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OPTIMAL CONTROL OF A FINITE-CAPACITY INVENTORY SYSTEM WITH SETUP COST AND LOST SALES*

Baimei Yang[1], Yiqian Huang[2] and Yifan Xu[2,†]

Abstract One of the most fundamental results in inventory theory is the optimality of (s, S) policy for inventory systems with setup cost. This result is established based on a key assumption of infinite production/ordering capacity. Several studies have shown that, when there is a finite production/ordering capacity, the optimal policy for the inventory system is very complicated and indeed, only partial characterization for the optimal policy is possible. In this paper, we consider a continuous review inventory system with finite production/ordering capacity and setup cost, and show that the optimal control policy for this system has a very simple structure. We also develop efficient algorithms to compute the optimal control parameters.

Keywords Inventory system, finite capacity, poisson demand, Markov decision process, lost sales.

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1. Introduction

One of the most fundamental results in inventory theory is the optimality of (s, S) policy for inventory systems with setup cost (Veinott [14]). A key assumption in this result is *infinite ordering capacity*. That is, regardless of how much is the ordered, the order will be ready after the fixed leadtime. This assumption is clearly not satisfied in many applications, especially in production systems; all production facility has a finite capacity. This problem is closely related to the optimal control of a production/inventory system.

Several studies have been conducted attempting to extend the analysis of production/inventory systems. Gavish and Graves [4] study one-product production/inventory problem with a fixed setup cost under continuous review policy, where the demand for the product is governed by a Poisson process and is backordered when it is out of stock, while the service time is deterministic. De Kok [9] deals with a one-product production/inventory model with lost sales, in which the production rate can be dynamically adjusted to cope with random fluctuations in demand, and derives approximations for the switch-over level. Efforts have also been made to analyze the structure of the optimal production/inventory control policy for the case with multiple products or production facilities. For example, Zheng and Zipkin [16] consider the case when two products competing for a single production

†the corresponding author. Email address: yfxu@fudan.edu.cn(Y. Xu)
[1]School of Business, Shanghai Dianji University, Shanghai, 201306, China
[2]School of Management, Fudan University, Shanghai, 200433, China
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facility, which can produce only one unit of either product at a time. They compared the first-come first-served(FCFS) discipline and the longest queue(LQ) discipline and demonstrated that the LQ discipline performs better in a certain sense than the FCFS discipline. Wein [15], Ha [6], Perez and Zipkin [12] investigate the production-inventory system with a single production facility to process a variety of different job classes, satisfy several demand classes or manufacture several products separately. Benjaafar etc. [1] study the problem of allocating demand in multiple products with multiple production facilities. They consider two types of demand allocation and two forms of inventory warehousing. At the same time, they highlight the effect of these various factors on demand allocation and inventory control decisions. Zhu etc. [18] investigate a problem of optimizing multi-period centralized production and inventory system with waste disposal subjected to uncertain demands. However, most of them consider the case without setup cost.

The inventory control model studied in this paper is a lost sale model. Inventory system with positive leadtimes and lost-sales is known to be a notorious difficult problem to analyze, and has gained momentum in the last several years. In the case the setup cost is 0, Karlin and Scarf [8]prove that when the lead time is one period, the optimal policy is *not* an order-up-to policy and that the optimal order quantity in any period is a decreasing function of the inventory on hand in that period and that the rate of decrease is less than 1. Morton [10] generalizes this result to an arbitrary lead time and shows that the optimal order quantity is a decreasing function of the *state vector*, i.e. a vector consisting of the amount of inventory on hand and the vector of quantities ordered in the past which have still not been delivered; moreover, the rate of decrease with respect to more recent orders is higher. Recently, Zipkin [19] has used the concept of *L-convexity* to provide an elegant proof for stronger versions of these results. Zipkin [20] reports a computational investigation on the performance of some of these heuristics. Janakiraman and Roundy [7] study lost sales inventory models with stochastic lead times with no order crossing. By focusing on the class of order-up-to policies (that are known to be sub-optimal), they show that the cost incurred by this system is a convex function of the order-up-to level used. Our paper also uses the operator algorithm in differential equation to make a further analysis. The results concerning this area can be found in works of Zhou [17], Pan and Zhang [11], Guo [5], Cheng and Ren [2].

In this paper, we consider a continuous review inventory control system with setup cost and lost sales. The production capacity is finite, for the production facility can only manufacture products one by one, at a given rate. The production times are random and are assumed to be exponentially distributed. Demand follows a Poisson process and any demand that arrives during a stockout period is lost. Each product sold generates a revenue. There is a production cost and a holding cost. There is also a setup cost every time the machine is set up to produce, but once the machine is set up, it can produce any number of products non-stop before turning off the machine or switching the machine for other tasks. This makes the model different from that of Gallego and Toktay [3], where a setup incurs in every period the machine produces. Our objective is to dynamically control the production process to maximize the long run average profit. Though this paper focuses on lost sale model, we point out that similar results can be obtained for the case with backlogs.

The main result in this paper is that the optimal policy is determined by two easily computable parameters r and S. Whenever the inventory drops to r, the machine is turned on to produce at the maximum rate, and the machine is turned off when the inventory level reaches S. The control parameters r and S are the solutions of a simple concave function $g(x)$. Various characterizations are presented on this stochastic inventory system. Our analysis is divided into two parts. In the first part, we focus on a subclass of admissions policies, that is the class of (r, S) policies, and identify, among this class of policies, the optimal one. Clearly, this problem is reduced to the search for the two optimal control parameters. Then, we prove that the policy obtained is actually optimal among all admissible policies.

The rest of this paper is organized as follows. Section 2 presents the problem formulation and outlines the main results of this paper. Section 3 analyzes the class of (r, S) policies and the associated auxiliary function, and two algorithms are developed to compute the optimal control parameters. Section 4 proves the global optimality of the policy obtained in Section 3 among all admissible policies. Finally, the paper concludes with a discussion in Section 5.

2. Problem Formulation and Main Results

Consider a make-to-stock production system for a single product with selling price p. The demand for the product follows a Poisson process with mean rate $\lambda > 0$, and each demand is for one unit of the product. The unit production cost is c, where $c < p$, and the production time is random which is exponentially distributed with mean $1/\mu$. Turning on the production requires a setup cost K. As indicated in the introduction, we only consider the case of lost-sales but point out that the analysis and results can be extended to the backlog case. The finished goods inventory has a holding cost b per product per unit of time. The goal is to dynamically adjust the production process to maximize the long run average profit.

Without loss of generality, we can redefine the time unit to assume $\lambda + \mu = 1$. We specify a dynamic control policy for the system by a production rate function $\mu(t) = \mu$ or 0 for all time t, where $\mu(t) = \mu$ represents that the machine is on and $\mu(t) = 0$ off. Let $u = \{\mu(t) : t > 0\}$, which is said to be *non-anticipatory* if $\mu(t)$ depends only on information up to t. Let $\mathcal{U}$ be the set of all such non-anticipatory control policies. Under a given policy $u \in \mathcal{U}$, let $N^u(t)$ be the cumulative total demand sold up to time t, $P^u(t)$ the total production by time $t, M^u(t)$ be the number of setups by time time t, and $x^u(t)$ be the finished goods inventory level at time t. Then, starting from an initial inventory level x, the long run average profit for a policy $u \in \mathcal{U}$ is defined to be

$$J^u(x) = \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T \Big(pdN^u(t) - cdP^u(t) - KdM^u(t) - bx^u(t)dt \Big) \Big| x^u(0) = x \right]. \tag{2.1}$$

A policy $u^* \in \mathcal{U}$ is said *optimal* if it maximizes the above long run average profit, i.e., it solves the following optimization problem for all x:

$$J^{u^*}(x) = \sup_{u\in\mathcal{U}} J^u(x). \tag{2.2}$$

In view of Puterman [13], the long-run average profit criterion for this Markovian decision process can be described by optimality equations. Let $(x(t), i(t))$ be the state of the system at time t, where $x(t)$ is the inventory level at time t and $i(t) = 0$ or 1 indicating whether production is off or on at time t. The production rate $\mu(t)$

is controllable with action space $\{0, \mu\}$. When the state is $(x, 0)$, action $\mu(t) = \mu$ turns the machine on with a setup cost K and the state is switched to $(x, 1)$, while $\mu(t) = 0$ keeps the machine in off state. Similarly, when the state is $(x, 1)$, action $\mu(t) = \mu$ means the machine continues to be on, while $\mu(t) = 0$ turns off the machine and the state is switched to $(x, 0)$. Hence, the optimality equations for the long-run average profit criterion are that, if there exist functions $J(x, i)$, $i = 0, 1$ and a number γ^* such that they satisfy, for state $(x, 0)$ with $x > 0$,

$$\begin{aligned} 0 = &-\gamma^* - bx + \max\{\lambda(J(x-1,0) - J(x,0) + p), \\ &\lambda(J(x-1,1) - J(x,0) + p - K) + \mu(J(x+1,1) - J(x,0) - c - K)\} \end{aligned} \tag{2.3}$$

and with $x = 0$,

$$0 = -\gamma^* + \max\left\{J(0,0), \lambda(J(0,1) - J(0,0) - K) + \mu(J(1,1) - J(0,0) - c - K)\right\}, \tag{2.4}$$

and for state $(x, 1)$ with $x > 0$,

$$\begin{aligned} 0 = &-\gamma^* - bx + \max\{\lambda(J(x-1,0) - J(x,1) + p) + \mu(J(x,0) - J(x,1)), \\ &\lambda(J(x-1,1) - J(x,1) + p) + \mu(J(x+1,1) - J(x,1) - c)\} \end{aligned} \tag{2.5}$$

and with $x = 0$,

$$0 = -\gamma^* + \max\{\mu(J(0,0) - J(0,1)), \mu(J(1,1) - J(0,1) - c)\} \tag{2.6}$$

then the production rate that optimizes the right-hand sides of (2.3) to (2.6) is an optimal policy, and in addition, γ^* is the optimal long-run average profit.

We shall only offer an explanation to (2.3). Due to the exponential service times, the continuous time semi-Markov decision process is transformed to a discrete time Markov decision process with time intervals being exponentially distributed with rate $\lambda + \mu = 1$. At state $(x, 0)$ with $x > 0$, the machine is off and the decision can be either to continue keep the machine off, or to set up the machine at a fixed cost K and start to produce. The state of the system at the next decision epoch depends on this decision. If the machine is kept off, the state at the next decision epoch is either $(x-1, 0)$ with a revenue p, which happens with probability λ which is due to arrival, or remains at $(x, 0)$, which happens with probability μ since the machine is off. On the other hand, if the machine is turned on, it first incurs a fixed cost K, then the state of the system at the next decision epoch will be either $(x-1, 1)$ with probability λ, together with a revenue p, or $(x+1, 1)$ with probability μ which represents a production completion that incurs a production cost c. Therefore, the average profit optimality equation is

$$\begin{aligned} J(x,0) = &-\gamma^* - bx + \max\{\lambda(J(x-1,0) + p) + \mu J(x,0), \\ &\lambda(J(x-1,1) - J(x,0) + p) + \mu(J(x+1,1) - J(x,0) - c) - K\} \end{aligned}$$

Moving $J(x, 0)$ to the right hand side we obtain (2.3).

In this paper, we obtain the optimal inventory control strategy and show that it is determined by two threshold parameters, r and S. We also present efficient computational algorithms for r and S. The main results are summarized in the following theorems.

Theorem 2.1. *The optimal production control policy is determined by two thresholds: r, S with $0 \le r < S$. An (r, S) policy manages the finished goods inventory in the following manner: Whenever the inventory level reaches S, the machine is turned off, and whenever the inventory drops to or below r, the machine is set up to produce until the inventory level reaches S.*

Theorem 2.2. *The optimal control parameters r and S are determined by the following concave function:*

$$g(x) = \begin{cases} \dfrac{1}{\beta(\mu-\lambda)}\Big(-\gamma + \lambda\alpha - b(x+1) + (\gamma - \mu\alpha)\beta^{x+2}\Big), & \lambda \neq \mu, \\ -\dfrac{\gamma}{\lambda} - \dfrac{\gamma - \lambda(p-c)}{\lambda}(x+1) - \dfrac{b}{2\lambda}(x+2)(x+1), & \lambda = \mu, \end{cases}$$

where $\alpha = p - c + b/(\mu - \lambda)$, $\beta = \lambda/\mu$, and γ is the optimal average profit to be calculated. The optimal control parameter r is the smallest nonnegative number such that $g(x) \ge 0$, while S is the smallest number greater than r such that $g(x) < 0$.

The optimal average profit γ is determined by

$$\sum_{x=r}^{S-1} g(x) = K.$$

Remark 2.1. Theorem 2.2 relies on the concavity of function $g(x)$ in x. This is clearly true for the case $\lambda = \mu$. For $\lambda \neq \mu$, $g(x)$ has three terms: a constant term, a linear term, and an exponential term. To see whether $g(x)$ is concave in x we only need to verify the coefficient of the exponential term, which is

$$\frac{\gamma - \mu\alpha}{\beta(\mu - \lambda)} \tag{2.7}$$

to be non-positive (note that λ can be greater or smaller than μ). It will be shown in the next section that in the range of interest of γ, (2.7) is always non-positive. See Proposition 3.4. Due to the quadratic form of g, r and S can be given in closed form for the case $\lambda = \mu$, and they are respectively given by

$$r = \left\lceil \frac{\lambda(p-c) - \gamma - b/2 - \sqrt{\lambda(p-c) - \gamma - b/2)^2 - 2b\gamma}}{b} - 1 \right\rceil^+,$$

$$S = \left\lceil \frac{\lambda(p-c) - \gamma - b/2 + \sqrt{\lambda(p-c) - \gamma - b/2)^2 - 2b\gamma}}{b} - 1 \right\rceil^+,$$

where $\lceil a \rceil^+$ is the smallest nonnegative integer greater than or equal to a.

In this paper we take the following approach. First, we identify a policy that maximizes the average profit within the class of (r, S) policies. Then, we prove that this policy is actually optimal among all admissible policies. For convenience, we refer to the *optimal policy within the class of* (r, S) policies the *optimal threshold policy.* Therefore, our agenda is to first find the optimal threshold policy, and then prove that the optimal threshold policy is actually the optimal policy among all policies. In the following section, we focus on the class of (r, S) policies and study the properties of this threshold policy. In Section 4, we prove that the policy obtained in Section 3 is optimal among all admissible policies.

3. Analysis of Value Function

3.1. Auxiliary Function

In this subsection, we analyze the class of (r, S) policies such that $0 \le r < S$. Under an (r, S) policy, whenever the inventory level reaches S, the machine is turned off and the state becomes $(S, 0)$. Demand arrives and gradually depletes the stock level. If the system starts from state $(x, 1)$ with $x \ge S$, then the system is turned off immediately, hence the state changes instantly from $(x, 1)$ to $(x, 0)$. When the inventory level drops to r, the machine is turned on, at a setup cost K, and production starts. If the system starts from a state $(x, 0)$ with $x \le r$, the machine is turned on immediately, with a setup cost K, and production process starts right away, thus state $(x, 0)$ is switched to $(x, 1)$ for $x \le r$. When the production is on, the inventory level will reach S after a random amount of time*, then the machine is turned off and the state is $(S, 0)$ again and a new cycle starts. Therefore, the process $(X(t), I(t))$ evolves as a Markov renewal process with renewal point $(S, 0)$.

Let $W(x, i)$ and $T(x, i)$ denote the expected remaining cumulative profit and expected remaining time until the cycle ends when state is currently (x, i). From the definition of the (r, S) policy, the machine is turned off as soon as the inventory level reaches S, thus

$$W(S, 1) = 0. \tag{3.1}$$

However, when starting from state $(x, 1)$, $0 < x < S$, we need to calculate $W(x, 1)$ by conditioning on which state the process will reach after exponentially distributed amount of time with mean 1. Thus

$$W(x, 1) = -bx + \lambda(W(x-1, 1) + p) + \mu(W(x+1, 1) - c), \quad 0 < x < S, \tag{3.2}$$

$$W(0, 1) = \lambda W(0, 1) + \mu(W(1, 1) - c). \tag{3.3}$$

Similarly, when the state of the system is $(x, 0)$, we have

$$W(x, 0) = -bx + \lambda(W(x-1, 0) + p) + \mu W(x, 0), \qquad x > r, \tag{3.4}$$

$$W(x, 0) = W(x, 1) - K, \qquad x \le r, \tag{3.5}$$

where the last equation follows from the fact that, in state $(x, 0)$ with $x \le r$, the machine is turned on immediately.

The expected time until the end of the cycle, $T(x, i)$, can be similarly derived.

$$T(S, 1) = 0, \tag{3.6}$$

$$T(x, 1) = 1 + \lambda T(x-1, 1) + \mu T(x+1, 1), \qquad 0 < x < S, \tag{3.7}$$

$$T(0, 1) = 1 + \lambda T(0, 1) + \mu T(1, 1); \tag{3.8}$$

$$T(x, 0) = 1 + \lambda T(x-1, 0) + \mu T(x, 0), \qquad x > r \tag{3.9}$$

$$T(x, 0) = T(x, 1), \qquad x \le r. \tag{3.10}$$

For a given γ, called *dummy profit parameter*, we define an *auxiliary value function* as

$$l_\gamma(x, i) = W(x, i) - \gamma T(x, i).$$

*This is true even if $\mu \le \lambda$ due to the lost-sale assumption, but for backlog models, we will need $\lambda < \mu$.

We often drop the subscript γ for convenience.

Applying (3.1) to (3.10), the recursive equations for $l_\gamma(x,i)$ can be derived as follows:

$$l_\gamma(S,1)=0; \tag{3.11}$$

for $x>r$,

$$0=-\gamma-bx+\lambda[l_\gamma(x-1,0)-l_\gamma(x,0)+p]; \tag{3.12}$$

for $0<x<S$,

$$0=-\gamma-bx+\lambda[l_\gamma(x-1,1)-l_\gamma(x,1)+p]+\mu[l_\gamma(x+1,1)-l_\gamma(x,1)-c]; \tag{3.13}$$

and

$$0=-\gamma+\mu[l_\gamma(1,1)-l_\gamma(0,1)-c]. \tag{3.14}$$

Finally, the auxiliary value function satisfies

$$l_\gamma(x,1)=l_\gamma(x,0), \qquad x\geq S, \tag{3.15}$$

$$l_\gamma(x,0)=l_\gamma(x,1)-K, \qquad 0\leq x\leq r. \tag{3.16}$$

To show the dependency of the auxiliary function on the policy parameters r and S, we sometimes write it as $l_\gamma^{r,S}(x,i)$ or simply $l^{r,S}(x,i)$.

We first present the following simple result.

Lemma 3.1. *A policy (r^*,S^*) is the optimal threshold policy with maximum average profit γ^* if and only if*

$$\max_{(r,S)} l_{\gamma^*}^{r,S}(S,0)=l_{\gamma^*}^{r^*,S^*}(S^*,0)=0. \tag{3.17}$$

Proof. If (3.17) is true, then

$$l_{\gamma^*}^{r^*,S^*}(S^*,0)=W(S^*,0)-\gamma^*T(S^*,0)=0,$$

and for any (r,S),

$$l_{\gamma^*}^{r,S}(S,0)=W(S,0)-\gamma^*T(S,0)\leq 0.$$

Thus

$$\frac{W(S,0)}{T(S,0)}\leq\gamma^*=\frac{W(S^*,0)}{T(S^*,0)}.$$

This shows that (r^*,S^*) is the optimal threshold policy.

On the other hand, if (r^*,S^*) is the optimal threshold policy, let $\gamma^*=\frac{W(S^*,0)}{T(S^*,0)}$. We show that (r^*,S^*) maximizes $l_{\gamma^*}^{r,S}(S,0)$. If this is not true, then for some (r,S) we have

$$l_{\gamma^*}^{r,S}(S,0)>l_{\gamma^*}^{r^*,S^*}(S^*,0)=0,$$

then

$$W(S,0)-\gamma^*T(S,0)>0$$

and

$$\frac{W(S,0)}{T(S,0)}>\gamma^*.$$

This shows that the average profit for (r, S) is higher than that of optimal policy (r^*, S^*), a contradiction. □

Lemma 3.1 suggests that, if we can find a γ^* such that the maximum value of (3.17) is 0, then the corresponding r and S is the optimal threshold policy. On the other hand, if

$$\max_{r,S} l_{\gamma^*}^{r,S} \neq 0,$$

then by successively modifying the dummy profit parameter, we can develop a computational procedure for obtaining the optimal threshold parameters.

3.2. Optimizing (r, S) Policies through Optimizing Auxiliary Function

In this subsection, we establish some properties that the optimal thresholds (r^*, S^*) must satisfy. We denote by $l^{r,S}(x, i)$ the auxiliary value function corresponding to (r, S) policy and for convenience, define two difference operators as follows: For $i = 0, 1$ and $x \geq 0$,

$$\Delta_i^r(x) = l^{r+1,S}(x, i) - l^{r,S}(x, i),$$
$$\Delta_i^S(x) = l^{r,S+1}(x, i) - l^{r,S}(x, i).$$

For a given dummy profit parameter γ, an (r, S) policy is said to *be better than* (r', S') policy if

$$l^{r,S}(S, 0) \geq l^{r',S'}(S', 0).$$

The following result will be useful in characterizing the optimal control parameters.

Lemma 3.2. *The following properties are satisfied by any given policy* (r, S)*:*

$$\Delta_1^S(x) = \Delta_1^S(S), \qquad x < S, \tag{3.18}$$
$$\Delta_1^r(x) = 0, \qquad x < S, \tag{3.19}$$
$$\Delta_0^S(x) = \Delta_0^S(r), \qquad x \geq r, \tag{3.20}$$
$$\Delta_0^S(r) = \Delta_1^S(r), \tag{3.21}$$
$$\Delta_0^r(x) = \Delta_0^r(r+1), \qquad x \geq r+1. \tag{3.22}$$

Proof. Both of the qualities follow from (3.12) and (3.14). Here we offer a proof based on the sample path argument. At any state $(x, 0)$ with $x > r$, under both policies (r, S) and $(r, S+1)$ the inventory level can only drop and that is due to demand arrivals. This occurs until the inventory level drops to r, and under both policies the machine is turned on to produce at rate μ and the state is changed to $(r, 1)$ at a cost K. Hence the difference of auxiliary value functions for polices (r, S) and $(r, S+1)$ starting at $(x, 0)$ is the same as that starting from $(r, 1)$. This proves (3.18). For (3.19), note that starting from state $(x, 1)$, the machine is currently producing, and independent of r, the production will not stop until the inventory level reaches S. Thus the sample paths for the two systems operating under (r, S) and $(r+1, S)$, when the starting state is $(x, 1)$ with $x < S$, are same until the end of the cycle. This proves (3.19). To prove (3.20), note that when starting from state $(x, 0)$ with $x > r$, the system operating under (r, S) and $(r, S+1)$ remains the same until the inventory level drops to r, at which point both systems are turned to produce at rate μ. Hence (3.20) follows. The arguments to prove (3.21) and (3.22) are similar. □

To present criteria for finding the optimal S and r, we need the following lemma. In the subsequently analysis we shall only present the results for the case $\lambda \neq \mu$. The corresponding results for the case $\lambda = \mu$ can be similarly derived.

Lemma 3.3. *Given a policy (r, S) and a dummy parameter γ, we have, for all $0 < x < S$,*

$$\begin{aligned} & l^{r,S}(x+1,1) - l^{r,S}(x,1) \\ =& \frac{\gamma + \mu c - \lambda p + bx}{\mu - \lambda} - \frac{b\lambda}{(\mu-\lambda)^2} - \left(\frac{\gamma - \mu(p-c)}{\mu - \lambda} - \frac{\mu b}{(\mu-\lambda)^2}\right)\left(\frac{\lambda}{\mu}\right)^{x+1}. \end{aligned} \tag{3.23}$$

In particular,

$$\begin{aligned} & l^{r,S}(S-1,1) \\ =& -\frac{\gamma + \mu c - \lambda p + b(S-1)}{\mu - \lambda} + \frac{b\lambda}{(\mu-\lambda)^2} + \left(\frac{\gamma - \mu(p-c)}{\mu - \lambda} - \frac{\mu b}{(\mu-\lambda)^2}\right)\left(\frac{\lambda}{\mu}\right)^{S}. \end{aligned} \tag{3.24}$$

Proof. It follows from successive applications of (3.13) and (3.14) that

$$\begin{aligned} & l^{r,S}(x+1,1) - l^{r,S}(x,1) \\ =& \frac{\gamma + bx + \mu c - \lambda p}{\mu} + \frac{\lambda}{\mu}\left[l^{r,S}(x,1) - l^{r,S}(x-1,1)\right] \\ =& \sum_{i=0}^{x-1} \frac{\gamma + \mu c - \lambda p + b(x-i)}{\mu}\left(\frac{\lambda}{\mu}\right)^i + \left(\frac{\lambda}{\mu}\right)^x (l^{r,S}(1,1) - l^{r,S}(0,1)) \\ =& \frac{\gamma + \mu c - \lambda p}{\mu}\sum_{i=0}^{x-1}\left(\frac{\lambda}{\mu}\right)^i + \sum_{i=0}^{x-1}\left(\frac{\lambda}{\mu}\right)^i \frac{b(x-i)}{\mu} + \frac{\gamma + ac}{\mu}\left(\frac{\lambda}{\mu}\right)^x \\ =& \frac{\gamma + \mu c - \lambda p + bx}{\mu - \lambda} - \frac{b\lambda}{(\mu-\lambda)^2} - \left(\frac{\gamma - \mu(p-c)}{\mu - \lambda} - \frac{\mu b}{(\mu-\lambda)^2}\right)\left(\frac{\lambda}{\mu}\right)^{x+1}. \end{aligned}$$

This proves (3.23). The second equation (4.15) follows from (3.23) and the observation that $l^{r,S}(S,1) = 0$. The proof is complete. □

The following useful relationship will be used repeatedly in the subsequent analysis, and it follows from the Lemma 3.3 and the definition of $g(x)$:

$$g(x) = l^{r,S}(x,1) - l^{r,S}(x+1,1) - \frac{\gamma + bx - \lambda p}{\lambda}. \tag{3.25}$$

Proposition 3.1. *For given γ, policy $(r, S+1)$ is better than policy (r, S) if and only if $g(S) \geq 0$.*

Proof. We need to prove that $l^{r,S+1}(S+1,0) \geq l^{r,S}(S,0)$ if and only if $g(S) \geq 0$. From (3.12), for $(r, S+1)$ policy,

$$0 = -\gamma - b(S+1) + \lambda p + \lambda[l^{r,S+1}(S,0) - l^{r,S+1}(S+1,0)].$$

Hence

$$\begin{aligned}
&l^{r,S+1}(S+1,0)-l^{r,S}(S,0)\\
=&[l^{r,S+1}(S+1,0)-l^{r,S+1}(S,0)]+[l^{r,S+1}(S,0)-l^{r,S}(S,0)]\\
=&\frac{-\gamma-b(S+1)+\lambda p}{\lambda}+\Delta_0^S(S)\\
=&\frac{-\gamma-b(S+1)+\lambda p}{\lambda}+\Delta_1^S(r),
\end{aligned}\tag{3.26}$$

where the last equality follows from (3.20) and (3.21) of Lemma 3.2.

To determine $\Delta_1^S(r)$, applying (3.18) of Lemma 3.2, we have

$$\Delta_1^S(r)=\Delta_1^S(S)=l^{r,S+1}(S,1),\tag{3.27}$$

where we have used the fact that $l^{r,S}(S,1)=0$. Applying (4.15) of Lemma 3.3 for policy $(r,S+1)$, we obtain

$$\begin{aligned}
l^{r,S+1}(S,1)=&\frac{-\gamma-\mu c+\lambda p-bS}{\mu-\lambda}+\frac{b\lambda}{(\mu-\lambda)^2}\\
&+\left(\frac{\gamma-\mu(p-c)}{\mu-\lambda}-\frac{\mu b}{(\mu-\lambda)^2}\right)\left(\frac{\lambda}{\mu}\right)^{S+1}.
\end{aligned}\tag{3.28}$$

Combining (3.26), (3.27) and (3.28), yields

$$\begin{aligned}
&l^{r,S+1}(S+1,0)-l^{r,S}(S,0)\\
=&\frac{-\gamma-b(S+1)+\lambda p}{\lambda}+\frac{-\gamma-\mu c+\lambda p-bS}{\mu-\lambda}+\frac{b\lambda}{(\mu-\lambda)^2}\\
&+\left(\frac{\gamma-\mu(p-c)}{\mu-\lambda}-\frac{\mu b}{(\mu-\lambda)^2}\right)\left(\frac{\lambda}{\mu}\right)^{S+1}\\
=&\frac{\mu}{\lambda(\mu-\lambda)}\left[-\gamma-b(S+1)+\lambda(p-c)+\frac{\lambda b}{\mu-\lambda}+\left(\gamma-\mu(p-c)-\frac{\mu b}{\mu-\lambda}\right)\left(\frac{\lambda}{\mu}\right)^{S+2}\right]\\
=&g(S).
\end{aligned}$$

Thus $l^{r,S+1}(S+1,0)\geq l^{r,S}(S,0)$ if and only if $g(S)\geq 0$. □

The next proposition presents a criterion for the optimal parameter r.

Proposition 3.2. *For given γ, policy $(r+1,S)$ is better than policy (r,S) if and only if $g(r)\leq 0$.*

Proof. We need to prove that $\Delta_0^r(S)=l^{r+1,S}(S,0)-l^{r,S}(S,0)\geq 0$ if and only if $g(r)\leq 0$. Applying (3.22) of Lemma 3.2 and (3.12), we have

$$\begin{aligned}
\Delta_0^r(S)&=\Delta_0^r(r+1)\\
&=l^{r+1,S}(r+1,0)-l^{r,S}(r+1,0)\\
&=l^{r+1,S}(r+1,0)-l^{r,S}(r,0)+\frac{\gamma+b(r+1)-\lambda p}{\lambda}\\
&=l^{r+1,S}(r+1,1)-l^{r,S}(r,1)+\frac{\gamma+b(r+1)-\lambda p}{\lambda}\\
&=\Delta_1^r(r+1)+[l^{r,S}(r+1,1)-l^{r,S}(r,1)]+\frac{\gamma+b(r+1)-\lambda p}{\lambda},
\end{aligned}\tag{3.29}$$

where the fourth equality follows from $l^{r,S}(r,0) = l^{r,S}(r,1) - K$ and $l^{r+1,S}(r+1,0) = l^{r,S}(r+1,1) - K$.

By (3.19) of Lemma 3.2 we have $\Delta_1^r(r+1) = 0$. By Lemma 3.3 we have

$$\begin{aligned} & l^{r,S}(r+1,1) - l^{r,S}(r,1) \\ =& \frac{\gamma + \mu c - \lambda p + br}{\mu - \lambda} - \frac{b\lambda}{(\mu-\lambda)^2} - \left(\frac{\gamma - \mu(p-c)}{\mu - \lambda} - \frac{\mu b}{(\mu-\lambda)^2}\right)\left(\frac{\lambda}{\mu}\right)^{r+1} \end{aligned} \tag{3.30}$$

Substituting (3.30) into (3.29) we obtain

$$\begin{aligned} \Delta^r(S,0) =& \frac{\gamma + b(r+1) - \lambda p}{\lambda} + \frac{\gamma + \mu c - \lambda p + br}{\mu - \lambda} - \frac{b\lambda}{(\mu-\lambda)^2} \\ & - \left(\frac{\gamma - \mu(p-c)}{\mu - \lambda} - \frac{\mu b}{(\mu-\lambda)^2}\right)\left(\frac{\lambda}{\mu}\right)^{r+1} \\ =& -g(r). \end{aligned}$$

This completes the proof of Proposition 3.2. □

The propositions above present criteria for searching for candidate policy parameters r and S for given γ. Will such r and S always exist? To answer this question, first recall that our goal is to find the optimal γ such that, with the corresponding optimal r_γ and S_γ, we have $l_\gamma^{r_\gamma,S_\gamma}(S_\gamma,0) = 0$. Hence in the search for the optimal γ, we first reduce the search space for γ by identifying upper and lower bounds for the optimal γ.

Proposition 3.3. *If $\lambda \le \mu$, then an upper bound for the optimal γ is*

$$\gamma_{\max} = \lambda(p-c),$$

and a lower bound for the optimal profit is

$$\gamma_{\min} = \frac{\mu(\lambda(p-c) - b)}{\lambda + \mu}.$$

If $\lambda > \mu$, then an upper bound for the optimal γ is

$$\gamma_{\max} = \mu(p-c),$$

and a lower bound for the optimal profit is

$$\gamma_{\min} = \mu(p-c) + \frac{b\mu}{\mu - \lambda}.$$

Proof. We first consider $\lambda \le \mu$. The average number of customers satisfied per period is bounded from above by the average number of arrivals per period, which is λ. Each satisfied job generates a revenue of p but costs c. Hence the upper bound follows.

To find a lower bound, we consider the policy with $r = 0$ and $S = 1$. Under this policy, the inventory level takes two values, 0 and 1, and it stays at each level for exponentially distributed amount of time, with mean $1/\lambda$ and $1/\mu$ respectively. The stationary probability for the inventory to be 1 is $\mu/(\lambda+\mu)$, thus the average revenue rate is $(p-c)\lambda\mu/(\lambda+\mu)$. On the other hand, the average number of inventory is also

$\mu/(\lambda+\mu)$ which gives an average holding cost rate $\mu b/(\lambda+\mu)$. Thus, the average profit under this policy is

$$(p-c)\frac{\lambda\mu}{\lambda+\mu} - b\frac{\mu}{\lambda+\mu} = \frac{\mu(\lambda(p-c)-b)}{\lambda+\mu},$$

which is a lower bound for optimal profit.

We then consider the case $\lambda > \mu$. Since the number of products produced per period is bounded from above by μ, the number of customers satisfied is also bounded by μ. Thus the maximum profit generated cannot be over $\mu(p-c)$. To find a lower bound, consider the policy that processes at all times. That is, never turn off the machine. The inventory level in the system under this policy is equivalent to an $M/M/1$ queue with arrival rate μ and service rate λ. The average inventory level is therefore $\mu/(\lambda-\mu)$, with an average holding cost $b\mu/(\lambda-\mu)$. The average number of customer satisfied per period is μ generating a profit $\mu(p-c)$. Hence a lower bound for the optimal profit is $\mu(p-c) - b\mu/(\lambda-\mu)$. □

Relationship (3.25) can be shown to be satisfied for both $\lambda \neq \mu$ and $\lambda = \mu$. For the latter case, by successive application of (3.25) and (3.13), we obtain by noting (3.14) that

$$\begin{aligned} g(x) &= l^{r,S}(x,1) - l^{r,S}(x+1,1) - \frac{\gamma + bx - \lambda p}{\lambda} \\ &= l(0,1) - l(1,1) - \sum_{i=1}^{x} \frac{\gamma + bi - \lambda(p-c)}{\lambda} - \frac{\gamma + bx - \lambda p}{\lambda} \\ &= -\frac{\gamma}{\lambda} - \frac{\gamma - \lambda(p-c)}{\lambda}(x+1) - \frac{b}{2\lambda}(x+2)(x+1). \end{aligned} \tag{3.31}$$

Proposition 3.4. *When $\gamma_{\min} \le \gamma \le \gamma_{\max}$, the function $g(x)$ is concave in x.*

Proof. The case $\lambda = \mu$ is easily seen from the quadratic expression (3.31) of $g(x)$. When $\lambda \neq \mu$, it suffices to show that the coefficient of the exponential term of $g(x)$ is nonpositive, or equivalently $(\gamma - \mu\alpha)/(\mu - \lambda) \le 0$.

First suppose $\lambda < \mu$. In this case it follows from the upper bound of γ that

$$\gamma \le \mu(p-c) \le \mu\Big(p - c + \frac{b}{\mu - \lambda}\Big) = \mu\alpha.$$

Hence $(\gamma - \mu\alpha)/(\mu - \lambda) \le 0$ when $\mu > \lambda$.

Then, consider the case $\lambda > \mu$. In this case it follows from the lower bound for γ that

$$\gamma \ge \mu\Big(p - c + \frac{b}{\mu - \lambda}\Big) = \mu\alpha.$$

Thus $(\gamma - \mu\alpha)/(\mu - \lambda) \le 0$ is also true when $\mu < \lambda$. □

Propositions 3.1 and 3.2 present criteria for the optimal parameters r and S. We will only be interested in γ with $g(x) \ge 0$ for at least some nonnegative x. Let y^*_γ be the maximizer of concave function $g(x)$ on $x \ge 0$, and define

$$S_\gamma = \min\{x \ge 0 : g(x) < 0\}, \tag{3.32}$$

$$r_\gamma = \min\{0 \le x < S : g(x) \ge 0\}. \tag{3.33}$$

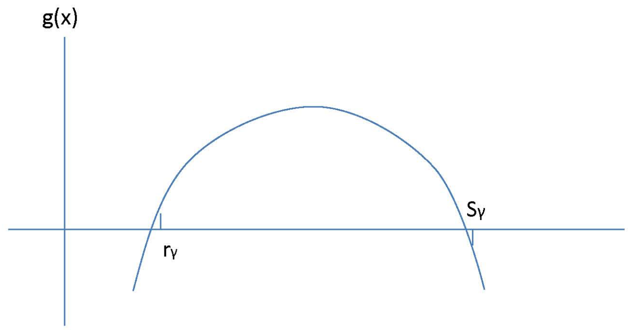

Figure 1. Illustration of $g(x)$, r_γ and S_γ

Proposition 3.5. *On the range* $\gamma_{\min} \le \gamma \le \gamma_{\max}$, r_γ *is increasing in* γ *and* S_γ *is decreasing in* γ.

Proof. Write $g(x)$ as $g_\gamma(x)$ and it can be expressed as

$$g_\gamma(x) = -\frac{1-\beta^{x+1}}{\beta(\mu-\lambda)}\gamma + \frac{1}{\beta(\mu-\lambda)}\Big(\lambda\alpha - bx - \mu\alpha\beta^{x+1}\Big).$$

The coefficient of γ is negative either when $\lambda < \mu$ when $\lambda > \mu$. Hence, when γ increase to γ', the concave function $g_{\gamma'}(x)$ is below $g_\gamma(x)$. Hence by the definition of r_γ and S_γ, we conclude that r_γ is increasing in γ and S_γ is decreasing in γ. □

3.3. Algorithms for (r, S)

To develop algorithms for computing (r, S), we need the auxiliary function $l(x, i)$. We solve $l(x, i)$ from equations through (3.12) to (3.14). Define a *shift operator* $\mathcal{D}$ and its inverse $\mathcal{D}^{-1}$ which satisfy $\mathcal{D}f(x) = f(x+1)$ and $\mathcal{D}^{-1}f(x) = f(x-1)$ for function $f(x)$ and integer x.

Using the operator $\mathcal{D}$, formula (3.12) at state $(x, 0)$ where $x > r$ can be written as

$$\gamma + bx - \lambda p = \lambda[\mathcal{D}^{-1} - I]l(x, 0).$$

Since the characteristic equation $y^{-1} - 1 = 0$ has a unique solution $y = 1$, the homogeneous solution of the equation is A_1, where A_1 is a constant to be determined. Let $B_1x + C_1x^2$ be a particular solution of the equation. Then, comparing the coefficients between two sides of $\gamma + bx - \lambda p = \lambda[\mathcal{D}^{-1} - I](B_1x + C_1x^2)$ yields

$$B_1 = p - \frac{b}{2\lambda} - \frac{\gamma}{\lambda}, \tag{3.34}$$

$$C_1 = -\frac{b}{2\lambda}. \tag{3.35}$$

For any $x > r$, the solution of (3.12) is

$$l(x, 0) = A_1 + \left(p - \frac{b}{2\lambda} - \frac{\gamma}{\lambda}\right)x - \frac{b}{2\lambda}x^2. \tag{3.36}$$

By operator $\mathcal{D}$, equation (3.13) can be written as

$$\gamma + bx + \mu c - \lambda p = [\lambda \mathcal{D}^{-1} - (\mu+\lambda)I + \mu\mathcal{D}]l(x,1)$$

for $0 < x < S$. Since characteristic equation $\lambda y^{-1} - (\mu+\lambda) + ay = 0$ has two solutions: $y_1 = 1$ and $y_2 = \lambda/\mu$, the homogeneous solution is $A_2(\lambda/\mu)^x + B_2$ where A_2 and B_2 are to be determined constants. Let $C_2x + D_2x^2$ be a particular solution of the equation. Comparing the coefficients between two sides of $\gamma + bx + \mu c - \lambda p = [\lambda\mathcal{D}^{-1} - (\mu+\lambda) + \mu\mathcal{D}](C_2x + D_2x^2)$ yields

$$C_2 = \frac{\gamma + \mu c - \lambda p}{\mu - \lambda} - \frac{b}{2(\mu-\lambda)^2}, \tag{3.37}$$

$$D_2 = \frac{b}{2(\mu-\lambda)}. \tag{3.38}$$

For any $0 < x < S$, the solution of equation (3.13) is

$$l(x,1) = A_2\left(\frac{\lambda}{\mu}\right)^x + B_2 + \left(\frac{g + \mu c - \lambda p}{\mu - \lambda} - \frac{b}{2(\mu-\lambda)^2}\right)x + \frac{b}{2(\mu-\lambda)}x^2. \tag{3.39}$$

And finally, at $x = 0$, equation (3.14) can be written as

$$\gamma + b(x) + \mu c = \mu[\mathcal{D} - 1]l(x,1).$$

Since $y = 1$ is the unique solution of characteristic equation $y - 1 = 0$, the homogeneous solution of the equation is A_3 which is to be determined. Let $B_3x + C_3x^2$ be a particular solution of the equation. Comparing the coefficients between two sides of $\gamma + \mu c + bx = \mu[\mathcal{D} - 1](B_3x + C_3x^2)$ yields

$$B_3 = \frac{\gamma + \mu c}{\mu} - \frac{b}{2\mu}, \tag{3.40}$$

$$C_3 = \frac{b}{2\mu}. \tag{3.41}$$

At $x = 0$,

$$l(x,1) = A_3 + \Big(\frac{\gamma + \mu c}{\mu} - \frac{b}{2\mu}\Big)x + \frac{b}{2\mu}x^2. \tag{3.42}$$

Now we determine coefficients $A_i, i = 1,2,3$ and B_2 in (3.36)-(3.40) by the boundary conditions. Consider both $l(x,1)$ is continuous at $x = 0, S$ and $\frac{d}{dx}l(x,1)$ at $x = 0$ respectively, as well as conditions $l(r,0) + K = l(r,1)$ and $l_g(S,1) = 0$. We obtain the following system.

$$\begin{cases} A_2 \ln\Big(\dfrac{\lambda}{\mu}\Big) = B_3 - C_2, \\ B_2 + A_2\Big(\dfrac{\lambda}{\mu}\Big)^S = -C_2S - D_2S^2, \\ A_3 - A_2 - B_2 = 0, \\ A_1 - A_2\Big(\dfrac{\lambda}{\mu}\Big)^r - B_2 = (C_2 - B_1)r + (D_2 - C_1)r^2 - K. \end{cases}$$

These equations are easily solved to obtain

$$A_2 = (B_3 - C_2)/\ln\Big(\frac{\lambda}{\mu}\Big),$$
$$B_2 = -C_2 S - D_2 S^2 - A_2\Big(\frac{\lambda}{\mu}\Big)^S,$$
$$A_3 = A_2 + B_2,$$
$$A_1 = A_2\Big(\frac{\lambda}{\mu}\Big)^r + B_2 + (C_2 - B_1)r + (D_2 - C_1)r^2 - K.$$

Thus we have obtained closed form solution for $l(x, i)$ for all $x \geq 0$ and $i = 1, 2$.

Since the difference equations for $T(x, i)$ can be obtained by that of $l(x, i)$ after letting

$$\gamma = -1, \quad b = c = p = K = 0,$$

we conclude that the same formula for $l(i, i)$ can be applied to compute $T(i, 0)$.

The previous analysis leads to the following procedures for the optimal parameters r and S.

Algorithm 1. (Successive Improvement Algorithm)

0. Let $\gamma = \gamma_{\min}$.
1. If $l_\gamma^{r_\gamma, S_\gamma}(S_\gamma, 0) \leq 0$, go to Step 2. Otherwise, let

$$\gamma := \gamma + \frac{l_\gamma(S, 0)}{T(S, 0)} \tag{3.43}$$

   and repeat Step 1.
2. Terminate the procedure with the maximum profit γ and optimal threshold (r, S).

Algorithm 2. (Bi-Section Algorithm)

0. Start with a lower bound and an upper bound, $\gamma_{\min}$ and $\gamma_{\max}$. Let $\gamma = \frac{\gamma_{max} + \gamma_{min}}{2}$.
1. If $l_\gamma^{r_\gamma, S_\gamma}(S_\gamma, 0) = 0$, go to Step 2. Otherwise, if $l_\gamma^{r_\gamma, S_\gamma}(S_\gamma, 0) > 0$ then set $\gamma_{min} := \gamma$ and if $l_\gamma^{r_\gamma, S_\gamma}(S_\gamma, 0) < 0$ then set $\gamma_{max} := \gamma$. Repeat Step 1.
2. Terminate the procedure with the maximum profit γ and optimal threshold (r, S).

Theorem 3.1. *Both Algorithms 1 and 2 give the optimal control parameters after finite iterations.*

Proof. First consider Algorithm 1. Since γ is the profit of a feasible strategy, say (r_0, S_0) if

$$\max_{r,S} l^{r,S}(S, 0) = W(S, 0) - \gamma T(S, 0) \leq 0,$$

then

$$\frac{W(S, 0)}{T(S, 0)} \leq \gamma$$

for all policy (r, S). Because $l^{r_0,S_0}(S_0, 0) = 0$, it follows from Lemma 3.1 this policy is optimal.

If $\max_{r,S} l^{r,S}(S, 0) > 0$, then the updated γ in (3.43), denoted by γ', is greater than the original γ. By Proposition 3.5, we have $S_{\gamma'} \le S_\gamma$, and it can be checked that $l^{r,S_\gamma}_{\gamma'}(S_\gamma, 0) = 0$. Therefore, if $S_{\gamma'} = S_\gamma$ then $l^{r,S_{\gamma'}}_{\gamma'}(S_{\gamma'}, 0) = 0$ and the optimal policy has been found, and otherwise $S_{\gamma'} < S_\gamma$ and we have reduced the feasible region by at least one. Since the starting feasible region for optimal S is finite and each step either stop or strictly reduce the feasible region, the Algorithm 1 must stop after a finite number of steps.

The convergence of Algorithm 2 follows from the fact that $l^{r_{\gamma^*},S_{\gamma^*}}_{\gamma}(S_{\gamma^*}, 0)$ is piece-wise linear in γ. Here the proof is omitted. □

4. Global Optimality of (r, S) Policy

In the section, we prove that the optimal threshold policy identified in the last section, which is optimal among the class of (r, S) policies, is optimal among all admissible policies. The optimality proof is based on some properties that are satisfied by the value function of the threshold policy obtained in Section 3. In the following analysis, $l(x, i)$ represent the value function corresponding to the optimal threshold policy (r, S). We shall only consider the case of $\lambda \neq \mu$.

Lemma 4.1. *For all $x < S$,*

$$l(x + 1, 1) - l(x, 1) - c = -\frac{\lambda}{\mu} g(x - 1). \tag{4.1}$$

Proof. From (3.13), we have

$$l(x + 1, 1) - l(x, 1) - c = \frac{1}{\mu}\Big[(\gamma + bx - \lambda p) + \lambda(l(x, 1) - l(x - 1, 1))\Big].$$

By (3.25), we have

$$g(x - 1) = \frac{1}{\lambda}\Big[\lambda(l(x - 1, 1) - l(x, 1)) - (\gamma + bx - \lambda p)\Big].$$

Thus the desired result follows. □

Lemma 4.2. *There holds that*

i) $l(x + 1, 1) - l(x, 1) \ge c$ *for* $x \le r$,

ii) $l(x + 1, 1) - l(x, 1) \le c$ *for* $r < x < S$ *and*

iii) $l(x + 1, 1) - l(x, 1) < c$ *on* $x \ge S$.

Proof. By Proposition 3.2, we have $g(x - 1) \le 0$ for $x \le r$. Hence,

$$l(x + 1, 1) - l(x, 1) - c = -\frac{\lambda}{\mu} g(x - 1) \ge 0.$$

This implies that for $x \le r$,

$$l(x + 1, 1) - l(x, 1) - c \ge 0.$$

So, i) is proved.

By Propositions 3.1 and 3.2, we have $g(x-1) \geq 0$ for $r < x < S$. This implies that

$$l(x+1,1) - l(x,1) - c = -\frac{\lambda}{\mu} g(x-1) \leq 0$$

holds for $r < x < S$. This proves ii).

By Lemma 3.1, the best (r, S) policy should satisfy both $g(S) < 0$ and $g(S-1) \geq 0$, i.e.,

$$\begin{aligned} &-\gamma + \lambda\alpha - b(S+1) + (\gamma - \mu\alpha)\beta^{S+2} < 0, \\ &-\gamma + \lambda\alpha - bS + (\gamma - \mu\alpha)\beta^{S+1} \geq 0. \end{aligned} \tag{4.2}$$

These deduce to

$$b + (\gamma - \mu\alpha)(1-\beta)\beta^{S+1} > 0. \tag{4.3}$$

From (3.12), we have

$$\begin{aligned} l(S+1,1) - l(S,1) &= l(S+1,0) - l(S,0) \\ &= p - \frac{\gamma + b(S+1)}{\lambda}. \end{aligned}$$

Hence, $l(S+1,1) - l(S,1) < c$ is equivalent to $\gamma - \lambda(p-c) + b(S+1) > 0$. By (4.3) and definition $\alpha = p - c + b/(\mu - \lambda)$, we obtain

$$\begin{aligned} \gamma - \lambda(p-c) + b(S+1) &> \frac{\lambda b}{\mu - \lambda} + (\gamma - \mu\alpha)\beta^{S+2} \\ &= \frac{\lambda}{\mu - \lambda}\left(b + (\gamma - \mu\alpha)(1-\beta)\beta^{S+1}\right) \\ &> 0, \end{aligned} \tag{4.4}$$

which implies $l(S+1,1) - l(S,1) < c$. For $x > S$, we have, by (3.12) and (3.15),

$$\begin{aligned} l(x+1,1) - l(x,1) - c &= l(x+1,0) - l(x,0) - c \\ &= \frac{1}{\lambda}\Big[-\gamma - b(x+1) + \lambda(p-c)\Big] \\ &\leq \frac{1}{\lambda}\Big[-\gamma - b(S+1) + \lambda(p-c)\Big] \\ &< 0. \end{aligned}$$

We finish the proof of iii). □

Lemma 4.3. *For $r \leq x < S$, there holds that*

$$l(x,1) - l(x,0) = \sum_{i=x}^{S-1} g(i). \tag{4.5}$$

In particular,

$$\sum_{i=r}^{S-1} g(i) = K. \tag{4.6}$$

Proof. Since $l(S,1) = l(S,0) = 0$, we have, by (3.12), that for any $r \le x \le S$,

$$l(x,0) = \sum_{i=x}^{S-1}[l(i,0) - l(i+1,0)] = \sum_{i=r}^{S-1} \frac{\gamma + b(i+1) - \lambda p}{\lambda},$$
$$l(x,1) = \sum_{i=x}^{S-1}[l(i,1) - l(i+1,1)].$$

Hence the result follows from (3.25).

That (4.6) holds follows from (4.5) and that $l(r,1) - l(r,0) = K$. □

We are now ready to prove the main results of this paper, Theorem 2.1 and 2.2.

Proof of Theorems 2.1 and 2.2. By the optimality equations (2.3) to (2.6), to prove that the optimal threshold policy (r,S) is optimal among all admissible policies, it suffices to prove that the value function of this policy $l(x,i)$, and the threshold (r,S) policy, satisfies (2.3) to (2.6). More specifically, we want to verify the following four inequalities:

1) For $x > r$, we have

$$\lambda[l(x-1,0) - l(x-1,1) + K] + \mu[l(x,0) - l(x+1,1) + c + K] \ge 0. \quad (4.7)$$

2) For $x \le r$, we have

$$\lambda[l(x-1,0) - l(x-1,1) + K] + \mu[l(x,0) - l(x+1,1) + c + K] \le 0. \quad (4.8)$$

3) For $x \ge S$,

$$\lambda[l(x-1,0) - l(x-1,1)] + \mu[l(x,0) - l(x+1,1) + c] \ge 0. \quad (4.9)$$

4) And finally, for $x < S$, we have

$$\lambda[l(x-1,0) - l(x-1,1)] + \mu[l(x,0) - l(x+1,1) + c] \le 0. \quad (4.10)$$

If 1) is satisfied, then the first term in the maximum $\{.,.\}$ of (2.3) is larger, implying that when the state is $(x,0)$, that is when the machine is off with inventory level $x > r$, it is optimal to keep the machine idle; when 2) is satisfied, then the second term in the maximum $\{.,.\}$ of (2.3) is larger, implying that when the inventory level drops to r it is optimal to turn on the machine and to start producing. Similarly, it can be seen from 3) and 4) that, when the machine is on and the inventory level is less than S, it is optimal to continue producing, and one should turn off the machine as soon as the inventory level reaches S.

For convenience let

$$L(x) = \lambda[l(x-1,0) - l(x-1,1)] + \mu[l(x,0) - l(x+1,1) + c] \quad (4.11)$$

for $x > 0$ and $L(0) = \mu[l(0,0) - l(1,1) + c]$. Then, 1) to 4) can be rewritten as:

a) $L(x) \le -K$ for $0 \le x \le r$;

b) $-K \le L(x) \le 0$ for $r < x < S$;

c) $L(x) \ge 0$ for $x \ge S$.

We first prove a). By (3.15) and (3.16), we have

$$l(x,1) - l(x,0) = K \tag{4.12}$$

for $x \le r$. By (3.13) and Lemma 4.1, for $x \le r$ we have

$$\begin{aligned} l(x+1,1) - l(x,0) &= [l(x+1,1) - l(x,1) - c] + K + c \\ &= -\frac{\lambda}{\mu} g(x-1) + K + c \\ &\ge K + c, \end{aligned} \tag{4.13}$$

where the last inequality follows from $g(x-1) \le 0$ for $x \le r$. Combining (4.12) and (4.13) we obtain a).

We then proceed to prove b). It follows from Lemma 4.3 and (3.12) we obtain that for $r < x < S$,

$$\begin{aligned} l(x+1,1) - l(x,0) &= [l(x+1,1) - l(x+1,0)] + [l(x+1,0) - l(x,0)] \\ &= \sum_{i=x+1}^{S-1} g(i) + \frac{-\gamma - b(x+1) + \lambda p}{\lambda}. \end{aligned}$$

Since $g(x) \ge 0$ on $r < x < S$, we conclude that $l(x+1,1) - l(x,0)$ is decreasing of x in $r < x < S$. Furhtermore, it follows from *ii*) of Lemma 4.2 and Lemma 4.3 that

$$\begin{aligned} l(r+2,1) - l(r+1,0) &= [l(r+2,1) - l(r+1,1)] + [l(r+1,1) - l(r+1,0)] \\ &\le c + \sum_{i=r+1}^{S-1} g(i) < c + \sum_{i=r}^{S-1} g(i) \\ &= c + K. \end{aligned}$$

This shows that for all $r < x < S$ we have

$$l(x,1) - l(x+1,0) + c \ge -K. \tag{4.14}$$

In addition, it follows from Lemma 4.3 and $g(i) \ge 0$ for any $r \le i < S$ that

$$0 \le l(x,1) - l(x,0) \le K \tag{4.15}$$

holds for any $r \le x < S$. Combining (4.14) and (4.15) we obtain b).

We finally prove c). If $x > S$, then by $l(x-1,0) = l(x-1,0)$ we obtain

$$L(x) = \mu[l(x,0) - l(x+1,1) + c] = \mu(l(x,1) - l(x+1,1) + c] \ge 0,$$

where the inequality follows from *iii*) of Lemma 4.2. For $x = S$, it follows from Lemma 4.3, $l(S+1,1) = l(S+1,0)$ and (3.12) that

$$\begin{aligned} L(S) &= \lambda[l(S-1,0) - l(S-1,1)] + \mu[l(S,0) - l(S+1,1) + c] \\ &= -\lambda g(S-1) + \frac{\mu}{\lambda}[\gamma + b(S+1) - \lambda(p-c)] \\ &\ge -\lambda g(S-1) + \frac{\mu}{\mu - \lambda}\Big[b + (\gamma - \mu\alpha)(1-\beta)\beta^{S+1}\Big] \\ &= -\lambda g(S-1) + \lambda(g(S-1) - g(S)) \\ &= -\lambda g(S), \end{aligned}$$

where the inequality follows from (4.4), and the third equality follows from

$$g(S-1) - g(S) = \frac{\mu}{\lambda(\mu-\lambda)} \left(b + (\gamma - \mu\alpha)(1-\beta)\beta^{S+1}\right).$$

Since $g(S) \leq 0$, we have proved that $L(S) \geq 0$. This completes the proof of c) and optimality of the (r, S) policy among all admissible policies.

Theorem 2.2 follows from the proof of Theorem 2.1 and (4.6) of Lemma 4.3.

5. Numerical examples

In this section, we review a number of numerical experiments for the case of a single product make-to-stock queue, in order to study the effects of parameters, such as the setup cost, the selling price, demand arrival rate and production rate, on optimal control strategy and maximum average profit.

In the subsequent numerical experiments , we use the following base settings: the production cost for each product is $c = 2$, while the selling price is $p = 10$ with the demand arrival rates $\lambda = 0.4$. According to our previous assumption, the production rate is $\mu = 1 - \lambda = 0.6$. The setup cost is $K = 10$. The holding cost per product is $b = 0.01$.

In particular, we consider different demand functions. There are two special cases of the function $D(p)$. One is $D(p) = \beta - \alpha p$ ($\alpha > 0, \beta > 0$) in the additive case and the other is $D(p) = \alpha p^{-\beta}$ ($\alpha > 0, \beta > 1$) in the multiplicative case. In Petruzzi and Data(1999), both are common in the economics literature. According to the definition of the demand functions, we assume the basic additive demand function we study is $\lambda = 1 - 0.06p$, while the basic multiplicative demand function is $\lambda = 40p^{-2}$.

5.1. Effects of K

In this subsection, we illustrate the optimal (r, S) policy for different values of K. The results are shown in Figure 2, where the x-axis is setup cost K and y-axis displays the optimal r and S. The value of K goes from 0 to 20, with the increment of 1. It is shown that S is increasing when K rises up, while r is decreasing.

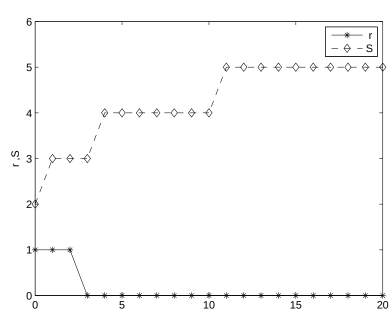

Figure 2. Optimal r and S for different K

Figure 3 shows the maximum average profit when K is different, while λ and p are under the same additive demand function $\lambda = 1 - 0.06p$. In this figure, the

x-axis is K and y-axis displays the maximum average profit γ^*. The value of K goes from 0 to 20 with the increment of 1. Figure 4 shows the maximum average profit γ^* when K is different, while λ and p are under the same multiplicative demand function $\lambda = 40p^{-2}$. In this figure, the x-axis is K and y-axis displays the maximum average profit. The value of K goes from 0 to 20 with the increment of 1.

From the two figures, the main conclusion is γ^* is decreasing on K. Moreover, when $\lambda < 0.5$, which means $\mu = 1 - \lambda$ is greater than 0.5, the maximum average profit γ^* drops more rapidly than the γ^* when $\lambda > 0.5$. The main result is that when $\lambda > 0.5$, it means the production rate μ is less than the demand arrival rate λ. Then the frequency of shutting down the machine decreases, which means less setup costs need to be covered. Hence, the effect of setup cost K on the maximum average profit is weakened.

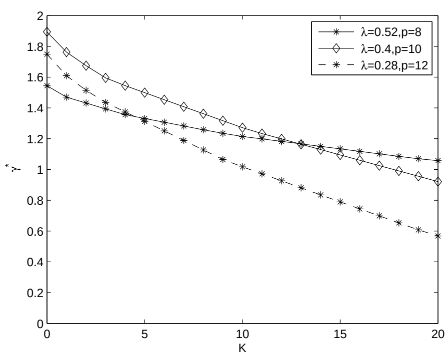

Figure 3. Optimal profit for different K when $\lambda = 1 - 0.06p$

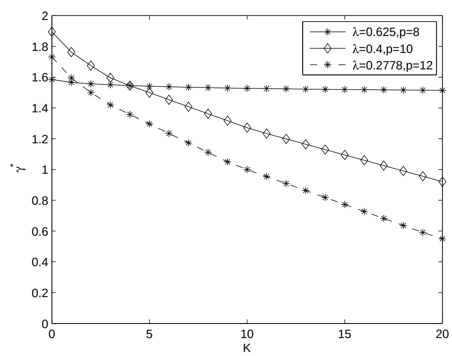

Figure 4. Maximum average profit for different K when $\lambda = 40p^{-2}$

5.2. Effects of p

Then we examine the effects of p in optimal (r, S) policy and compare the maximum average profit under different demand functions.

First of all, we compute r and S for different values of p under additive demand function. The results are shown in Figure 5, where the x-axis is p and y-axis displays the optimal r and S. The value of p goes from 5 to 15, with the increment of 1. Moreover, we compare different maximum average profit under different additive demand functions. Figure 6 shows the results, where the x-axis is p and y-axis displays the maximum average profit γ^*. The value of p goes from 5 to 15, with the increment of 1. It is shown that both r and S are decreasing on p, while γ^* is concave on p.

When the demand function is multiplicative, we compute r and S for different values of p. The results are shown in Figure 7, where the x-axis is p and y-axis displays the optimal r and S. The value of p goes from 7 to 20, with the increment of 1. Moreover, we compare different maximum average profit under different multiplicative demand functions. Figure 8 shows the results, where the x-axis is p and y-axis displays the maximum average profit γ^*. The value of p goes from 8 to 20, with the increment of 1. It is also shown that both r and S are decreasing on p.

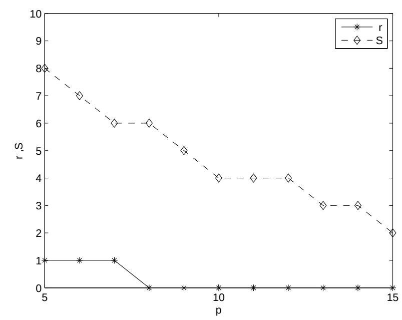

Figure 5. Optimal r and S for different p when $\lambda_1 = 1 - 0.06p_1$

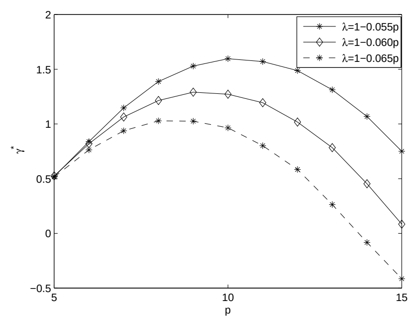

Figure 6. Maximum average profit for different p under additive demand functions

Although γ^* is not concave on p, it still exists thatγ^* will rise up when p is small. When p is great enough, γ^* will also drop.

6. Conclusion

It is well known that the when there is a finite production/ordering capacity, (s, S) policy is no longer optimal. In this paper we consider a continuous review inventory system with finite production capacity and lost-sales, and show that the optimal control policy for the system still has a very simple structure of (r, S). We also develop simple algorithms to compute the optimal parameters.

From the numerical tests, we obtain more results. When other parameters are fixed, the shut down point S is increasing on setup cost K, while shut on point r is decreasing on K. At the same time, the maximum average profit γ^* is decreasing on K. Moreover, when $\lambda < 0.5$, the maximum average profit γ^* drops more rapidly than the γ^* when $\lambda > 0.5$.

When the selling price p is decision variable, it would be more complicate, for the selling price will influence both the demand arrival rate λ and the production rate μ. It is shown that both r and S are decreasing on p no matter what kind of the demand function is. However, when the demand function is additive, the maximum average profit γ^* is concave on p. When the demand function is multiplicative, γ^* will rise up when p is small enough, while γ^* will drop when p is large.

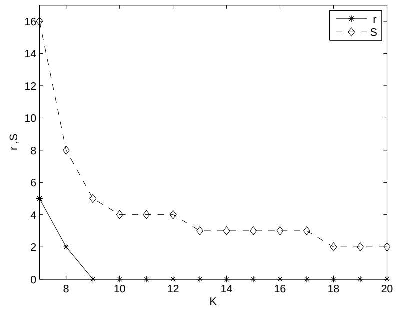

Figure 7. Optimal r and S for different p_1 when $\lambda_1 = 1.8p_1^{-1.5}$

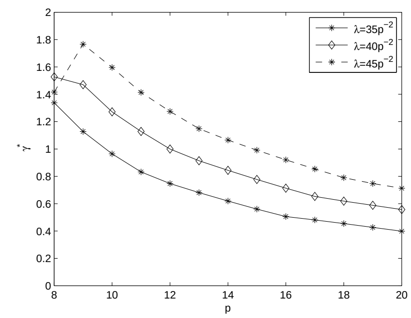

Figure 8. Maximum average profit for different p under multiplicative demand functions

The model considered in this paper is clearly rather simplistic and it will be of interest to explore more general systems for which a simple policy remains optimal. Extensions of the model studied in this paper include arbitrary production time distribution, and batch Poisson demand processes. An important question to answer is, given that the optimal control policy for finite capacity production/inventory system with setup cost is known to not have a simple form, which classes of systems still inherit the special simple structure of the optimal control policy? In other words, it would be particularly interesting to characterize the inventory systems that have finite capacity and setup cost but nevertheless, the simple (s, S) type of policy remains optimal. We leave this as a future research topic.

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