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SOLITON SOLUTIONS OF SHALLOW WATER WAVE EQUATIONS BY MEANS OF G'/G EXPANSION METHOD

Marwan Alquran[1,†] and Aminah Qawasmeh[1]

Abstract In this work, we investigate the traveling wave solutions for some generalized nonlinear equations: The generalized shallow water wave equation and the Whitham-Broer-Kaup model for dispersive long waves in the shallow water small-amplitude regime. We use the G'/G expansion method to determine different soliton solutions of these models. The conditions of existence and uniqueness of exact solutions are also presented.

Keywords Shallow water wave equations, G'/G expansion method.

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1. Introduction

Exact solutions to nonlinear partial differential equations (PDEs) play an important role in nonlinear theory, since they can provide much information and more insight into the aspects of the problem and thus lead to further applications. In the literature, many significant methods have been proposed for obtaining exact solutions of nonlinear partial differential equations such as the tanh method, trigonometric and hyperbolic function methods, the rational sine-cosine method, the extended tanh-function method, the Exp-function method, the Hirota's method, Hirota bilinear forms, the tanh-sech method, the first integral method, the Riccati equation method and so on [1–11,20,24,25,28–34]. The main aim of this paper is to apply the G'/G-expansion method [13, 16, 21, 26, 39] with the help of symbolic computation to obtain soliton solutions of the generalized shallow water wave equation which reads [27]:

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - \gamma u_{xx} = 0, \tag{1.1}$$

and the Whitham-Broer-Kaup model [35] for dispersive long waves in the shallow water small-amplitude regime:

$$\begin{aligned} &u_t + u u_x + v_x + \beta u_{xx} = 0, \\ &v_t - (uv)_x - \beta v_{xx} + \alpha u_{xxx} = 0. \end{aligned} \tag{1.2}$$

The paper is organized as follows. In section 2, a brief background and construction of the proposed method G'/G is given. In section 3, related works regards the generalized shallow water wave equation are discussed and our findings are concluded. In section 4, the Whitham-Borer-Kaup system is introduced and discussed. Section 5 is devoted to the conclusion.

[†]the corresponding author.Email address:marwan04@just.edu.jo(M. Alquran)
[1]Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box(3030), Irbid 22110, Jordan

2. Construction and analysis of G'/G method

Consider the following nonlinear partial differential equation:

$$P(u, u_t, u_x, u_{tt}, u_{xt}, ...) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. By the wave variable $\zeta = x - ct$ the PDE (2.1) is then transformed to the ordinary differential equation (ODE)

$$P(u, -cu', u', c^2u'', -cu'', u'', ...) = 0, \tag{2.2}$$

where $u = u(\zeta)$. Suppose that the solution of ODE (2.2) can be expressed by a polynomial in G'/G as follows

$$u(\zeta) = \alpha_m (\frac{G'}{G})^m + ..., \tag{2.3}$$

where $G = G(\zeta)$ satisfies the second order differential equation in the form

$$G'' + \lambda G' + \mu G = 0, \tag{2.4}$$

$\alpha_0, \alpha_1, ..., \alpha_m, \lambda$ and μ are constants to be determined later, provided that $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in the ODE (2.2).

Now, if we let

$$Y = Y(\zeta) = \frac{G'}{G}, \tag{2.5}$$

then by the help of (2.4) we get

$$Y' = \frac{GG'' - G'^2}{G^2} = \frac{G(-\lambda G' - \mu G) - G'^2}{G^2} = -\lambda Y - \mu - Y^2. \tag{2.6}$$

Or, equivalently

$$Y' = -Y^2 - \lambda Y - \mu. \tag{2.7}$$

By result (2.7) and implicit differentiation, one can derive the following two formulas

$$Y'' = 2Y^3 + 3\lambda Y^2 + (2\mu + \lambda^2)Y + \lambda\mu, \tag{2.8}$$
$$Y''' = -6Y^4 - 12\lambda Y^3 - (7\lambda^2 + 8\mu)Y^2 - (\lambda^3 + 8\lambda\mu)Y - (\lambda^2\mu + 2\mu^2). \tag{2.9}$$

Combining equations (2.3), (2.5) and (2.7-2.9), then it results in a polynomial of powers of Y. Then, collecting all terms of same order of Y and equating to zero, yields a set of algebraic equations for $\alpha_0, \alpha_1, ..., \alpha_m, \lambda$, and μ.

It is known that the solution of equation (2.4) is a linear combination of sinh and cosh or of sine and cosine, respectively, if $\triangle = \lambda^2 - 4\mu > 0$ or $\triangle < 0$. Without lost of generality, we consider the first case and therefore

$$G(\zeta) = e^{-\frac{\lambda\zeta}{2}} \left(A \ \sinh(\frac{\sqrt{\lambda^2 - 4\mu}\ \zeta}{2}) + B \ \cosh(\frac{\sqrt{\lambda^2 - 4\mu}\ \zeta}{2}) \right), \tag{2.10}$$

where A and B are any constants.

3. The generalized shallow water wave equation

The generalized shallow water wave equation is given by

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - \gamma u_{xx} = 0, \tag{3.1}$$

where subscripts indicate partial derivatives, u is a real scalar function of the two independent variables x and t, while $\alpha,\ \beta,\ \gamma$ are all model parameters and they are arbitrary, nonzero constants. This equation can be derived from the classical shallow water theory in the so-called Boussinesq approximation [36]. Two special cases of (3.1) have been discussed in the literature, $\alpha = \beta$ and $\alpha = 2\beta$. Hietarinta [18] discussed the GSWW equation and he showed that it can be expressed in Hirota bilinear form [19] if and only if either $\alpha = \beta$ or $\alpha = 2\beta$. Gao and Tian [17] obtained soliton like solutions of (3.1) for $\alpha = \beta = -3$ and $\gamma = 0$ by the generalized tanh method. Yan and Zhang [38] obtained new families of soliton like solutions of (3.1) for $\alpha = -4,\ \beta = -2$ and $\gamma = 0$, which is named $(2+1)$-dimensional breaking soliton equation, by using computerized symbolic computation. Recently, Elwakil et al. [14] presented solitary wave solutions of (3.1) by using homogeneous balance method and auto-Backlund transformation and they also applied modified extended tanh-function method for obtaining new exact traveling wave solutions.

By the wave variable $\zeta = x - ct$, the above PDE is transformed into the ODE

$$-cu''' - \frac{c}{2}(\alpha+\beta)u'^2 + (c-\gamma)u' = 0. \tag{3.2}$$

Assume the solution of (3.2) is

$$u(\zeta) = a_m(\frac{G'}{G})^m + ... \tag{3.3}$$

Then, by using formula (2.4) and (2.5), one can derive

$$u'^2(\zeta) = m^2 a_m^2(\frac{G'}{G})^{2(m+1)} + ..., \tag{3.4}$$

and

$$u'''(\zeta) = m(m+1)(m+2)a_m(\frac{G'}{G})^{m+3} + ... \,. \tag{3.5}$$

Balancing the orders in (3.4) and (3.5), we require that $2(m+1) = m+3$. Thus, $m = 1$, and therefore (3.3) can be rewritten as

$$u(\zeta) = \ a_1(\frac{G'}{G}) + a_0 \ = \ a_1 Y + a_0. \tag{3.6}$$

Now, we substitute equations (2.7) and (2.9) in (3.2) to get the following algebraic system:

$$\begin{aligned}
0 &= -a_1c\mu + a_1\gamma\mu + a_1c\lambda^2\mu + 2a_1c\mu^2 - \frac{1}{2}a_1^2c\alpha\mu^2 - \frac{1}{2}a_1^2c\beta\mu^2,\\
0 &= -a_1c\lambda + a_1\gamma\lambda + a_1c\lambda^3 + 8a_1c\lambda\mu - a_1^2c\alpha\lambda\mu - a_1^2c\beta\lambda\mu,\\
0 &= -a_1c + a_1\gamma + 7a_1c\lambda^2 - \frac{1}{2}a_1^2c\alpha\lambda^2 - \frac{1}{2}a_1^2c\beta\lambda^2 + 8a_1c\mu - a_1^2c\alpha\mu - a_1^2c\beta\mu,\\
0 &= 12a_1c\lambda - a_1^2c\alpha\lambda - a_1^2c\beta\lambda,\\
0 &= 6a_1c - \frac{1}{2}a_1^2c\alpha - \frac{1}{2}a_1^2c\beta.
\end{aligned} \tag{3.7}$$

Solving the above system yields the following cases:

$$\mu = \frac{-c+\gamma+c\lambda^2}{4c}, \quad a_1 = \frac{12}{\alpha+\beta}, \tag{3.8}$$

or

$$\mu = 0, \quad \lambda = \pm\frac{\sqrt{c-\gamma}}{\sqrt{c}}, \quad a_1 = \frac{12}{\alpha+\beta}, \tag{3.9}$$

or

$$\mu = \frac{-c+\gamma}{4c}, \quad a_1 = \frac{12}{\alpha+\beta}, \quad \lambda = 0. \tag{3.10}$$

We should mention here that the above obtained cases yields the same $\triangle$ which is $\triangle = 1 - \frac{\gamma}{c}$ and therefore the solutions of equation (3.1) are

$$u_1(x,t) = a_0 + 6\frac{\left((A\sqrt{1-\frac{\gamma}{c}} - B\lambda) + (B\sqrt{1-\frac{\gamma}{c}} - A\lambda)\tanh(\sqrt{\frac{c-\gamma}{4c}}(x-ct))\right)}{(\alpha+\beta)\left(B + A\ \tanh(\sqrt{\frac{c-\gamma}{4c}}(x-ct))\right)}, \tag{3.11}$$

for the case in (3.8), provided that $\frac{\gamma}{c} < 1$ and $A \neq \pm B$ and $\alpha+\beta \neq 0$. Or

$$u_2(x,t) = a_0 - 6(A-B)\sqrt{1-\frac{\gamma}{c}}\frac{\left(1-\tanh(\sqrt{\frac{c-\gamma}{4c}}(x-ct))\right)}{(\alpha+\beta)\left(B + A\ \tanh(\sqrt{\frac{c-\gamma}{4c}}(x-ct))\right)}, \tag{3.12}$$

for the case in (3.9) with $\lambda > 0$, provided that $\frac{\gamma}{c} < 1$ and $A \neq \pm B$ and $\alpha+\beta \neq 0$. Or

$$u_3(x,t) = a_0 + 6(A+B)\sqrt{1-\frac{\gamma}{c}}\frac{\left(1+\tanh(\sqrt{\frac{c-\gamma}{4c}}(x-ct))\right)}{(\alpha+\beta)\left(B + A\ \tanh(\sqrt{\frac{c-\gamma}{4c}}(x-ct))\right)}, \tag{3.13}$$

for the case in (3.9) with $\lambda < 0$, provided that $\frac{\gamma}{c} < 1$ and $A \neq \pm B$ and $\alpha+\beta \neq 0$. Or

$$u_4(x,t) = a_0 + 6\sqrt{1-\frac{\gamma}{c}}\frac{\left(1+\tanh(\sqrt{\frac{c-\gamma}{4c}}(x-ct))\right)}{(\alpha+\beta)\left(B + A\ \tanh(\sqrt{\frac{c-\gamma}{4c}}(x-ct))\right)}, \tag{3.14}$$

for the case in (3.10), provided that $\frac{\gamma}{c} < 1$ and $A \neq B$ and $\alpha+\beta \neq 0$. The above obtained solutions are of types kink or soliton depend on the choices of A and B.

4. The Whitham-Broer-Kaup model

The Whitham-Broer-Kaup model for dispersive long waves in the shallow water small-amplitude regime is given by

$$\begin{aligned} &u_t + uu_x + v_x + \beta u_{xx} = 0,\\ &v_t - (uv)_x - \beta v_{xx} + \alpha u_{xxx} = 0, \end{aligned} \tag{4.1}$$

where $u = u(x,t)$ is the field of horizontal velocity, $v = v(x,t)$ is the height that deviates from the equilibrium position of the liquid and $\beta,\ \alpha$ are constants which are represented in different diffusion powers. The WBK model (4.1) was derived by Whitham [37] and Broer [12], and was shown to be completely integrable by Kaup [22] and Kupershmidt [23]. The WBK model (4.1) is a very good model to describe dispersive waves. It characterizes the dispersive long waves in the shallow water small-amplitude regime. If $\beta \neq 0,\ \alpha = 0$, then the system represents the classical long wave system that describes a shallow water wave with diffusion. If $\beta = 0,\ \alpha = 1$, then the system represents the variant Boussinesq equation. The WBK equation was investigated via many different approaches. Some of the obtained solutions are given as polynomials of tanh, sech, tan and cot. Other approaches focused on the determination of one soliton solution by using the tanh method, symmetry analysis and the Backlund transformation method. Kaup [22] studied the inverse transformation solution for the WBK equation. Kupershmidt [23] studied the symmetries and conservation laws for this equation. Fan [15] used the Backlund transformation method to obtain three families of solutions for WBK equations, one of which is the family of solitary wave solutions.

By the wave variable $\zeta = x - ct$, the above PDEs are transformed into the following system of ODE

$$\begin{aligned} &-cu' + uu' + v' + \beta u'' = 0 \\ &-cv' + (uv)' - \beta v'' + \alpha u''' = 0. \end{aligned} \tag{4.2}$$

Solving the first equation of (4.2) for the function $v = v(\zeta)$ we get the relation

$$v = cu - \frac{1}{2}u^2 - \beta u'. \tag{4.3}$$

Substitute (4.3) in the second equation of (4.2) yields the following ODE in the function $u = u(\zeta)$

$$(\beta^2 + \alpha)u'' - \frac{1}{2}u^3 + \frac{3}{2}u^2 - c^2 u = 0. \tag{4.4}$$

Balancing the order of the linear term u'' and the nonlinear term u^3 requires the relation $m + 2 = 3m$ and thus $m = 1$. Therefore, the solution of equation (4.4) is

$$u = u(\zeta) = a_1 Y + a_0. \tag{4.5}$$

Now, we need both equations(4.5) and (2.8) to be inserted in (4.4) to obtain the following system.

$$\begin{aligned} 0 &= -\frac{a_0^3}{2} + \frac{3a_0^2 c}{2} - a_0 c^2 + a_1 k\lambda\mu \\ 0 &= -\frac{3a_0 a_1^2}{2} + \frac{3a_1^2 c}{2} + 3a_1 k\lambda \\ 0 &= -\frac{a_1^3}{2} + 2a_1 k \\ 0 &= -\frac{3a_0^2 a_1}{2} + 3a_0 a_1 c - a_1 c^2 + a_1 k\lambda^2 + 2a_1 k\mu, \end{aligned} \tag{4.6}$$

where $k = (\beta^2 + \alpha)$. Solving the above system yields

$$\mu = \frac{a_0(a_0 - 2c)}{4k}, \quad \lambda = \pm\, \frac{a_0 - c}{\sqrt{k}}, \quad a_1 = \pm\, 2\sqrt{k}. \tag{4.7}$$

Substitute (4.7) and (2.10) in (3.6), then the general formula of the function $u(x,t)$ is given by

$$u(x,t)=\frac{c(A+B)\left(1+\tanh\left(\frac{c(x-ct)}{2\sqrt{\beta^2+\alpha}}\right)\right)}{\left(B+A\ \tanh\left(\frac{c(x-ct)}{2\sqrt{\beta^2+\alpha}}\right)\right)}, \tag{4.8}$$

provided that $A\neq\pm B$ and $A,\ B,\ c$ are free parameters. Accordingly, the formula of the function $v(x,t)$ can be obtained by relation (4.3) and it is given by

$$v(x,t)=\frac{c^2(A+B)^2(\sqrt{\beta^2+\alpha}-\beta)}{\left(B\ \cosh\left(\frac{c(x-ct)}{2\sqrt{\beta^2+\alpha}}\right)+A\ \sinh\left(\frac{c(x-ct)}{2\sqrt{\beta^2+\alpha}}\right)\right)^2}. \tag{4.9}$$

Figure 1 and 2 represent two different solution sets of the WBK model based on some selected values of the free parameters.

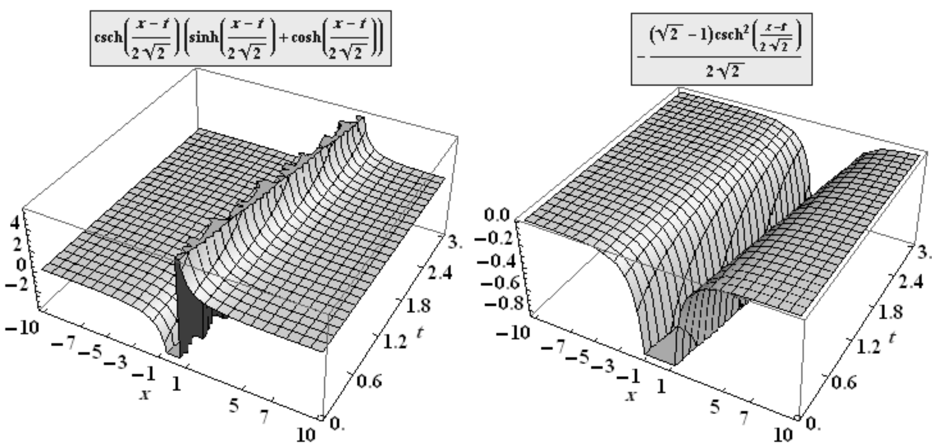

Figure 1: Plot of the first obtained solutions of WBK: $u(x,t)$ on the left and $v(x,t)$ on the right, where $c=1,\ A=1,\ B=0,\ \alpha=1,\ \beta=1$.

5. Conclusion

In summary, the G'/G expansion method with symbolic computation is conducted for a reliable treatment of two models arise in shallow water waves theory. The obtained solutions with free parameters may be important to explain some physical phenomena. The paper shows that the revised algorithm is effective and can be used for many other nonlinear PDEs in mathematical physics.

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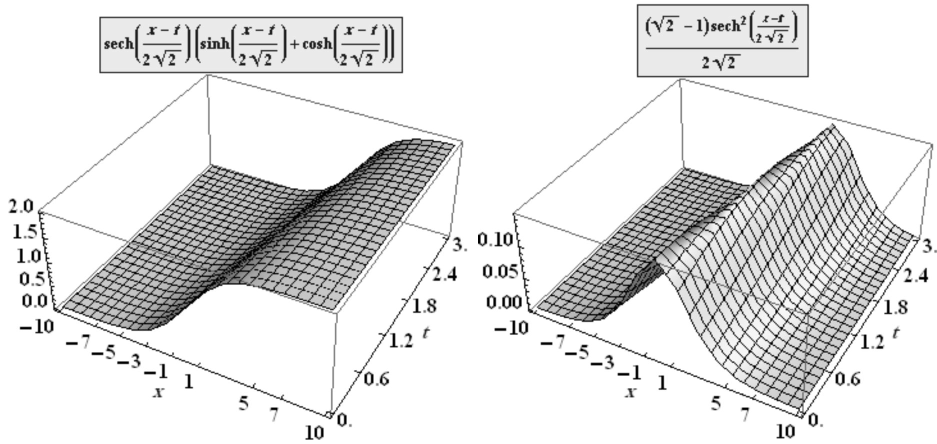

Figure 2: Plot of the second obtained solutions of WBK: $u(x,t)$ on the left and $v(x,t)$ on the right, where $c = 1,\ A = 0,\ B = 1,\ \alpha = 1,\ \beta = 1$.

References

[1] M. Alquran and K. Al-Khaled, *Mathematical methods for a reliable treatment of the (2+1)-dimensional Zoomeron equation*, Mathematical Sciences, 6:11(2012).

[2] M. Alquran and K. Al-Khaled, *Sinc and solitary wave solutions to the generalized Benjamin-Bona-Mahony-Burgers equations,* Physica Scripta, 83 (2011).

[3] M. Alquran and K. Al-Khaled, *The tanh and sine-cosine methods for higher order equations of Korteweg-de Vries type,* Physica Scripta, 84 (2011).

[4] M. Alquran, *Solitons and periodic solutions to nonlinear partial differential equations by the Sine-Cosine method*, Applied Mathematics and Information Sciences *AMIS*, 6(1) (2012), 85-88.

[5] M. Alquran, *Bright and dark soliton solutions to the Ostrovsky-Benjamin-Bona-Mahony (OS-BBM) equation,* J. Math. Comput. Sci., 2(1) (2012), 15-22.

[6] M. Alquran, K. Al-Khaled and H. Ananbeh, *New soliton solutions for systems of nonlinear evolution equations by the rational Sine-Cosine method,* Studies in Mathematical Sciences, 3(1) (2011), 1-9.

[7] M. Alquran, M. Ali and K. Al-Khaled, *Solitary wave solutions to shallow water waves arising in fluid dynamics,* Nonlinear Studies, 19(4) (2012), 555-562.

[8] M. Alquran and A. Qawasmeh, *Classifications of solutions to some generalized nonlinear evolution equations and systems by the sine-cosine method,* Nonlinear Studies, 20(2) (2013), 261-270.

[9] M. Alquran, R. Al-Omary and Q. Katatbeh, *New explicit solutions for homogeneous KdV equations of third order by trigonometric and hyperbolic function methods,* Applications and Applied Mathematics, 7(1) (2012), 211-225.

[10] A. Bekir and O. Unsal, *Exact solutions for a class of nonlinear wave equations by using first integral method*, International Journal of Nonlinear Science, 15(2)(2013), 99-110.

[11] A. Bekir and O. Unsal, *Exact solutions for a class of nonlinear wave equations by using first integral,* International Journal of Nonlinear Science, 15(2) (2013), 99-110.

[12] L.J. Broer, *Approximate equations for long water waves,* Appl. Sci. Res., 31 (1975), 377-395.

[13] F. Chand and A.K. Malik, *Exact traveling wave solutions of some nonlinear equations using G'/G-expansion method,* International Journal of Nonlinear Science, 14(4) (2012), 416-424.

[14] S.A. Elwakil, S.K. El-labany, M.A. Zahran and R. Sabry, *Exact travelling wave solutions for the generalized shallow water wave equation,* Chaos, Solitons and Fractals, 17 (2003), 121-126.

[15] E. Fan and H. Zhang, *Backlund transformation and exact solutions for Whitham-Broer-Kaup equations in shallow water,* Appl. Math. Mech., 19 (1998), 713-716.

[16] H. Gao and R.X. Zhao, *New application of the G'/G-expansion method to high-order nonlinear equations,* Appl Math Computation, 215 (2009), 2781-2786.

[17] Y.T. Gao and B. Tian, *Generalized tanh method with symbolic computation and generalized shallow water wave equation,* Computers Math. Applic., 33 (1997), 115-118.

[18] J. Hietarinta, *Partially Integrable Evolution Equations in Physics,* Dordrecht: Kluwer, 1990.

[19] R. Hirota, *Solitons,* Springer: Berlin, 1980.

[20] H. Jaradat, S. Al-Shara, F. Awawdeh and M. Alquran, *Variable coefficient equations of the Kadomtsev-Petviashvili hierarchy: multiple soliton solutions and singular multiple soliton solutions,* Phys. Scr., 85 (2012) 035001.

[21] H. Kkeiri, M.R. Moghaddam and V. Vafafi, *Application of the G'/G-expansion method for the Burgers, Burgers-Huxley and modified Burgers-KdV equations,* PRAMANA-journal of physics, 76(6) (2011), 831-842.

[22] D.J. Kaup, *A higher-order water-wave equation and the method for solving it,* Prog. Theor. Phys., 54 (1975), 396-408.

[23] B.A. Kupershmidt, *Mathematics of dispersive waves,* Commun. Math. Phys., 99 (1985), 51-73.

[24] X. Li and M. Wang, *A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equation with high order nonlinear terms*, Physics Letters A., 361(2007), 115-118.

[25] X. Li, J. Han and F. Wang, *The extended Riccati equation method for travelling wave solutions of ZK equation,* Journal of Applied Analysis and Computation, 2(4) (2012), 423-430.

[26] X. Liu, W. Zhang and Z. Li, *Application of improved G'/G-expansion method to traveling wave solutions of two nonlinear evolution equations,* Adv. Appl. Math. Mech., 4(1) (2012), 122-130.

[27] I. Mustafa and E. Mahmut, *Periodic wave solutions for the generalized shallow water equation by the improved Jaccobi elliptic function method,* Applied Mathematics E-Notes, 5(2005), 89-96.

[28] K. Nakkeeran, *Bright and dark optical solitons in fiber media with higher order effects*, Chaos, Solitons and Fractals, 13(2002), 673-679.

[29] S.L. Palacios, *Two simple ansatze for obtaining exact solutions of high dispersive nonlinear schrodinger equations*, Chaos, Solitons and Fractals, 19(2004), 203-207.

[30] A. Qawasmeh, *Soliton solutions of (2+1)-Zoomeron equation and duffing equation and SRLW equation,* J. Math. Comput. Sci., 3(6) (2013), 1475-1480.

[31] L. Qiao, H. Fu and S. Tang, *Peakon soliton solutions of* $K(2,-2,4)$ *equation*, Journal of Applied Analysis and Computation, 3(4) (2013), 399-403.

[32] A.M. Wazwaz, *The extended Tanh method for Abundant solitary wave solutions of nonlinear wave equations,* Applied Mathematics and Computation, 187(2007), 1131-1142.

[33] A.M. Wazwaz, *Nonlinear variants of KdV and KP equations with compactons, solitons and periodic solutions,* Communications in Nonlinear Science and Numerical Simulation, 10 (2005), 451-463.

[34] A.M. Wazwaz, *The extended Tanh method for Abundant solitary wave solutions of nonlinear wave equations,* Applied Mathematics and Computation, 187 (2007), 1131-1142.

[35] A.M. Wazwaz, *Multiple soliton solutions for the Whitham-Broer-Kaup model in the shallow water small-amplitude regime,* Phys. Scr., 88 (2013).

[36] G.B. Whitham, *Linear and Nonlinear Waves,* New York: Wiley, 1974.

[37] G.B. Whitham, *Variational methods and applications to water waves,* Proc. R. Soc. Lond. A, 299 (1967), 6-25.

[38] Z. Yan and H.Q. Zhang, *Constructing families of soliton-like solutions to a* (2+1)*-dimensional breaking soliton equation using symbolic computation,* Computers Math. Applic., 44 (2002), 1439-1444.

[39] H. Zhang, *New application of the* G'/G*-expansion method,* Commun Nonlinear Sci Numer Simulat, 14 (2009), 3220-3225.