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ENTROPY SOLUTIONS TO NONLINEAR ELLIPTIC ANISOTROPIC PROBLEM WITH VARIABLE EXPONENT

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Abstract In this work, we give an existence result of entropy solutions for nonlinear anisotropic elliptic equation of the type

$$-\mathrm{div}\big(a(x,u,\nabla u)\big)+g(x,u,\nabla u)+|u|^{p_0(x)-2}u=f-\mathrm{div}\phi(u), \quad \text{in } \Omega,$$

where $-\mathrm{div}\big(a(x,u,\nabla u)\big)$ is a Leray-Lions operator, $\phi\in C^0(I\!R, I\!R^N)$. The function $g(x,u,\nabla u)$ is a nonlinear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition and the datum f belongs to $L^1(\Omega)$.

Keywords Anisotropic Sobolev spaces, variable exponent, nonlinear elliptic problem, entropy solutions.

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1. Introduction

In the last decades, one of the topics from the field of partial differential equations that continuously gained interest is the one concerning the Sobolev space with variable exponents, $W^{1,p(.)}(\Omega)$ (where p is a function depending on x), see for example the book by Diening et *als* [11] and [1–3, 5]. The most studied problems was that involving the $p(.)$-Laplace operators

$$\Delta_{p(.)}u=\mathrm{div}\left(|\nabla u|^{p(x)-2}\,\nabla u\right),$$

which generalize work wich involves the p-Laplace operators with p constant (see for example [4, 10] where the authors proved existence of entropy solution in weighted Sobolev spaces) At the same time, due to the development of the theory regarding the anisotropic Sobolev space $W^{1,\vec{p}(.)}$ (where $\vec{p}(.)=(p_1,...,p_N)$ is a constant vector),

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a new operator takes its place in the mathematical literature (see [7–9, 15, 16, 19]), namely

$$\Delta_{\vec{p}(.)}u = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p(x)-2} \partial_{x_i} u \right).$$

In this paper, we consider Ω a bounded open subset of $I\!R^N$ $(N \geq 2)$ and $p_i(x) \in C_+(\bar{\Omega})$ for $i = 0, 1, \ldots, N$, with

$$\underline{p}(x) = \min\{p_i(x), \quad i = 0, 1, 2, \ldots, N\}, \ \forall x \in \Omega. \tag{1.1}$$

Our aim is to prove the existence of an entropy solution to the following nonlinear anisotropic elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i \ a_i(x, u, \nabla u) + g(x, u, \nabla u) + |u|^{p_0(x)-2}u = f - \mathrm{div}\phi(u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the right-hand side $f \in L^1(\Omega)$.

We assume that for $i = 1, \ldots, N$ the function $a_i : \Omega \times I\!R \times I\!R^N \to I\!R$ is Carathéodory (measurable with respect to x in Ω for every (s, ξ) in $I\!R \times I\!R^N$ and continuous with respect to (s, ξ) in $I\!R \times I\!R^N$ for almost every x in Ω) and satisfy the following conditions:

$$|a_i(x, s, \xi)| \leq \beta \left(K_i(x) + |s|^{p_i(x)-1} + |\xi_i|^{p_i(x)-1}\right), \qquad \text{for} \quad i = 1, \ldots, N, \tag{1.3}$$

$$a_i(x, s, \xi)\xi_i \geq \alpha|\xi_i|^{p_i(x)}, \qquad \text{for} \quad i = 1, \ldots, N, \tag{1.4}$$

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0, \qquad \text{for} \quad \xi_i \neq \xi_i', \tag{1.5}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in I\!R \times I\!R^N$, where $K_i(x)$ is a nonnegative function lying in $L^{p_i'(.)}(\Omega)$ and $\alpha, \beta > 0$.

The nonlinear term $g(x, s, \xi)$ is a Carathéodory function which satisfies

$$g(x, s, \xi)s \ \geq \ 0, \tag{1.6}$$

$$|g(x, s, \xi)| \leq b(|s|)\Big(c(x) + \sum_{i=1}^{N} |\xi_i|^{p_i(x)}\Big), \tag{1.7}$$

where $b : I\!R^+ \to I\!R^+$ is a continuous nondecreasing function, $c \in L^1(\Omega)$ and

$$f \in L^1(\Omega) \text{ with } \phi \in C^0(I\!R, I\!R^N). \tag{1.8}$$

2. Preliminary

2.1. Lebesgue and Sobolev spaces with variable exponents

As the exponent $p_i(.)$ appearing in (1.3), (1.4) and (1.7) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents. Set

$$\mathcal{C}_+(\overline{\Omega}) = \{p \in \mathcal{C}(\overline{\Omega}), \ 1 < p(x) < +\infty \ , \text{ for any } \ x \in \bar{\Omega}\}.$$

For any $p \in \mathcal{C}_+(\overline{\Omega})$, we define,

$$p^+ = \max_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- = \min_{x\in\Omega} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to I\!R$ for which the convex modular

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e. if $p^+ < +\infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv\ dx\right| \leq (\frac{1}{p^-} + \frac{1}{p'^-})\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Let

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|.\|_{1,p(\cdot)})$ is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result:

Proposition 2.1. *(see [12, 20]) If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p^+ < +\infty$, then the following properties hold true:*

(i) $\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|^{p^-}_{p(\cdot)} < \rho_{p(\cdot)}(u) < \|u\|^{p^+}_{p(\cdot)}$;

(ii) $\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|^{p^+}_{p(\cdot)} < \rho_{p(\cdot)}(u) < \|u\|^{p^-}_{p(\cdot)}$;

(iii) $\|u\|_{p(\cdot)} < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{p(\cdot)}(u) < 1$ *(respectively* $= 1; > 1$*)*;

(iv) $\|u_n\|_{p(\cdot)} \to 0$ *(respectively* $\to +\infty$*)* $\Leftrightarrow \rho_{p(\cdot)}(u_n) \to 0$ *(respectively* $\to +\infty$*)* ;

(v) $\rho_{p(\cdot)}(u/\|u\|_{p(\cdot)}) = 1$.

Next, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ and

$$p^*(\cdot) = \begin{cases} \dfrac{Np(\cdot)}{N - p(\cdot)}, & \text{for } p(\cdot) < N, \\ \infty, & \text{for } p(\cdot) \geq N. \end{cases}$$

Proposition 2.2. *(see [12], [13])*

(i) *If $q \in \mathcal{C}_+(\bar{\Omega})$ and $q(\cdot) < p^*(\cdot)$ for any $x \in \overline{\Omega}$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ is compact and continuous.*

(ii) *Poincaré inequality: there exists a constant $C > 0$, such that*

$$\|u\|_{p(\cdot)} \leq C \, \|\nabla u\|_{p(\cdot)}, \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

Therefore, $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms in $W_0^{1,p(\cdot)}(\Omega)$.

(iii) *Sobolev-Poincaré inequality: there exists another constant $C > 0$, such that*

$$\|u\|_{p*(\cdot)} \leq C \, \|\nabla u\|_{p(\cdot)}, \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

Remark 2.1. As shown by Zhikov [21,22] the smooth functions are in general not dense in $W^{1,p(\cdot)}(\Omega)$, but if the variable exponent $p \in \mathcal{C}_+(\overline{\Omega})$ is log-Hölder continuous, that is, there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{-log|x-y|} \text{ for every } x, y \in \Omega \text{ with } |x-y| \leq \frac{1}{2}, \tag{2.2}$$

then the smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$, the Sobolev embedding, convergence of mollifiers regularization and identification of $W_0^{1,p(\cdot)}(\Omega)$ with $W_0^{1,1}(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ are established.

2.2. Anisotropic Sobolev spaces with variable exponents

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of problem (1.2).

Let $p_0(x), p_1(x), ..., p_N(x)$ be $N+1$ variable exponents in $C_+(\overline{\Omega})$. We denote

$$\vec{p}(\cdot) = \{p_0(\cdot), \dots, p_N(\cdot)\}, \; D^0 u = u \; \text{ and } D^i u = \frac{\partial u}{\partial x_i}, \quad \text{for } \; i = 1, \dots, N$$

and

$$\forall x \in \Omega, \quad \underline{p}(x) = \min\{p_i(x), \quad i = 0, 1, 2, \dots, N\}.$$

Then $\underline{p}(.) \in C^+(\overline{\Omega})$.

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follow

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \; \in \; L^{p_0(\cdot)}(\Omega), \quad D^i u \; \in \; L^{p_i(\cdot)}(\Omega), \quad i = 1, 2, \dots, N\}$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i(\cdot)}(\Omega)}. \tag{2.3}$$

We define also $W_0^{1,\vec{p}(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm (2.3). The dual of $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is denote by $W^{-1,\vec{p}(\cdot)'}(\Omega)$, where $\vec{p}(\cdot)' = \{p_0'(\cdot), \dots, p_N'(\cdot)\}$, $\frac{1}{p_i'(\cdot)} + \frac{1}{p_i(\cdot)} = 1$, (cf. [6] for the constant exponent case).

The space $\left(W_0^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)}\right)$ is a reflexive Banach space (cf [18]).

Let us introduce the following notations:

$$p_-^+ = \max\{p_1^-, ..., p_N^-\}; \quad p_-^* = \frac{N}{\displaystyle\sum_{i=1}^N \frac{1}{p_i^-} - 1}; \quad p_{-,\infty} = \max\{p_-^+, p_-^*\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^N \frac{1}{p_i^-} > 1. \tag{2.4}$$

We have the following result (cf. [18])

Theorem 2.1. *Assume $\Omega \subset I\!R^N (N \geq 3)$ is a bounded domain with smooth boundary. Assume that (2.4) is fulfilled. For any $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < p_{-,\infty}, \quad \textit{for all } x \in \overline{\Omega},$$

the embedding

$$W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \quad \textit{is continuous and compact.}$$

Definition 2.1. We denote the dual of the anisotropic Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ by $W^{-1,\vec{p}(\cdot)'}(\Omega)$ and for each $F \in W^{-1,\vec{p}(\cdot)'}(\Omega)$ there exists $f_i \in L^{p_i'(\cdot)}(\Omega)$ for $i = 0, 1, \ldots, N$, such that $F = f_0 - \displaystyle\sum_{i=1}^N D^i f_i$. Moreover, for all $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\langle F, u\rangle = \sum_{i=0}^N \int_\Omega f_i \, D^i u \, dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p}(\cdot)'} \simeq \sum_{i=0}^N \|f_i\|_{p_i'(\cdot)}.$$

Set

$$\mathcal{T}_0^{1,\vec{p}(.)}(\Omega)$$
$$:= \{u : \Omega \to I\!R, \text{ measurable such that } T_k(u) \in W_0^{1,\vec{p}(.)}(\Omega), \text{ for any } k > 0\}.$$

Proposition 2.3. *Let $u \in \mathcal{T}_0^{1,\vec{p}(.)}(\Omega)$. Then, there exists a unique measurable function $v_i : \Omega \longmapsto I\!R$ such that*

$$D^i T_k(u) = v_i . \chi_{\{|u|<k\}} \textit{ for a.e. } x \in \Omega, \forall k > 0, i \in \{1, \ldots, N\},$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial gradients of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional gradient of u, that is, $v_i = D^i u$.

3. Existence of Entropy Solutions

Let $p_i(.) \in C_+(\bar{\Omega})$ for $i = 0, 1, \ldots, N$, where

$$\forall x \in \Omega, \qquad p_0(x) \geq \max\{p_i(x), \quad i = 1, 2, \ldots, N\}. \tag{3.1}$$

We can give a simpler definition of an entropy solution of (1.2) as follows.

Definition 3.1. A function u is called an entropy solution of the strongly nonlinear anisotropic elliptic problem (1.2) if $u \in \mathcal{T}_0^{1,\vec{p}(.)}(\Omega)$, $g(x, u, \nabla u) \in L^1(\Omega)$, $\phi_i(u) \in L^{p_i'(.)}(\Omega)$ for $i = 1, ..., N$ and

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) D^i T_k(u - v)\, dx + \int_\Omega g(x, u, \nabla u) T_k(u - v)\, dx \\ &\quad + \int_\Omega |u|^{p_0(x)-2} u \; T_k(u - v)\, dx \\ \leq \quad & \int_\Omega f T_k(u - v)\, dx + \sum_{i=1}^{N} \int_\Omega \phi_i(u) D^i T_k(u - v)\, dx, \end{aligned} \tag{3.2}$$

for every $v \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Our objective is to prove the following existence theorem.

Theorem 3.1. *Assuming that* (1.3) − (1.7) *holds,* $f \in L^1(\Omega)$ *and* $\phi \in C^0(\mathbb{R}, \mathbb{R}^N)$, *then the problem* (1.2) *has at least one entropy solution.*

Remark 3.1. The assumption (3.1) is essential to ensure that $|a_i(x, u, \nabla u)|$ belongs to $L^{p_i'(x)}(\Omega)$. In the case of $Au = -\sum_{i=1}^{N} D^i a_i(x, \nabla u)$ the existence of entropy solution is guaranteed, without using this assumption.

First, we give the following results which will be used in the proof of our main result.

3.1. Useful results

Lemma 3.1. *(see [14], Theorem 13.47) Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $L^1(\Omega)$ *such that* $u_n \to u$ *a.e.,* u_n, $u \geq 0$ *a.e. and* $\int_\Omega u_n\, dx \to \int_\Omega u\, dx$, *then* $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 3.2. *(see [5]) Let* $g \in L^{p(.)}(\Omega)$ *and* $g_n \in L^{p(.)}(\Omega)$ *with* $\|g_n\|_{p(.)} \leq C$ *for* $1 < p(x) < \infty$.

If $g_n(x) \to g(x)$ *a.e. in* Ω, *then* $g_n \rightharpoonup g$ *in* $L^{p(.)}(\Omega)$.

Lemma 3.3. *Assuming that* (1.3) − (1.5) *hold, and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $W_0^{1,\vec{p}(.)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,\vec{p}(.)}(\Omega)$ *and*

$$\begin{aligned} &\int_\Omega (|u_n|^{p_0(x)-2} u_n - |u|^{p_0(x)-2} u)(u_n - u)\, dx \\ &+ \sum_{i=1}^{N} \int_\Omega \Big(a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u)\Big)(D^i u_n - D^i u)\, dx \longrightarrow 0, \end{aligned} \tag{3.3}$$

then $u_n \longrightarrow u$ *in* $W_0^{1,\vec{p}(.)}(\Omega)$ *for a subsequence.*

Proof. Let

$$\begin{aligned} S_n &= \sum_{i=1}^{N}(a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u))(D^iu_n-D^iu)\\ &\quad +(|u_n|^{p_0(x)-2}u_n-|u|^{p_0(x)-2}u)(u_n-u). \end{aligned}$$

Thanks to (1.5), we deduce that S_n is a positive function and, by (3.3), $S_n\to 0$ in $L^1(\Omega)$ *as* $n\to\infty$.

Since $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}(.)}(\Omega)$, using the compact embedding we obtain $u_n\to u$ in $L^{\underline{p}(.)}(\Omega)$, and since $S_n\to 0$ a.e. in Ω, there exists a subset B in Ω with measure zero such that $\forall x\in\Omega\backslash B$

$$|u(x)|<\infty,\quad |D^iu(x)|<\infty,\quad K_i(x)<\infty,\quad u_n\to u \ \text{ and } \ S_n\to 0.$$

We have

$$\begin{aligned} S_n(x) &= \sum_{i=1}^{N}(a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u))(D^iu_n-D^iu)\\ &\quad +\ (|u_n|^{p_0(x)-2}u_n-|u|^{p_0(x)-2}u)(u_n-u)\\ &= \sum_{i=1}^{N}\Big(a_i(x,u_n,\nabla u_n)D^iu_n+a_i(x,u_n,\nabla u)D^iu\\ &\quad -a_i(x,u_n,\nabla u)D^iu_n-a_i(x,u_n,\nabla u_n)D^iu\Big)\\ &\quad +\ |u_n|^{p_0(x)}+|u|^{p_0(x)}-|u_n|^{p_0(x)-2}u_nu-|u|^{p_0(x)-2}uu_n\\ &\geq \underline{\alpha}\sum_{i=0}^{N}|D^iu_n|^{p_i(x)}+\underline{\alpha}\sum_{i=0}^{N}|D^iu|^{p_i(x)}\\ &\quad -\beta\sum_{i=1}^{N}(K_i(x)+|u_n|^{p_i(x)-1}+|D^iu|^{p_i(x)-1})|D^iu_n|\\ &\quad -\ \beta\sum_{i=1}^{N}(K_i(x)+|u_n|^{p_i(x)-1}+|D^iu_n|^{p_i(x)-1})|D^iu|\\ &\quad -|u_n|^{p_0(x)-1}|u|-|u|^{p_0(x)-1}|u_n|\\ &\geq \underline{\alpha}\sum_{i=0}^{N}|D^iu_n|^{p_i(x)}-C_x\sum_{i=1}^{N}(1+|D^iu_n|^{p_i(x)-1}+|D^iu_n|), \end{aligned}$$

with $\underline{\alpha}=\min(\alpha,1)$ and C_x depending only on x. It follows that

$$S_n(x)\geq\sum_{i=0}^{N}|D^iu_n|^{p_i(x)}\big(\underline{\alpha}-\frac{C_x}{|D^iu_n|^{p_i(x)}}-\frac{C_x}{|D^iu_n|}-\frac{C_x}{|D^iu_n|^{p_i(x)-1}}\big).$$

By a standard argument, $(D^iu_n)_{n\in I\!N}$ is bounded almost everywhere in Ω for all $i=0,1,\ldots,N$. Indeed, if $|D^iu_n|\to\infty$ in a measurable subset $E\subset\Omega$ then

$$\begin{aligned} &\lim_{n\to\infty}\int_\Omega S_n(x)\,dx\\ &\geq \lim_{n\to\infty}\sum_{i=0}^{N}\int_E|D^iu_n|^{p_i(x)}\big(\underline{\alpha}-\frac{C_x}{|D^iu_n|^{p_i(x)}}-\frac{C_x}{|D^iu_n|}-\frac{C_x}{|D^iu_n|^{p_i(x)-1}}\big)\,dx=\infty, \end{aligned}$$

which is a contradiction since $S_n \to 0$ in $L^1(\Omega)$.

Let ξ_i^* an accumulation point of $(D^i u_n)_{n\in I\!N}$ for $i = 1, \dots, N$. We have $|\xi_i^*| < \infty$ and by the continuity of $a(x,\cdot,\cdot)$, we obtain

$$\Big(a_i(x,u_n,\xi^*) - a_i(x,u_n,\nabla u)\Big)(\xi_i^* - D^i u) = 0, \qquad \text{for} \quad i = 1,\dots,N.$$

Thanks to (1.5), we have $\xi^* = \nabla u$ and the uniqueness of the accumulation point implies that $\nabla u_n \to \nabla u$ a.e in Ω.

Since $(a_i(x,u_n,\nabla u_n))_{n\in I\!N}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$ and $a_i(x,u_n,\nabla u_n) \to a_i(x,u,\nabla u)$ a.e in Ω, by the Lemma 3.2, we can establish that

$$a_i(x,u_n,\nabla u_n) \rightharpoonup a_i(x,u,\nabla u) \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega), \quad \text{for} \quad i = 1,\dots,N.$$

Using (3.3) and the Lemma 3.1, we deduce that

$$|u_n|^{p_0(x)} \longrightarrow |u|^{p_0(x)}, \qquad \text{in} \quad L^1(\Omega) \tag{3.4}$$

and

$$a_i(x,u_n,\nabla u_n)D^i u_n \longrightarrow a_i(x,u,\nabla u)D^i u, \qquad \text{in} \quad L^1(\Omega). \tag{3.5}$$

According to the condition (1.4), we have

$$\alpha|D^i u_n|^{p_i(x)} \;\le\; a_i(x,u_n,\nabla u_n)D^i u_n, \quad \text{for} \quad i = 1,\dots,N.$$

Let $y_n^i = \frac{1}{\alpha}a_i(x,u_n,\nabla u_n)D^i u_n$ and $y^i = \frac{1}{\alpha}a_i(x,u,\nabla u)D^i u$, in view of the Fatou's Lemma, we get

$$\int_\Omega 2y^i\,dx \le \liminf_{n\to\infty}\int_\Omega (y_n^i + y^i - |D^i u_n - D^i u|^{p_i(x)})dx,$$

which implies that $0 \le -\limsup\limits_{n\to\infty}\int_\Omega |D^i u_n - D^i u|^{p_i(x)}\,dx$ and since

$$0 \le \liminf_{n\to\infty}\int_\Omega |D^i u_n - D^i u|^{p_i(x)}\,dx \le \limsup_{n\to\infty}\int_\Omega |D^i u_n - D^i u|^{p_i(x)}\,dx \le 0,$$

it follows that

$$\int_\Omega |D^i u_n - D^i u|^{p_i(x)}\,dx \longrightarrow 0 \text{ as } n \to \infty.$$

Consequently

$$D^i u_n \longrightarrow D^i u \quad \text{in} \quad L^{p_i(\cdot)}(\Omega), \quad \text{for} \quad i = 1,\dots,N.$$

In view of (3.4) we deduce that

$$u_n \longrightarrow u \qquad \text{in} \quad W_0^{1,\vec{p}(\cdot)}(\Omega).$$

□

3.2. Proof of Theorem 3.1

Step 1: Approximate problems.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $W^{-1,\vec{p}(.)'}(\Omega)\cap L^1(\Omega)$ such that $f_n\to f$ in $L^1(\Omega)$ and $|f_n|\le|f|$. We consider the approximate problem

$$\begin{cases} A_nu_n+g_n(x,u_n,\nabla u_n)+|u_n|^{p_0(x)-2}u_n=f_n-\operatorname{div}\phi_n(u_n),\\ u_n\in W_0^{1,\vec{p}(.)}(\Omega),\end{cases}\tag{3.6}$$

with

$$\phi_n(s)=\phi(T_n(s)),\qquad A_nv=-\sum_{i=1}^N\frac{\partial}{\partial x_i}a_i(x,T_n(v),\nabla v)$$
$$\text{and}\qquad g_n(x,s,\xi)=\frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}.$$

Note that

$$g_n(x,s,\xi)s\ge0,\quad |g_n(x,s,\xi)|\le|g(x,s,\xi)|\quad\text{and}\quad |g_n(x,s,\xi)|\le n,\quad\forall n\in\mathbb{N}^*.$$

We define the operator $G_n:W_0^{1,\vec{p}(.)}(\Omega)\longrightarrow W^{-1,,\vec{p}(.)'}(\Omega)$ by

$$\langle G_nu,v\rangle=\int_\Omega g_n(x,u,\nabla u)v\,dx+\int_\Omega|u|^{p_0(x)-2}uv\,dx,\qquad\forall v\in W_0^{1,\vec{p}(.)}(\Omega).$$

Thanks to the generalized Hölder type inequality, we have for all $u,v\in W_0^{1,\vec{p}(.)}(\Omega)$,

$$\begin{aligned}&\left|\int_\Omega g_n(x,u,\nabla u)v\,dx+\int_\Omega|u|^{p_0(x)-2}u\ v\,dx\right|\\ \le&\frac{1}{p_0^-}+\frac{1}{(p_0')^-})\left(\|g_n(x,u,\nabla u)\|_{p_0'(x)}+\||u|^{p_0(x)-1}\|_{p_0'(x)}\right)\|v\|_{p_0(x)}\\ \le&(\frac{1}{p_0^-}+\frac{1}{(p_0')^-})\left((\int_\Omega n^{p_0'(x)}\,dx+1)^{\frac{1}{(p_0')^-}}+(\int_\Omega|u|^{p_0(x)}\,dx+1)^{\frac{1}{(p_0')^-}}\right)\|v\|_{1,\vec{p}(x)}\\ \le&(\frac{1}{p_0^-}+\frac{1}{(p_0')^-})\left((n^{(p_0')^+}\operatorname{meas}(\Omega)+1)^{\frac{1}{(p_0')^-}}+(\int_\Omega|u|^{p_0(x)}\,dx+1)^{\frac{1}{(p_0')^-}}\right)\|v\|_{1,\vec{p}(x)}\\ \le&C_0\|v\|_{1,\vec{p}(.)}\,.\end{aligned}\tag{3.7}$$

Furthermore, we define the operator $R_n:W_0^{1,\vec{p}(.)}(\Omega)\longrightarrow W^{-1,\vec{p}(.)'}(\Omega)$ by

$$\langle R_n(u),v\rangle=\langle\operatorname{div}\phi_n(u),v\rangle=-\int_\Omega\phi_n(u)\nabla v dx,\qquad\forall u,v\in W_0^{1,\vec{p}(.)}(\Omega),$$

with $\phi_n(u)=(\phi_{i,n}(u),\dots,\phi_{N,n}(u))$. Using the generalized Hölder type inequality, we deduce that

$$
\begin{aligned}
&\Big|\int_\Omega \phi_n(u)\nabla v\,dx\Big|\\
\leq&\sum_{i=1}^N\int_\Omega |\phi_{i,n}(u)|\,|D^iv|\,dx\\
\leq&\sum_{i=1}^N\big(\frac{1}{p_i^-}+\frac{1}{(p_i')^-}\big)\,\big\|\phi_{i,n}(u)\big\|_{p_i'(x)}\big\|D^iv\big\|_{p_i(x)}\\
\leq&\sum_{i=1}^N\big(\frac{1}{p_i^-}+\frac{1}{(p_i')^-}\big)\,\big(\int_\Omega |\phi_{i,n}(u)|^{p_i'(x)}\,dx+1\big)^{\frac{1}{(p_i')^-}}\|v\|_{1,\vec{p}(x)}\\
\leq&\sum_{i=1}^N\big(\frac{1}{p_i^-}+\frac{1}{(p_i')^-}\big)\,\big(\sup_{|s|\leq n}(|\phi_i(s)|+1)^{(p_i')^+}\operatorname{meas}(\Omega)+1\big)^{\frac{1}{(p_i')^-}}\|v\|_{1,\vec{p}(x)}\\
\leq&C_1\|v\|_{1,\vec{p}(.)}.
\end{aligned}
\tag{3.8}
$$

In view of Lemma 4.1 (see Appendix), there exists at least one solution $u_n \in W_0^{1,\vec{p}(.)}(\Omega)$ of the problem (3.6) (see [17]).

Step 2: A priori estimates.

Let n large enough. We choose $T_k(u_n)$ as a test function in (3.6) to get

$$
\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)\,dx\\
&+\int_\Omega |u_n|^{p_0(x)-2}u_nT_k(u_n)\,dx\\
=&\int_\Omega f_nT_k(u_n)\,dx+\sum_{i=1}^N\int_\Omega \phi_{i,n}(u_n)D^iT_k(u_n)\,dx.
\end{aligned}
\tag{3.9}
$$

From (1.4) and Young's inequality, we obtain

$$
\begin{aligned}
&\underline{\alpha}\sum_{i=0}^N\int_\Omega |D^iT_k(u_n)|^{p_i(x)}\,dx\\
\leq&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)\,dx\\
&+\int_\Omega |u_n|^{p_0(x)-2}u_nT_k(u_n)\,dx\\
\leq&\int_\Omega f_nT_k(u_n)\,dx+\sum_{i=1}^N\int_\Omega \phi_{i,n}(u_n)D^iT_k(u_n)\,dx.
\end{aligned}
\tag{3.10}
$$

Taking $\Phi_{i,n}(t) = \displaystyle\int_0^t \phi_{i,n}(\tau)\,d\tau$, then $\Phi_{i,n}(0) = 0$ and $\Phi_{i,n} \in C^1(I\!R)$, in view of the Green formula, we obtain

$$\begin{aligned} \int_\Omega \phi_{i,n}(u_n) D^i T_k(u_n)\,dx &= \int_\Omega D^i \Phi_{i,n}(T_k(u_n))\,dx \\ &= \int_{\partial\Omega} \Phi_{i,n}(T_k(u_n)) \cdot n_i \, d\sigma = 0, \end{aligned} \tag{3.11}$$

since $u_n = 0$ on $\partial\Omega$, with $\vec{n} = (n_1, n_2, \ldots, n_N)$ the normal vector on $\partial\Omega$. It follows from (3.10) that

$$\sum_{i=0}^{N} \|D^i T_k(u_n)\|_{p_i(x)}^{\underline{p}^-} \le \frac{k}{\underline{\alpha}} \|f\|_1 + C_2,$$

where $\underline{p}^- = \min\limits_{x\in\Omega}(\underline{p}(x))$. Consequently,

$$\|T_k(u_n)\|_{1,\vec{p}(x)} \le C_3 k^{\frac{1}{\underline{p}^-}} \qquad \text{for all} \quad k \ge 1. \tag{3.12}$$

Now, we will show that $(u_n)_{n\in I\!N}$ is a Cauchy sequence in measure. Indeed, by Combining the generalized Hölder type inequality and (3.12), it follows that

$$\begin{aligned} k \text{ meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\,dx \le \int_\Omega |T_k(u_n)|\,dx \\ &\le \Big(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\Big) \|1\|_{p_0'(x)} \|T_k(u_n)\|_{p_0(x)} \\ &\le \Big(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\Big) \ (\text{meas}(\Omega)+1)^{\frac{1}{(p_0')^-}} \|T_k(u_n)\|_{1,\vec{p}(x)} \\ &\le C_4 k^{\frac{1}{\underline{p}^-}}, \end{aligned} \tag{3.13}$$

which yields

$$\text{meas}\{|u_n| > k\} \le C_4 \frac{1}{k^{1-\frac{1}{\underline{p}^-}}} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{3.14}$$

Moreover, we have, for every $\delta > 0$,

$$\begin{aligned} &\text{meas}\{|u_n - u_m| > \delta\} \\ \le &\text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned}$$

Let $\varepsilon > 0$, using (3.14) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \le \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \le \frac{\varepsilon}{3}. \tag{3.15}$$

Moreover, since the sequence $(T_k(u_n))_{n\in I\!N}$ is bounded in $W_0^{1,\vec{p}(.)}(\Omega)$, then there exists a subsequence still denoted $(T_k(u_n))_{n\in I\!N}$ such that

$$T_k(u_n) \rightharpoonup \eta_k \quad \text{in} \quad W_0^{1,\vec{p}(.)}(\Omega) \ \text{ as } \ n \to \infty$$

and by the compact embedding, we have

$$T_k(u_n) \longrightarrow \eta_k \quad \text{in} \quad L^{\underline{p}(x)}(\Omega) \text{ and a.e. in } \Omega.$$

Consequently, we can assume that $(T_k(u_n))_{n\in I\!N}$ is a Cauchy sequence in measure.

Thus,

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3}, \qquad \text{for all } m, n \geq n_0(\delta, \varepsilon). \tag{3.16}$$

Finally, from (3.15) and (3.16), we obtain that

$$\begin{aligned} &\forall\delta, \varepsilon > 0, \quad \text{there exists} \quad n_0 = n_0(\delta, \varepsilon) \ \text{ such that} \\ &\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon, \qquad \forall n, m \geq n_0(\delta, \varepsilon), \end{aligned}$$

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Therefore,

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u), & \text{in} \quad W_0^{1,\vec{p}(\cdot)}(\Omega) \\ T_k(u_n) \to T_k(u), & \text{in} \quad L^{\underline{p}(\cdot)}(\Omega) \quad \text{and a.e. in} \quad \Omega. \end{cases} \tag{3.17}$$

Step 3: Convergence of the gradient.

In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \dots$ a various functions of real numbers which converges to 0 as n tends to infinity.

Let $\varphi_k(s) = s.\exp(\gamma s^2)$, where $\gamma = \left(\frac{b(k)}{2\alpha}\right)^2$.

It is obvious that

$$\varphi_k'(s) - \frac{b(k)}{\alpha}|\varphi_k(s)| \geq \frac{1}{2}, \qquad \forall s \in I\!R.$$

We also consider $h > k > 0$ and $M = 4k + h$ and we set

$$\omega_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)).$$

By taking $\varphi_k(\omega_n)$ as a test function in (3.6), we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)\varphi_k'(\omega_n)D^i\omega_n \, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k(\omega_n) \, dx \\ &+ \int_\Omega |u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n) \, dx \\ = &\int_\Omega f_n\varphi_k(\omega_n) \, dx + \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_n)\varphi_k'(\omega_n)D^i\omega_n \, dx. \end{aligned}$$

It is easy to see that $D^i\omega_n = 0$ on $\{|u_n| > M\}$ and since $g_n(x, u_n, \nabla u_n)\varphi_k(\omega_n) \geq 0$ on $\{|u_n| > k\}$, then

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n)D^i\omega_n \, dx \\ &+ \int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi_k(\omega_n) \, dx + \int_{\{|u_n|\leq k\}} |u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n) \, dx \\ \leq &\int_\Omega f_n\varphi_k(\omega_n) \, dx + \int_{\{|u_n|\leq M\}} \phi_{i,n}(T_M(u_n))\varphi_k'(\omega_n)D^i\omega_n \, dx. \end{aligned} \tag{3.18}$$

First estimate :

We have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)D^i\omega_n\,dx\\
=&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)D^iT_{2k}(u_n-T_k(u))\,dx \qquad (3.19)\\
&+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)D^i\omega_n\,dx.
\end{aligned}$$

On one hand, since $|u_n - T_k(u)| \le 2k$ on $\{|u_n| \le k\}$, then

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)D^iT_{2k}(u_n-T_k(u))\,dx\\
=&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx \qquad (3.20)\\
&+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)D^iT_k(u)\,dx,
\end{aligned}$$

since $1 \le \varphi_k'(\omega_n) \le \varphi_k'(2k)$, then

$$\begin{aligned}
&\Big|\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)D^iT_k(u)\,dx\Big|\\
\le&\varphi_k'(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}} |a_i(x,T_k(u_n),\nabla T_k(u_n))|\,|D^iT_k(u)|\,dx,
\end{aligned}$$

and since $(|a_i(x,T_k(u_n),\nabla T_k(u_n))|)_{n\in\mathbb{N}}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists $\vartheta \in L^{p_i'(\cdot)}(\Omega)$ such that $|a(x,T_k(u_n),\nabla T_k(u_n))| \rightharpoonup \vartheta$ in $L^{p_i'(\cdot)}(\Omega)$. Therefore,

$$\int_{\{|u_n|>k\}} |a(x,T_k(u_n),\nabla T_k(u_n))|\,|D^iT_k(u)|\,dx \longrightarrow \int_{\{|u|>k\}} \vartheta\,|D^iT_k(u)|\,dx = 0.$$

It follows that

$$\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)D^iT_k(u)\,dx = \varepsilon_0(n). \qquad (3.21)$$

On the other hand, for the second term on the right hand side of (3.19), taking

$z_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, then

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n|>k\}} a_i(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) D^i\omega_n \, dx \\
=& \sum_{i=1}^{N} \int_{\{|u_n|>k\}\cap\{|z_n|\le 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) \\
&\cdot D^i(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx \\
=& \sum_{i=1}^{N} \int_{\{|u_n|>k\}\cap\{|z_n|\le 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) D^i(u_n - T_k(u)).\chi_{\{|u_n|>h\}} \, dx \\
&- \int_{\{|u_n|>k\}\cap\{|z_n|\le 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) D^i T_k(u).\chi_{\{|u_n|\le h\}} \, dx \\
\ge& -\varphi_k'(2k) \sum_{i=1}^{N} \int_{\{|u_n|>k\}} |a_i(x, T_M(u_n), \nabla T_M(u_n))| \, |D^i T_k(u)| \, dx.
\end{aligned}$$

In the same way as for (3.21), we can prove that

$$\varphi_k'(2k) \sum_{i=1}^{N} \int_{\{|u_n|>k\}} |a_i(x, T_M(u_n), \nabla T_M(u_n))| \, |D^i T_k(u)| \, dx = \varepsilon_1(n). \tag{3.22}$$

After adding (3.19) – (3.22), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) D^i\omega_n \, dx \\
\ge& \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n))\varphi_k'(\omega_n)(D^i T_k(u_n) - D^i T_k(u)) \, dx + \varepsilon_2(n),
\end{aligned} \tag{3.23}$$

which is equivalent to say

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\
&\cdot (D^i T_k(u_n) - D^i T_k(u)) \, \varphi_k'(\omega_n) \, dx \\
\le& \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n))\varphi_k'(\omega_n) D^i\omega_n \, dx \\
&- \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u))(D^i T_k(u_n) - D^i T_k(u)) \, \varphi_k'(\omega_n) \, dx - \varepsilon_2(n), \\
\le& \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \, \varphi_k'(\omega_n) \, D^i\omega_n \, dx \\
&+ \varphi_k'(2k) \sum_{i=1}^{N} \int_{\Omega} |a_i(x, T_k(u_n), \nabla T_k(u))| \, |D^i T_k(u_n) - D^i T_k(u)| \, dx - \varepsilon_2(n).
\end{aligned} \tag{3.24}$$

Concerning the second term on the right-hand side of (3.24), by applying the Lebesgue convergence theorem, we have $T_k(u_n) \to T_k(u_n)$ in $L^{p_0(x)}(\Omega)$, then, $|a_i(x, T_k(u_n), \nabla T_k(u))| \to |a_i(x, T_k(u), \nabla T_k(u))|$ in $L^{p_i'(x)}(\Omega)$, and since $D^i T_k(u_n)$ converges to $D^i T_k(u)$ weakly in $L^{p_i(x)}(\Omega)$, we obtain

$$\varepsilon_3(n) = \varphi_k'(2k) \sum_{i=1}^{N} \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))||D^i T_k(u_n) - D^i T_k(u)|\, dx \longrightarrow 0, \quad \text{as } n \to \infty. \tag{3.25}$$

We conclude that

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ &\cdot (D^i T_k(u_n) - D^i T_k(u))\, \varphi_k'(\omega_n)\, dx \\ \leq &\sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n))\, \varphi_k'(\omega_n)\, D^i \omega_n\, dx + \varepsilon_4(n). \end{aligned} \tag{3.26}$$

Second estimate :

Now, we treat the second term on the left-hand side of (3.18).

From (1.7), we obtain

$$\begin{aligned} &\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \varphi_k(\omega_n)\, dx \right| \\ \leq &\int_{\{|u_n| \leq k\}} b(|u_n|)(c(x) + \sum_{i=1}^{N} |D^i T_k(u_n)|^{p_i(x)}) |\varphi_k(\omega_n)|\, dx \\ \leq & b(k) \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)|\, dx \\ &+ \frac{b(k)}{\alpha} \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u_n) |\varphi_k(\omega_n)|\, dx \\ \leq & b(k) \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)|\, dx \\ &+ \frac{b(k)}{\alpha} \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ &\cdot (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)|\, dx \\ &+ \frac{b(k)}{\alpha} \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)|\, dx \\ &+ \frac{b(k)}{\alpha} \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)|\, dx. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{b(k)}{\alpha} \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ &\cdot (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)|\, dx \end{aligned} \tag{3.27}$$

$$
\begin{aligned}
\geq & \left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \varphi_k(\omega_n)\, dx \right| - b(k) \int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)|\, dx \\
& - \frac{b(k)}{\alpha} \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)|\, dx \\
& - \frac{b(k)}{\alpha} \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)|\, dx.
\end{aligned}
$$

We have $\varphi_k(\omega_n) \rightharpoonup \varphi_k(T_{2k}(u - T_h(u)))$ weak-$\star$ in $L^\infty(\Omega)$ as $n \to +\infty$, then

$$
\int_{\{|u_n| \leq k\}} c(x) |\varphi_k(\omega_n)|\, dx \longrightarrow \int_{\{|u| \leq k\}} c(x) |\varphi_k(T_{2k}(u - T_h(u)))|\, dx = 0. \tag{3.28}
$$

Concerning the third term on the right-hand side of (3.27), thanks to (3.25), we have

$$
\begin{aligned}
& \left| \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)|\, dx \right| \\
\leq & \varphi_k(2k) \sum_{i=1}^{N} \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))|\, |D^i T_k(u_n) - D^i T_k(u)|\, dx \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.29}
$$

For the last term of the right-hand side of (3.27), we have that

$$
(a_i(x, T_k(u_n), \nabla T_k(u_n)))_{n \in I\!N} \text{ is bounded in } L^{p_i'(.)}(\Omega),
$$

then there exists $\psi_i \in L^{p_i'(.)}(\Omega)$ such that $a_i(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \psi_i$ in $L^{p_i'(.)}(\Omega)$ and, using the fact that

$$
D^i T_k(u)\, |\varphi_k(\omega_n)| \rightharpoonup D^i T_k(u)\, |\varphi_k(T_{2k}(u - T_h(u)))| \qquad \text{in } L^{p_i(.)}(\Omega),
$$

it follows that

$$
\begin{aligned}
& \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)|\, dx \\
\longrightarrow & \int_\Omega \psi_i D^i T_k(u) |\varphi_k(T_{2k}(u - T_h(u)))|\, dx = 0.
\end{aligned} \tag{3.30}
$$

By combining (3.27) – (3.30), we get

$$
\begin{aligned}
& \frac{b(k)}{\alpha} \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\
& \cdot (D^i T_k(u_n) - D^i T_k(u)) |\varphi_k(\omega_n)|\, dx \\
\geq & \left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \varphi_k(\omega_n)\, dx \right| + \varepsilon_5(n).
\end{aligned} \tag{3.31}
$$

Third estimate :

We have

$$
\begin{aligned}
&\int_{\{|u_n|\le k\}} |u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n)\,dx\\
=&\int_\Omega |T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n)-T_k(u))\exp(\gamma\omega_n^2)\,dx\\
&\int_{\{|u_n|>k\}} |T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n)-T_k(u))\exp(\gamma\omega_n^2)\,dx\\
=&\int_\Omega \Big(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u)\Big)(T_k(u_n)-T_k(u))\exp(\gamma\omega_n^2)\,dx\\
&+\int_\Omega |T_k(u)|^{p_0(x)-2}T_k(u)(T_k(u_n)-T_k(u))\exp(\gamma\omega_n^2)\,dx\\
&-\int_{\{|u_n|>k\}} |T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n)-T_k(u))\exp(\gamma\omega_n^2)\,dx\\
\ge&\int_\Omega \Big(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u)\Big)(T_k(u_n)-T_k(u))\,dx\\
&-\exp(\gamma(2k)^2)\int_\Omega |T_k(u)|^{p_0(x)-1}\;|T_k(u_n)-T_k(u)|\,dx\\
&-\exp(\gamma(2k)^2)\int_{\{|u_n|>k\}} k^{p_0(x)-1}\;|T_k(u_n)-T_k(u)|\,dx
\end{aligned}
$$

and as $T_k(u_n)\to T_k(u)$ in $L^{p_0(\cdot)}(\Omega)$, then the second and the last term on the right-hand side of the inequality above converges to 0 as n goes to infinity. Therefore, we get

$$
\begin{aligned}
&\int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\\
&\le\int_{\{|u_n|\le k\}} |u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n)\,dx+\varepsilon_6(n).
\end{aligned}
\tag{3.32}
$$

Thanks to (3.26), (3.31) and (3.32), we obtain

$$
\begin{aligned}
&\frac12\sum_{i=1}^N\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\\
&\cdot(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&+\int_\Omega(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\\
\le&\sum_{i=1}^N\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)D^i\omega_n\,dx\\
&-\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx\right|\\
&+\int_{\{|u_n|\le k\}}|u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n)\,dx+\varepsilon_7(n),\\
\le&\int_\Omega f_n\varphi_k(\omega_n)\,dx+\sum_{i=1}^N\int_{\{|u_n|\le M\}}\phi_{i,n}(T_M(u_n))\varphi_k'(\omega_n)\nabla\omega_n\,dx+\varepsilon_7(n).
\end{aligned}
\tag{3.33}
$$

We have

$$\int_\Omega f_n \varphi_k(\omega_n)\,dx = \int_\Omega f\varphi_k(T_{2k}(u - T_h(u)))\,dx + \varepsilon_8(n) \tag{3.34}$$

and for n large enough (for example $n \geq M$), we can write

$$\int_\Omega \phi_{i,n}(T_M(u_n))\varphi_k'(\omega_n)D^i\omega_n\,dx = \int_\Omega \phi_i(T_M(u_n))\varphi_k'(\omega_n)D^i\omega_n\,dx.$$

It follows that

$$\begin{aligned}&\int_\Omega \phi_{i,n}(T_M(u_n))\varphi_k'(\omega_n)D^i\omega_n\,dx\\ &= \int_\Omega \phi_i(T_M(u))\varphi_k'(T_{2k}(u - T_h(u)))D^iT_{2k}(u - T_h(u))\,dx + \varepsilon_9(n).\end{aligned} \tag{3.35}$$

By combining (3.33), (3.34) and (3.35), we obtain

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u))\,dx\\ &\quad + \int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n) - T_k(u))\,dx\\ &\leq 2\int_\Omega f\varphi_k(T_{2k}(u - T_h(u)))\,dx\\ &\quad + 2\sum_{i=1}^N \int_\Omega \phi_i(T_M(u))\varphi_k'(T_{2k}(u - T_h(u)))D^iT_{2k}(u - T_h(u))\,dx + \varepsilon_{10}(n).\end{aligned} \tag{3.36}$$

Taking $\Psi_i(t) = \displaystyle\int_0^t \phi_i(\tau)\varphi_k'(\tau - T_h(\tau))\,d\tau$, then $\Psi_i(0) = 0$ and $\Psi_i \in C^1(\mathbb{R})$.

In view of the Green formula, we have

$$\begin{aligned}&\int_\Omega \phi_i(T_M(u))\varphi_k'(T_{2k}(u - T_h(u)))D^iT_{2k}(u - T_h(u))\,dx\\ &= \int_{\{h<|u|\leq 2k+h\}} \phi(u)\varphi_k'(u - T_h(u))D^iu\,dx\\ &= \int_{\{|u|\leq 2k+h\}} \phi_i(T_{2k+h}(u))\varphi_k'(T_{2k+h}(u) - T_h(u))D^iT_{2k+h}(u)\,dx\\ &\quad - \int_{\{|u|\leq h\}} \phi_i(T_h(u))\varphi_k'(T_h(u) - T_h(u))D^iT_h(u)\,dx\\ &= \int_\Omega D^i\Psi_i(T_{2k+h}(u))\,dx - \int_\Omega D^i\Psi_i(T_h(u))\,dx\\ &= \int_{\partial\Omega} \Psi_i(T_{2k+h}(u)).n_i\,dx - \int_{\partial\Omega} \Psi_i(T_h(u)).n_i\,dx = 0.\end{aligned}$$

Then, by letting h and n goes to infinity in (3.36), we get

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u))\,dx\\ &+ \int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n) - T_k(u))\,dx \longrightarrow 0.\end{aligned} \tag{3.37}$$

Using the Lemma 3.3, we deduce that

$$T_k(u_n) \longrightarrow T_k(u), \qquad \text{in} \quad W_0^{1,\vec{p}(.)}(\Omega). \tag{3.38}$$

Therefore,

$$D^i u_n \longrightarrow D^i u \quad \text{a.e. in } \Omega.$$

Step 4: The equi-integrability of g_n.

In order to pass to the limit in the approximate equation, we show that

$$g_n(x, u_n, \nabla u_n) \longrightarrow g(x, u, \nabla u) \quad \text{ strongly in } L^1(\Omega), \text{ as } n \to +\infty.$$

By using Vitali's Theorem, it suffices to prove that $g_n(x, u_n, \nabla u_n)$ is uniformly equi-integrable. Indeed, taking $T_1(u_n - T_h(u_n))$ as a test function in (3.6), and using (1.4) since $T_1(u_n - T_h(u_n))$ have the same sign as u_n, we obtain

$$\begin{aligned} &\int_{\{h<|u_n|\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_h(u_n))\, dx \\ &\leq \int_{\{h<|u_n|\}} f_n T_1(u_n - T_h(u_n))\, dx + \sum_{i=1}^{N} \int_{\{h<|u_n|\leq h+1\}} \phi_{i,n}(u_n) D^i u_n\, dx. \end{aligned} \tag{3.39}$$

Taking $\Phi_{i,n}(t) = \displaystyle\int_0^t \phi_{i,n}(\tau)\, d\tau$, then $\Phi_{i,n}(0) = 0$ and $\Phi_n \in C^1(I\!R)$. Thanks to the Green formula, we have

$$\begin{aligned} &\int_{\{h<|u_n|\leq h+1\}} \phi_{i,n}(u_n) D^i u_n\, dx \\ &= \int_\Omega \phi_{i,n}(T_{h+1}(u_n)) D^i T_{h+1}(u_n)\, dx - \int_\Omega \phi_{i,n}(T_h(u_n)) D^i T_h(u_n)\, dx \\ &= \int_\Omega D^i \Phi_{i,n}(T_{h+1}(u_n))\, dx - \int_\Omega D^i \Phi_{i,n}(T_h(u_n))\, dx = 0. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\{h+1\leq|u_n|\}} |g_n(x, u_n, \nabla u_n)|\, dx &= \int_{\{h+1\leq|u_n|\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_h(u_n))\, dx \\ &\leq \int_{\{h\leq|u_n|\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_h(u_n))\, dx \\ &\leq \int_{\{h\leq|u_n|\}} f_n T_1(u_n - T_h(u_n))\, dx \\ &\leq \int_{\{h\leq|u_n|\}} |f|\, dx, \end{aligned}$$

thus, for all $\eta > 0$, there exists $h(\eta) > 0$ such that

$$\int_{\{h(\eta)\leq|u_n|\}} |g_n(x, u_n, \nabla u_n)|\, dx \leq \frac{\eta}{2}. \tag{3.40}$$

On the other hand, for any measurable subset $E \subset \Omega$, we have

$$\int_E |g_n(x,u_n,\nabla u_n)|\,dx \le \int_{E\cap\{|u_n|<h(\eta)\}} b(h(\eta))(c(x)+\sum_{i=1}^N |D^i u_n|^{p_i(x)})\,dx \\ + \int_{\{|u_n|\ge h(\eta)\}} |g_n(x,u_n,\nabla u_n)|\,dx. \tag{3.41}$$

From (3.38), there exists $\beta(\eta)>0$ such that

$$\int_{E\cap\{|u_n|<h(\eta)\}} b(h(\eta))(c(x)+\sum_{i=1}^N |D^i u_n|^{p_i(x)})\,dx \le \frac{\eta}{2}, \\ \text{for all } E \text{ such that } \operatorname{meas}(E)\le\beta(\eta). \tag{3.42}$$

Finally, by combining (3.40), (3.41) and (3.42), one easily has

$$\int_E |g_n(x,u_n,\nabla u_n)|\,dx \le \eta \quad \text{for all } E \text{ such that meas} \le \beta(\eta), \tag{3.43}$$

we then deduce that $(g_n(x,u_n,\nabla u_n))_n$ is equi-integrable, and by Vitali's Theorem we deduce that

$$g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \qquad \text{in} \quad L^1(\Omega). \tag{3.44}$$

Step 5: Passing to the limit.

By using $T_k(u_n-\varphi)$ as a test function in (3.6), with $\varphi \in W_0^{1,\vec{p}(.)}(\Omega)\cap L^\infty(\Omega)$, and putting $M = k + \|\varphi\|_\infty$, we get

$$\sum_{i=1}^N \int_\Omega a_i(x,T_n(u_n),\nabla u_n) D^i T_k(u_n-\varphi)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n) T_k(u_n-\varphi)\,dx \\ + \int_\Omega |u_n|^{p_0(x)-2} u_n T_k(u_n-\varphi)\,dx \\ = \int_\Omega f_n T_k(u_n-\varphi)\,dx + \sum_{i=1}^N \int_\Omega \phi_{i,n}(u_n)\nabla T_k(u_n-\varphi)\,dx. \tag{3.45}$$

On one hand, if $|u_n|>M$ then $|u_n-\varphi| \ge |u_n| - \|\varphi\|_\infty > k$, therefore $\{|u_n-\varphi|\le k\} \subseteq \{|u_n|\le M\}$, which implies that

$$\int_\Omega a_i(x,T_n(u_n),\nabla u_n) D^i T_k(u_n-\varphi)\,dx \\ = \int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx \\ = \int_\Omega (a_i(x,T_M(u_n),\nabla T_M(u_n)) - a_i(x,T_M(u_n),\nabla\varphi)) \\ \cdot (D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx \\ + \int_\Omega a_i(x,T_M(u_n),\nabla\varphi)(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx. \tag{3.46}$$

According to Fatou's Lemma, we obtain

$$
\begin{aligned}
&\liminf_{n\to+\infty}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\
\geq &\int_\Omega (a_i(x,T_M(u),\nabla T_M(u))-a_i(x,T_M(u),\nabla\varphi))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&+\lim_{n\to+\infty}\int_\Omega a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx.
\end{aligned}
\tag{3.47}
$$

The second term in the right hand side of (3.47) is equal to

$$
\int_\Omega a_i(x,T_M(u),\nabla\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}dx.
$$

Therefore, we get

$$
\begin{aligned}
&\liminf_{n\to+\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\
\geq &\sum_{i=1}^N\int_\Omega a_i(x,T_M(u),\nabla T_M(u))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
= &\sum_{i=1}^N\int_\Omega a_i(x,u,\nabla u)(D^iu-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
= &\sum_{i=1}^N\int_\Omega a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,dx.
\end{aligned}
$$

On the other hand, we have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak-$\star$ in $L^\infty(\Omega)$ and $g_n(x,u_n,\nabla u_n)\longrightarrow g(x,u,\nabla u)$ in $L^1(\Omega)$. Then,

$$
\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega g(x,u,\nabla u)\ T_k(u-\varphi)\,dx \tag{3.48}
$$

and

$$
\int_\Omega f_nT_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega f\ T_k(u-\varphi)\,dx. \tag{3.49}
$$

Again, since $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ in $W_0^{1,\vec{p}(.)}(\Omega)$ and $\phi_{i,n}(u_n)=\phi_i(T_M(u_n))$ in $\{|u_n-\varphi|\leq k\}$ for $n\geq M$, we have

$$
\int_\Omega \phi_{i,n}(u_n)\nabla T_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega\phi_i(u)\nabla T_k(u-\varphi)\,dx \tag{3.50}
$$

and

$$
\int_\Omega |u_n|^{p_0(x)-2}u_nT_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega |u|^{p_0(x)-2}uT_k(u-\varphi)\,dx. \tag{3.51}
$$

Which completes the proof of Theorem 3.1.

4. Appendix

Lemma 4.1. *The operator $B_n = A_n + G_n + R_n$ from $W_0^{1,\vec{p}(.)}(\Omega)$ into $W^{-1,\vec{p}(.)'}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,\vec{p}(.)}} \longrightarrow +\infty \quad \text{if} \quad \|v\|_{1,\vec{p}(.)} \longrightarrow +\infty, \qquad \forall v \in W_0^{1,\vec{p}(.)}(\Omega).$$

Proof. Using the Hölder's inequality and the growth condition (1.3), we can show that the operator A_n is bounded, and by (3.7) and (3.8) we conclude that B_n is bounded. For the coercivity, we have for all $u \in W_0^{1,\vec{p}(.)}(\Omega)$,

$$\begin{aligned}
\langle B_n u, u\rangle =& \langle A_n u, u\rangle + \langle G_n u, u\rangle + \langle R_n u, u\rangle \\
=& \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u)\; D^i u\, dx + \int_\Omega g_n(x, u, \nabla u) u\, dx + \int_\Omega |u|^{p_0(x)}\, dx \\
& - \int_\Omega \phi_n(u) \nabla u\, dx \\
\geq& \underline{\alpha} \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i(x)}\, dx - \sum_{i=1}^{N} \int_\Omega |\phi_{i,n}(u)|\; |D^i u|\, dx \\
\geq& \underline{\alpha}\; \|u\|_{1,\vec{p}(.)}^{\underline{p}^-} - \underline{\alpha}(N+1) - C_1 \|u\|_{1,\vec{p}(.)},
\end{aligned}$$

then, we obtain

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,\vec{p}(.)}} \longrightarrow +\infty \quad \text{if } \|u\|_{1,\vec{p}(.)} \longrightarrow +\infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in I\!N}$ be a sequence in $W_0^{1,\vec{p}(.)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u, & \text{in } W_0^{1,\vec{p}(.)}(\Omega), \\ B_n u_k \rightharpoonup \chi, & \text{in } W^{-1,\vec{p}(.)'}(\Omega), \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases} \tag{4.1}$$

We will prove that

$$\chi = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \text{ as } k \to +\infty.$$

Firstly, since $W_0^{1,\vec{p}(.)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}(.)}(\Omega)$, then

$$u_k \to u \quad \text{in } L^{\underline{p}(.)}(\Omega) \quad \text{for a subsequence denoted again } (u_k)_{k\in I\!N}.$$

As $(u_k)_{k\in I\!N}$ is a bounded sequence in $W_0^{1,\vec{p}(.)}(\Omega)$, then by the growth condition $(a_i(x, T_n(u_k), \nabla u_k))_{k\in I\!N}$ is bounded in $L^{p_i'(.)}(\Omega)$.

Therefore there exists a function $\varphi_i \in L^{p_i'(.)}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \quad \text{in} \quad L^{p_i'(.)}(\Omega) \text{ as } k \to \infty. \tag{4.2}$$

Similarly, it is easy to see that $(g_n(x, u_k, \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{\underline{p}'}(\Omega)$, then there exists a function $\psi_n \in L^{\underline{p}'}(\Omega)$ such that

$$g_n(x, u_k, \nabla u_k) \rightharpoonup \psi_n \quad \text{in} \quad L^{\underline{p}'}(\Omega) \quad \text{as} \quad k \to \infty. \tag{4.3}$$

Finally, since $\phi_n = \phi \circ T_n$ is a bounded continuous function and $u_k \to u$ in $L^{\underline{p}(\cdot)}(\Omega)$, by using the Lebesgue dominated convergence theorem, we deduce that

$$\phi_{i,n}(u_k) \longrightarrow \phi_{i,n}(u) \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega), \quad \text{for} \quad i = 1, \dots, N. \tag{4.4}$$

It is clear that, for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we get

$$\begin{aligned}
&\langle \chi, v\rangle \\
&= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v \, dx + \lim_{k\to\infty} \int_\Omega g_n(x, u_k, \nabla u_k) v \, dx \\
&\quad + \lim_{k\to\infty} \int_\Omega |u_k|^{p_0(x)-2} u_k \ v \, dx - \lim_{k\to\infty} \sum_{i=1}^N \int_\Omega \phi_{i,n}(u_k) D^i v \, dx \\
&= \sum_{i=1}^N \int_\Omega \varphi_i \ D^i v \, dx + \int_\Omega \psi_n v \, dx + \int_\Omega |u|^{p_0(x)-2} u \ v \, dx - \sum_{i=1}^N \int_\Omega \phi_{i,n}(u) D^i v \, dx.
\end{aligned} \tag{4.5}$$

From relations (4.1) and (4.5), we have

$$\begin{aligned}
&\limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle \\
&= \limsup_{k\to\infty} \Big\{ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx + \int_\Omega g_n(x, u_k, \nabla u_k) u_k \, dx \\
&\quad + \int_\Omega |u_k|^{p_0(x)} \, dx - \sum_{i=1}^N \int_\Omega \phi_{i,n}(u_k) D^i u_k \, dx \Big\} \\
&\le \sum_{i=1}^N \int_\Omega \varphi_i \ D^i u \, dx + \int_\Omega \psi_n u \, dx + \int_\Omega |u|^{p_0(x)} \, dx - \sum_{i=1}^N \int_\Omega \phi_{i,n}(u) D^i u \, dx.
\end{aligned} \tag{4.6}$$

Thanks to (4.3) and (4.4), we have

$$\begin{aligned}
&\int_\Omega g_n(x, u_k, \nabla u_k) u_k \, dx \longrightarrow \int_\Omega \psi_n u \, dx \\
\text{and} \qquad &\sum_{i=1}^N \int_\Omega \phi_{i,n}(u_k) D^i u_k \, dx \longrightarrow \sum_{i=1}^N \int_\Omega \phi_{i,n}(u) D^i u \, dx.
\end{aligned} \tag{4.7}$$

Therefore

$$\begin{aligned}
&\limsup_{k\to\infty} \Big\{ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx + \int_\Omega |u_k|^{p_0(x)} \, dx \Big\} \\
&\le \sum_{i=1}^N \int_\Omega \varphi_i \ D^i u \, dx + \int_\Omega |u|^{p_0(x)} \, dx.
\end{aligned} \tag{4.8}$$

On the other hand, by (1.5), we get

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx \\ &+\int_{\Omega}(|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dx\geq 0, \end{aligned} \tag{4.9}$$

then

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx+\int_{\Omega}|u_k|^{p_0(x)}\,dx \\ \geq&\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu\,dx+\int_{\Omega}|u_k|^{p_0(x)-2}u_ku\,dx \\ &+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u)(D^iu_k-D^iu)\,dx+\int_{\Omega}|u|^{p_0(x)-2}u(u_k-u)\,dx. \end{aligned}$$

In view of Lebesgue dominated convergence theorem, we have $T_n(u_k)\to T_n(u)$ in $L^{p_i(\cdot)}(\Omega)$ then $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$, and by (4.2), we get

$$\begin{aligned} &\liminf_{k\to\infty}\left\{\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx+\int_{\Omega}|u_k|^{p_0(x)}\,dx\right\} \\ \geq&\sum_{i=1}^{N}\int_{\Omega}\varphi_i\ D^iu\,dx+\int_{\Omega}|u|^{p_0(x)}\,dx. \end{aligned}$$

This implies by using (4.8) that

$$\begin{aligned} &\lim_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx+\int_{\Omega}|u_k|^{p_0(x)}\,dx \\ =&\sum_{i=1}^{N}\int_{\Omega}\varphi_i\ D^iu\,dx+\int_{\Omega}|u|^{p_0(x)}\,dx. \end{aligned} \tag{4.10}$$

According to (4.5), (4.7) and (4.10), we obtain

$$\langle B_nu_k,u_k\rangle\longrightarrow\langle\chi,u\rangle\quad\text{as}\quad k\to+\infty.$$

Now, by (4.10) we can prove that

$$\begin{aligned} &\lim_{k\to+\infty}\left\{\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx\right. \\ &\left.+\int_{\Omega}(|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dx\right\}=0. \end{aligned}$$

and so, by vertue of Lemma 3.3, we get

$$u_k\longrightarrow u\ \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega)\quad\text{ and }\quad D^iu_k\longrightarrow D^iu \text{ a.e. in } \Omega,$$

then

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup a_i(x, T_n(u), \nabla u) \text{ and } \phi_{i,n}(u_k) \longrightarrow \phi_{i,n}(u) \quad \text{in} \quad L^{p'_i(\cdot)}(\Omega),$$
$$\text{for} \quad i = 1, \dots, N$$

and

$$g_n(x, u_k, \nabla u_k) \rightharpoonup g_n(x, u, \nabla u) \qquad \text{in} \quad L^{p'_0(\cdot)}(\Omega),$$

which implies that $\chi = B_n u$. □

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