k(U))(t). \tag{2.28}$$

Now for $t > t_i$, by (2.26) we have

$$U_i(t) = E_i^* = T_i(F^k(E^*))(t) \geq T_i(F^k(U))(t). \tag{2.29}$$

(2.28) and (2.29) therefore imply that $U(t)$ is an upper solution. □

By the system (2.16), Part (a) of Lemma 2.2 we can easily prove the the following lemma on the lower solution.

Lemma 2.5. *Let* $0 << U_0 \in \mathbb{R}^n$. *If* $F(U_0, \eta^* U_0) \geq 0$, *then*

$$U_0(t) = U_0, \qquad t \geq \delta - c\sigma$$

is a lower solution.

Proof. First by the definition of the operator $T = T^{U_0}$ given in (2.19) we have

$$U_0 = T(kU_0)(t), \quad t \geq \delta - c\sigma \tag{2.30}$$

for any constant function $U_0(t) = U_0$. If $F(U_0, \eta^* U_0) \geq 0$, then $F^k(U_0) \geq kU_0$ by the definition of F^k. Hence

$$U_0(t) = U_0 = T(kU_0)(t) \leq T(F^k(U_0))(t), \quad t \geq \delta - c\sigma$$

by the monotonicity of T. That is, $U_0(t) \equiv U_0$ is a lower solution of (2.21). □

Corollary 2.1. *Suppose that (2.21) has an upper solution $U(t)$ with*

$$U_0 \leq U(t) \leq E^*, \qquad t \geq \delta - c\sigma.$$

If in addition that $F(U_0, \eta^* U_0) \geq 0$, then (2.15) has a solution $U^\delta(t)$ which is increasing function of t.

Proof. By Lemma 2.6, $V^0(t) \equiv U_0$ for $t \geq \delta - c\sigma$ is a lower solution of (2.21). Hence, a monotone iteration argument implies that the sequence $\{V^m\}$ with

$$\begin{aligned} &0 << U_0 \leq V^m(t) = T(F^k(V^{m-1}))(t) \leq V^{m+1}(t) \leq U(t), \\ &t \geq \delta - c\sigma \end{aligned} \tag{2.31}$$

is a monotone increasing sequence bounded above by $U(\cdot)$. It follows that $V^m(\cdot)$ converges to some function as $m \to \infty$. We let

$$\lim_{m\to\infty} V^m(\cdot) = U^\delta(\cdot) \in C([\delta - c\sigma, \infty), \mathcal{R}).$$

Then it is clear that $U^\delta(\cdot)$ is a solution (2.21), so is a solution of (2.15). We claim that $U^\delta(t)$ is an increasing function of t for $t \geq \delta$. To confirm this claim, it will be sufficient to show that $V^m(t)$ is increasing with respect to t for each $m \geq 1$. This can be done by using induction argument. Let $U_0 = (a_1, \cdots, a_n)$ and $V^m(t) = \big(V_1^m(t), \cdots, V_n^m(t)\big)$. Then by the definition of the operator T and the function $F^k(U)$, we have

$$\begin{aligned} &V^{m+1}(\theta) = a_i, \qquad \theta \in [\delta - c\sigma, \delta], \\ &\ddot{V}_i^{m+1}(t) - c_i \dot{V}_i^{m+1}(t) - kV_i^{m+1}(t) = -f_i^m(t), \quad t \geq \delta. \end{aligned} \tag{2.32}$$

where

$$f_i^m(t) = F_i^k(V^m)(t) = kV_i^m(t) + \frac{1}{d_i} F_i\Big(V^m(t), \int_{-\sigma}^0 d\eta(\theta) V^m(t + c\theta)\Big).$$

Noticing that $V^0(t) \equiv U_0$, we have

$$f_i^0(t) \equiv ka_i + \frac{1}{d_i} F_i\big(U_0, \eta^* U_0\big) \geq ka_i, \tag{2.33}$$

for $F(U_0, \eta^* U_0) \geq 0$. Hence, with the application of Lemma 2.3, we conclude that $\dot{V}^1(t) \geq 0$ for $t \geq \delta$. Now for $m \geq 1$, if $V^m(t)$ is an increasing function of t for $t \geq \delta$, then $f_i(t) = F_i^k(V^m)(t)$ is monotone increasing with respect to $t \geq \delta$ and $f_i(t) \geq F_i^k(V^0) \geq ka_i$. Hence again by Lemma 2.3 and (2.32) that $\dot{V}^{m+1}(t) \geq 0$. Hence $V^m(t)$ is increasing for all m. □

3. Proof of Theorem 1.1

We are now in the position to prove Theorem 1.1.
Proof. For each positive integer N with $-N < \min\{t_1, \cdots, t_n\}$, the Assumptions (a1) and **H4** imply that there is an $s > 0$ and a strictly positive vector p such that

$$F(sp, s\eta^* p) \geq 0, \quad U(\theta) \geq sp, \qquad \theta \in [-N - c\sigma, -N].$$

If we let $U_0 = sp$ and $\delta = -N$, then Corollary 2.7 implies that the equation (2.15) has a monotone increasing solution $U^N(t)$ defined for $t \geq -N - c\sigma$. Moreover, $U_0 \leq U^N(t) \leq E^*$. Since $U^N(t)$ is increasing and bounded, $\lim_{t\to\infty} U^N(t)$ exists and $U^N(\infty)$ must be an equilibrium of (1.4). Hence

$$F(U^N(\infty), \eta^* U^N(\infty)) = 0. \tag{3.1}$$

It follows from (3.1) and Assumption (a3) that $U^N(\infty) = E^*$.

Let $U^N(t) = (U_1^N(t), \ldots, U_n^N(t))$. Since $U_1^N(t_0) \leq U_1(t_0) < E_1^*$ and $u_1^N(\infty) = E_1^*$. Without loss of generality we let $t_0 = 0$. So there is a $t_N > t_0 = 0$ such that

$$0 < U_1^N(t_N) = U_1(t_0) < 1.$$

If we let

$$u^N(t) = U^N(t + t_N).$$

Then $u^N(t)$ is again a monontone increasing solution of (2.15) defined for $t \geq -(N + t_N) - c\sigma$ with

$$u_1^N(0) = U_1(0) \tag{3.2}$$

for all N. It is obvious that the sequence of functions $\{u^N(t) : t \in [-(N + t_N)] - c\sigma, \infty)\}$ is uniformly bounded. We also can show that it is equicontinous. Hence, by Ascoli-Arzela theorem, $u^N(t)$ contains a subsequence $\{u^{N_k}(t)\}$ that converges to a function $U^c(t)$ uniformly on any compact subset of R. Thus one easily deduceds that $U^c(t)$ is a solution of (1.4) for $t \in R$. In addition, $U^c(t)$ is a monotone increasing function since u^N is monotone increasing for each N. Let

$$U^c(-\infty) = E_-, \quad U^c(\infty) = E_+.$$

Then we must have

$$0 \leq E_- \leq E_+ \leq E^*.$$

Also it is apparent that E_- and E_+ must be equilibrium points of (1.4). We claim that $E_- = 0$ and $E_+ = E$. First, $u_1^N(0) = U_1(0) > 0$ for all N implies that

$$U_1^c(0) = U_1(0) > 0.$$

Hence

$$0 < U^c(0) \leq E_+.$$

So that $E_+ = E^*$ by Assumption (a3). Similarly $E_- < E^*$ yields that $E_- = 0$. Therefore, $U^c(t)$ is a traveling wave solution of (1.2) connecting 0 and E^*. □

Remark 3.1. In the proof of Theorem 1.1 we assume that all diffusion coefficients d_i's are strictly positive. This restriction can be released by only assuming some of coefficients are positive. Without loss of generality, suppose that

$$d_i = 0, \quad i = 1, \cdots, l < n, \qquad d_j > 0, j = l + 1, \cdots, n.$$

Then the equation for U_i, $i = 1, \cdots, l$, in the system (2.15) can be written as

$$\begin{aligned} &\dot{U}_i(t) + kU_i = kU_i + \frac{1}{c}F_i\Big(U(t), \int_{-\sigma}^{0} d\eta(\theta)U(t + c\theta)\Big), \quad t \geq \delta, \\ &U_i(\theta) = a_i, \qquad \theta \in [\delta - c\sigma, \delta]. \end{aligned} \tag{3.3}$$

One easily deduces that (3.3) is equivalent to the integral equation

$$U_i(t) = a_i^{-k(t-\delta)} + \int_{\delta}^{t} e^{-k(t-s)} F_i^k(U)(s)ds, \quad t \geq \delta, \tag{3.4}$$

where

$$F_i^k(U)(t) = kU_i + \frac{1}{c}F_i\Big(U(t), \int_{-\sigma}^{0} d\eta(\theta)U(t + c\theta)\Big). \tag{3.5}$$

Hence, if for $i = 1, \cdots, l$, we replace the operator T_i defined in (2.19) by the operator

$$T_i(f) = a_i^{-k(t-\delta)} + \int_{\delta}^{t} e^{-k(t-s)} f(s)ds, \quad t \geq \delta$$

and redefine the function F_i^k in (2.20) by (3.5). Then one is able to verify that all arguments used to prove Theorem 1.1 remains valid.

Remark 3.2. The Assumption (a3) in Theorem 1.1 can be replaced by a weaker condition

(a3)' $F(V_0, \eta^* V_0) \neq 0$ for all $V_0 \in \mathrm{Int}\mathcal{R}$ (the interior of $\mathcal{R}$) and $F(\cdot, \eta^* \cdot)$ has at most finitely many zeros in the boundary of $\mathcal{R}$.

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