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WELL-POSEDNESS AND LONG TIME BEHAVIOR OF A HYPERBOLIC CAGINALP PHASE-FIELD SYSTEM

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Abstract The aim of this paper is to prove the continuity of exponential attractors for a hyperbolic perturbed Caginalp system to an exponential attractor for the limit parabolic-hyperbolic Caginalp system. The symmetric distance between the perturbed and unperturbed exponential attractors in terms of the perturbation parameter is obtained.

Keywords Caginalp system, well-posedness, dissipativity, global attractor, exponential attractors, asymptotic expansions.

MSC(2000) 35B40, 35B41, 35B45.

1. Introduction

The study of the long time behavior of systems arising from mechanics and physics is important, as it is essential, for practical purposes, to understand and predict the asymptotic behavior of the systems. Several objects have been introduced for this study.

A first object is the global attractor, which is a compact invariant set which attracts uniformly the bounded sets of the phase space. The global attractor presents two major defaults. It can attract the trajectories slowly and it can be sensitive to perturbations.

In order to overcome these difficulties, Foias, Sell and Temam [14] have introduced the notion of an inertial manifold. An inertial manifold is a smooth, finite dimensional, hyperbolic (and thus robust) positively invariant manifold which contains the global attractor and attracts exponentially the trajectories. Unfortunately, all constructions of inertial manifolds are based on a very restrictive condition: the so-called spectral gap condition. Consequently, the existence of inertial manifolds is not known for many physically important systems.

Eden, Foias, Nicolaenko and Temam have introduced in [5] the notion of exponential attractor which is an intermediate object between the two ideal objects that the global attractor and an inertial manifold are. Indeed, an exponential attractor is a compact positively invariant set which contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories. An exponential attractor is more robust under perturbations and numerical approximations than the global attractor. We note that, contrary to the global attractor, an exponential attractor is not necessarily unique.

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Exponential attractors have first been constructed for systems in Hilbert spaces only by using orthogonal projectors with finite rank in order to prove the so-called squeezing property. Recently, Efendiev, Miranville and Zelik gave in [6] and [7] a construction of exponential attractors that is no longer based on the squeezing property and that is valid in a Banach space. So, exponential attractors are as general as global attractors.

The main goal of this paper is to construct a robust family of exponential attractors which is continuous as the perturbation parameter ϵ goes to 0 when the unperturbed system is the parabolic-hyperbolic Caginalp system and the perturbed system is the hyperbolic Caginalp system.

The Caginalp phase-field system has been proposed by Caginalp(see [2]) to model phase transition phenomena in certain classes of materials. Some generalizations of the original system have been proposed and studied in bounded domains as well as in unbounded domains (see for instance Miranville & Quintanilla [12] and [13], Conti & Gatti [3], Cherfils & Miranville [4]).

We consider the singular perturbation of a Caginalp system in a smooth bounded domain $\Omega \subset \mathbb{R}^n, 1 \leq n \leq 3$. The perturbed system is hyperbolic with a perturbation parameter $\epsilon > 0$ and the unperturbed system is parabolic-hyperbolic ($\epsilon = 0$).

The perturbed system reads

$$\epsilon\partial_t^2 u + \partial_t u - \Delta u + f(u) = \partial_t \alpha, \qquad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

$$\partial_t^2 \alpha - \partial_t \Delta\alpha - \Delta\alpha = -\partial_t u, \qquad \text{in } \Omega \times (0, \infty), \tag{1.2}$$

$$u = \alpha = 0, \qquad \text{on } \partial\Omega, \tag{1.3}$$

$$u(0, x) = u_0,\ \alpha(0, x) = \alpha_0, \tag{1.4}$$

$$\partial_t u|_{t=0} = u_1,\ \partial_t \alpha|_{t=0} = \alpha_1, \tag{1.5}$$

where $u(t, x)$ is the phase field or order parameter and $\alpha(t, x)$ the thermal displacement variable.

In these systems, $u(t, x)$ and $\alpha(t, x)$ are unknown functions, and $f(s) = s^3 - s$. Then f is of class C^2 and satisfies

$$-1 \leq f'(s), \forall s \in \mathbf{R} \quad \text{and} \tag{1.6}$$

$$-1 \leq 4F(s) \leq f(s)s,\ \forall s \in \mathbf{R}, \tag{1.7}$$

$$\text{where} \quad F(s) = \int_0^s f(\tau)d\tau,$$

$$\text{and} \quad |f'(s)| \leq 3(s^2 + 1), \forall s \in \mathbf{R}. \tag{1.8}$$

We introduce, for every $\kappa \geq 0$, the standard energy norm

$$\| \zeta_v(t) \|^2_{\varepsilon^\kappa(\epsilon)} = \| v(t) \|^2_{H^{\kappa+1}} + \epsilon \| \partial_t v(t) \|^2_{H^\kappa} + \| \partial_t v(t) \|^2_{H^{\kappa-1}},$$

where $\zeta_v(t) = [v(t), \partial_t v(t)]$.

Thus, the energy spaces $\varepsilon^\kappa(\epsilon)$ coincide with $\Big[H^{\kappa+1}(\Omega) \times H^\kappa(\Omega)\Big] \cap \{\zeta|_{\partial\Omega} = 0\}$ for all $\epsilon > 0$ and with $[H^{\kappa+1}(\Omega) \times H^{\kappa-1}(\Omega)] \cap \{\zeta|_{\partial\Omega} = 0\}$ if $\epsilon = 0$, where $H^\kappa(\Omega)$ denotes the standard Sobolev space, the boundary conditions being added only for the κ for which they make sense. We will write in the sequel $\varepsilon(\epsilon)$ instead of $\varepsilon^0(\epsilon)$.

We recall that $H^1_0(\Omega) \subset L^p(\Omega)\ \forall p \geq 1$ if $n = 1, 2$ and $p = 6$, if $n = 3$, and $f : H^1_0(\Omega) \to L^2(\Omega)$ is Lipschitz continuous on each bounded subset of $H^1_0(\Omega)$.

We denote by $(.,.)$ and $\| \, . \, \|$ the usual scalar product and the associated norm in the Hilbert space $L^2(\Omega)$. We shall note $(.,.)_X$ and $\| \, . \, \|_X$ the scalar product and the associated norm in the Hilbert space X. The constant c_0 is the Poincaré constant in Ω.

In section 1, we prove the existence of solutions when $\epsilon = 0$ by using classical arguments, and give some estimates we need in the sequel. In section 2, as above the existence of solutions is proved and some estimates are given when $\epsilon > 0$. In section 3 we obtain several estimates on the difference of solutions of system $(1.1) - (1.5)$ with $\epsilon > 0$ and $\epsilon = 0$ which are necessary to construct robust exponential attractors. Finally, in section 4, we apply the abstract result of Fabrie & Galusinski(see [8]) to construct a continuous (as $\epsilon \longrightarrow 0^+$) family of exponential attractors for system $(1.1) - (1.5)$. Two uniform (as $\epsilon \to 0^+$) estimates on the linear problems are given in an appendix.

2. The limit parabolic-hyperbolic Caginalp system

In this section, we consider the limit parabolic-hyperbolic system

$$\partial_t u - \Delta u + f(u) = \partial_t \alpha, \tag{2.1}$$

$$\partial_t^2 \alpha - \partial_t \Delta \alpha - \Delta \alpha = -\partial_t u, \tag{2.2}$$

$$u = \alpha = 0 \text{ on } \partial\Omega, \tag{2.3}$$

$$u(0,x) = u_0, \quad \zeta_\alpha|_{t=0} = [\alpha_0, \alpha_1], \tag{2.4}$$

Theorem 2.1. *We assume that $(u_0, \alpha_0, \alpha_1) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$. Then, the system (2.1)-(2.4) possesses at least one solution (u, α) such that $u \in L^\infty(\mathbf{R}_+; H_0^1(\Omega))$ $\cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, $\alpha \in L^\infty(\mathbf{R}_+; H_0^1(\Omega))$, $\partial_t u \in L^2(0,T; L^2(\Omega))$ and $\partial_t \alpha \in$ $L^\infty(\mathbf{R}_+; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)), \forall T > 0$.*

Proof. Multiply (2.1) by $2\partial_t u$ and (2.2) by $2\partial_t \alpha$, integrate over Ω and sum the two resulting equations. We obtain

$$\frac{d}{dt} E_1 + 2 \| \partial_t u \|^2 + 2 \| \partial_t \alpha \|_{H^1}^2 = 0, \tag{2.5}$$

where

$$E_1 = \| u \|_{H^1}^2 + \| \alpha \|_{H^1}^2 + \| \partial_t \alpha \|^2 + 2(F(u), 1).$$

The existence of a solution is based on the estimate (2.5) and a standard Galerkin scheme. Multiplying (2.1) by $-2\Delta u$ and integrating over Ω, we have

$$\frac{d}{dt} \| u \|_{H^1}^2 + \| u \|_{H^2}^2 \leq 2 \| u \|_{H^1}^2 + 2 \| \partial_t \alpha \|_{H^1}^2, \tag{2.6}$$

which yields $u \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$. $\square$

Theorem 2.2 (Uniqueness)**.** *Let the assumptions of Theorem* 2.1 *hold. Then, the system* $(2.1)-(2.4)$ *possesses unique solution* (u, α) *such that* $u, \alpha \in L^\infty(\mathbf{R}_+; H_0^1(\Omega))$, $\partial_t u \in L^2(0,T; L^2(\Omega))$ *and* $\partial_t \alpha \in L^\infty(\mathbf{R}_+; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, $\forall T > 0$.

Proof. Let $(u^{(1)}, \alpha^{(1)})$ and $(u^{(2)}, \alpha^{(2)})$ be two solutions of $(2.1) - (2.4)$ with initial data $(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$, respectively, and set $u = u^{(1)} - u^{(2)}$ and

$\alpha = \alpha^{(1)} - \alpha^{(2)}$. Then (u, α) satisfies

$$\partial_t u - \Delta u + f(u^{(1)}) - f(u^{(2)}) = \partial_t \alpha, \tag{2.7}$$

$$\partial_t^2 \alpha - \partial_t \Delta \alpha - \Delta \alpha = -\partial_t u, \tag{2.8}$$

We multiply (2.7) by $2\partial_t u$ and (2.8) by $2\partial_t \alpha$, integrate over Ω and sum the two resulting equations to obtain

$$\frac{d}{dt}\Big(\parallel u \parallel_{H^1}^2 + \parallel \alpha \parallel_{H^1}^2 + \parallel \partial_t \alpha \parallel^2 \Big) \leq C(\parallel u^{(1)} \parallel_{H^1}^2 + \parallel u^{(2)} \parallel_{H^1}^2 + 1) \parallel u \parallel_{H^1}^2, \tag{2.9}$$

which yields the uniquess, as well as the continuous dependence with respect to the initial data. □

Theorem 2.3. *We assume that* $(u_0, \alpha_0, \alpha_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ *. Then, the system* $(2.1) - (2.4)$ *possesses unique solution* (u, α) *such that* $u, \alpha \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t u \in L^2(0, T; H_0^1(\Omega))$ *and* $\partial_t \alpha \in L^\infty(0, T; H_0^1(\Omega))$ $\cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t^2 \alpha \in L^2(0, T; L^2(\Omega)), \forall T > 0$.

Proof. Following Theorem 2.1, the system (2.1)-(2.4) possesses one solution (u, α) such that $u \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$. We multiply (2.1) by $-2\Delta \partial_t u$ and (2.2) by $-2\Delta \partial_t \alpha$ and integrate over Ω, sum the two resulting equations. We have, using the fac that $u \in L^2(0, T; H_0^1(\Omega))$,

$$\frac{d}{dt}\Big(\parallel u \parallel_{H^2}^2 + \parallel \alpha \parallel_{H^2}^2 + \parallel \partial_t \alpha \parallel_{H^1}^2 \Big) + \parallel \partial_t u \parallel_{H^1}^2 + 2 \parallel \partial_t \alpha \parallel_{H^2}^2 \leq C \parallel u \parallel_{H^2}^2, \tag{2.10}$$

which implies $u, \alpha \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t u \in L^2(0, T; H_0^1(\Omega))$ and $\partial_t \alpha \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$. □

Corollary 2.1. *Under the assumptions of Theorem 2.3,* $\partial_t^2 \alpha \in L^2(0, T; L^2(\Omega))$.

Proof. We multiply (2.2) by $2\partial_t^2 \alpha$ and integrate over Ω to find the following

$$\frac{d}{dt} \parallel \partial_t \alpha \parallel_{H^1}^2 + \parallel \partial_t^2 \alpha \parallel^2 \quad \leq \quad 2 \parallel \alpha \parallel_{H^2}^2 + 2 \parallel \partial_t u \parallel^2, \tag{2.11}$$

which implies $\partial_t^2 \alpha \in L^2(0, T; L^2(\Omega))$. □

The phase spaces have the form $\Phi_\kappa = H^{\kappa+1}(\Omega) \times \varepsilon^\kappa(1)$ with $\kappa = 0, 1$. The standard energy norms for the unperturbed system are

$$\parallel (u, \zeta_\alpha) \parallel_{\Phi_\kappa}^2 = \parallel u \parallel_{H^{\kappa+1}}^2 + \parallel \zeta_\alpha \parallel_{\varepsilon^\kappa(1)}^2 .$$

Theorems 2.1 and 2.3 allow to define for $\kappa = 0, 1$ the solving semigroup $S_t(0)$ associated with system (2.1)-(2.2) by

$$\begin{aligned} &S_t(0) : \Phi_\kappa \longrightarrow \Phi_\kappa \\ &(u_0, \zeta_{\alpha_0}) \longmapsto (u(t), \zeta_\alpha(t)), \end{aligned}$$

where (u, ζ_α) is such that (u, α) is the unique solution of (2.1)-(2.4) with initial data $(u_0, \zeta_{\alpha_0}) = (u(0), \zeta_\alpha(0)) \in \Phi_\kappa$.

Theorem 2.4. *Let the assumptions of Theorem 2.1 hold and* (u, α) *be a solution of the system (2.1)-(2.4) with initial data* $(u(0), \zeta_\alpha(0)) \in \Phi_0$*. Then, the following* *estimate is valid*

$$\| u(t) \|_{H^1}^2 + \| \zeta_\alpha(t) \|_{\varepsilon(1)}^2 + \int_0^t e^{-\beta(t-s)}\Big(\| \partial_t u(s) \|^2 + \|\partial_t\alpha\|^2 + \| \partial_t\alpha(s) \|_{H^1}^2 \Big)ds$$
$$\leq Q(\| (u(0), \zeta_\alpha(0)) \|_{\Phi_0})e^{-\beta t} + C, \tag{2.12}$$

where Q is a monotonic function.

Proof. Multiply (2.1) by $2u$ and (2.2) by 2α and integrate over Ω. We have the following estimates:

$$\frac{d}{dt} \| u \|^2 + \| u \|_{H^1}^2 \leq C + c_0 \| \partial_t\alpha \|^2, C > 0, \tag{2.13}$$

$$\frac{d}{dt}\Big(\| \alpha \|_{H^1}^2 + 2(\alpha, \partial_t\alpha)\Big) + \| \alpha \|_{H^1}^2 \leq c_0 \| \partial_t u \|^2 + 2 \| \partial_t\alpha \|^2 . \tag{2.14}$$

Multiplying (2.2) by $2(-\Delta)^{-1}\partial_t\alpha$ and integrating over Ω, we have

$$\frac{d}{dt}\Big(\| \alpha \|^2 + \| \partial_t\alpha \|_{H^{-1}}^2 \Big) + \| \partial_t\alpha \|^2 \leq C_1 \| \partial_t u \|^2 . \tag{2.15}$$

Summing (2.5), ϵ_1(2.13), ϵ_2(2.14) and ϵ_3(2.15) where ϵ_1, ϵ_2 and $\epsilon_3 > 0$ are chosen small enough such that

$$\epsilon_3 - 2\epsilon_2 - \epsilon_1 c_0 > 0,$$
$$1 - \epsilon_2 c_0 - \epsilon_3 C_1 > 0,$$

we have

$$\frac{d}{dt}E_2 + \beta E_2 + \| \partial_t u \|^2 + \epsilon_3\|\partial_t\alpha\|^2 + \| \partial_t\alpha \|_{H^1}^2 \leq C', \tag{2.16}$$

where $\beta > 0$ and

$$E_2 = E_1 + \epsilon_1 \| u \|^2 + \epsilon_2\Big(\| \alpha \|_{H^1}^2 + 2(\alpha, \partial_t\alpha)\Big) + \epsilon_3\Big(\| \alpha \|^2 + \| \partial_t\alpha \|_{H^{-1}}^2 \Big).$$

Applying Gronwall's inequality and the fact that for a sufficiently small $\epsilon_2 > 0$ there exists $k_2 > 0$ such that

$$k_2^{-1}(\| u(t) \|_{H^1}^2 + \| \zeta_\alpha(t) \|_{\varepsilon(1)}^2) \leq E_2(t) \leq k_2(\| u(t) \|_{H^1}^2 + \| \zeta_\alpha(t) \|_{\varepsilon(1)}^2),$$

we obtain

$$\| u(t) \|_{H^1}^2 + \| \zeta_\alpha(t) \|_{\varepsilon(1)}^2 + \int_0^t e^{-\beta(t-s)}\Big(\| \partial_t u(s) \|^2 + \partial_t\alpha\|^2 + \| \partial_t\alpha(s) \|_{H^1}^2 \Big)ds$$
$$\leq Q(\| u(0), \zeta_\alpha(0) \|_{\Phi_0})e^{-\beta t} + C. \tag{2.17}$$

This finishes the proof. □

Theorem 2.5. *Let the assumptions of Theorem 2.3 hold and (u, α) be a solution of the system (2.1)-(2.4) with initial data $(u(0), \zeta_\alpha(0)) \in \Phi_1$. Then, the following estimate is valid*

$$\| u(t) \|_{H^2}^2 + \| \zeta_\alpha(t) \|_{\varepsilon^1(1)}^2$$
$$+ \int_0^t e^{-\beta(t-s)}\Big(\| \partial_t u(s) \|_{H^1}^2 + \| \partial_t\alpha(s) \|_{H^2}^2 + \| \partial_t\alpha(s) \|_{H^1}^2 \Big)ds$$
$$\leq Q(\| (u(0), \zeta_\alpha(0)) \|_{\Phi_1})e^{-\beta t} + C, \tag{2.18}$$

where Q is a monotone function.

Proof. Multiply (2.1) by $-2\Delta u$ and (2.2) by $-2\Delta\alpha$ and integrate over Ω. We have

$$\frac{d}{dt}\parallel u\parallel_{H^1}^2+\parallel u\parallel_{H^2}^2\leq C_1+2\parallel\partial_t\alpha\parallel^2, C_1>0, \tag{2.19}$$

$$\frac{d}{dt}\Big(\parallel\alpha\parallel_{H^2}^2+2(\nabla\alpha,\nabla\partial_t\alpha)\Big)+\parallel\alpha\parallel_{H^2}^2\leq\parallel\partial_t u\parallel^2+2\parallel\partial_t\alpha\parallel_{H^1}^2. \tag{2.20}$$

Sum (2.16), ϵ_4(2.10), ϵ_5(2.19) and ϵ_6(2.20), where ϵ_4,ϵ_5 and $\epsilon_6>0$) are chosen small enough such that

$$1-\epsilon_6>0,$$
$$\epsilon_5-\epsilon_4C>0,$$
$$1-2\epsilon_5c_0-2\epsilon_6>0,$$

we have

$$\frac{d}{dt}E_3+\beta E_3+C_1\parallel\partial_t u\parallel_{H^1}^2+C_1\parallel\partial_t\alpha\parallel_{H^2}^2+C_4\parallel\partial_t\alpha\parallel_{H^1}^2\leq C'', \tag{2.21}$$

where

$$\begin{aligned}E_3=&E_2+\epsilon_4(\parallel u\parallel_{H^2}^2+\parallel\alpha\parallel_{H^2}^2+\parallel\partial_t\alpha\parallel_{H^1}^2)+\epsilon_5\parallel u\parallel_{H^1}^2\\&+\epsilon_6\Big(\parallel\alpha\parallel_{H^2}^2+2(\nabla\alpha,\nabla\partial_t\alpha)\Big).\end{aligned}$$

Applying Gronwall's inequality and the fact that for a sufficiently small $\epsilon_6>0$ there exists $k_3>0$ such that

$$k_3^{-1}(\parallel u(t)\parallel_{H^2}^2+\parallel\zeta_\alpha(t)\parallel_{\varepsilon^1(1)}^2)\leq E_3(t)\leq k_3(\parallel u(t)\parallel_{H^2}^2+\parallel\zeta_\alpha(t)\parallel_{\varepsilon^1(1)}^2), \tag{2.22}$$

we obtain

$$\begin{aligned}&\parallel u(t)\parallel_{H^2}^2+\parallel\zeta_\alpha(t)\parallel_{\varepsilon^1(1)}^2\\&+\int_0^t e^{-\beta(t-s)}\Big(\parallel\partial_t u(s)\parallel_{H^1}^2+\parallel\partial_t\alpha(s)\parallel_{H^2}^2+\parallel\partial_t\alpha(s)\parallel_{H^1}^2\Big)ds\\\leq&Q(\parallel(u(0),\zeta_\alpha(0))\parallel_{\Phi_1})e^{-\beta t}+C.\end{aligned}$$

□

Theorem 2.6. *The semigroup $S_t(0)$ associated with system (2.1)-(2.4) is dissipative in Φ_0, i.e., it possesses a bounded absorbing set $\mathcal{B}_0$ in Φ_0.*

Proof. Let B be a bounded subset of Φ_0, and $R>0$ such that $\parallel(u_0,\zeta_{\alpha_0})\parallel_{\Phi_0}\leq R,\forall(\zeta_{u_0},\zeta_{\alpha_0})\in B$. It follows from (2.12) an inequality of the form

$$\parallel(u(t),\zeta_\alpha(t))\parallel_{\Phi_0}^2\leq c(R)e^{-\beta t}+c'', \tag{2.23}$$

where $c(R)=Q(\parallel(u_0,\zeta_{\alpha_0})\parallel_{\Phi_0})$ and c'' is independent of R. Taking $t_0=t_0(R)=-\frac{1}{\beta}\ln\left(\frac{c''}{c(R)}\right)$, we have $\forall t\geq t_0$,

$$\parallel(u(t),\zeta_\alpha(t))\parallel_{\Phi_0}^2\leq 2c''. \tag{2.24}$$

□

Theorem 2.7. *The semigroup $S_t(0)$ associated with system (2.1)-(2.4) possesses a bounded absorbing set $\mathcal{B}_1$ which is bounded in Φ_1.*

Proof. Let B be a bounded subset of Φ_1 and R be such that $\| (u_0, \zeta_{\alpha_0}) \|_{\Phi_1}^2 \leq R^2, \forall (u_0, \zeta_{\alpha_0}) \in B$. Thanks to (2.18), we have $\forall t \geq 0$

$$\| (u(t), \zeta_\alpha(t)) \|_{\Phi_1}^2 \leq C + Q(\|(u_0, \zeta_{\alpha_0})\|_{\Phi_1}).$$

Hence, $B_{\Phi_1}(0, \sqrt{2C + Q(\|(u_0, \zeta_{\alpha_0})\|_{\Phi_1}})$ is a bounded absorbing set in Φ_1 for semigroup $S_t(0)$. $\square$

Theorem 2.8. *Under the assumptions of Theorem 2.1, the semigroup $S_t(0)$ associated to (2.1)-(2.4) possesses the global attractor $\mathcal{A}$ which is bounded in Φ_0.*

Proof. We know that $S_t(0)$ possesses a bounded absorbing set $\mathcal{B}_0$ in Φ_0. It is sufficient to decompose the solution $(u, \alpha) \in \mathcal{B}_0$ in the form

$$(u, \alpha) = (\nu, \eta) + (\omega, \xi),$$

where (ν, η) solves

$$\partial_t \nu - \Delta \nu = \partial_t \eta, \tag{2.25}$$

$$\partial_t^2 \eta - \partial_t \Delta \eta - \Delta \eta = -\partial_t \nu, \tag{2.26}$$

$$\nu = \eta = 0 \quad \text{on} \quad \partial\Omega,$$

$$\nu|_{t=0} = u_0, \quad \zeta_\eta|_{t=0} = \zeta_\alpha(0),$$

and (ω, ξ) solves

$$\partial_t \omega - \Delta \omega + f(u) = \partial_t \xi, \tag{2.27}$$

$$\partial_t^2 \xi - \partial_t \Delta \xi - \Delta \xi = -\partial_t \omega, \tag{2.28}$$

$$\omega = \xi = 0 \quad \text{on} \quad \partial\Omega$$

$$\omega|_{t=0} = \zeta_\xi|_{t=0} = 0,$$

and to show that

$$\| (\nu(t), \zeta_\eta(t)) \|_{\Phi_0} \ \text{ tends to } 0, \text{ as } t \longrightarrow +\infty,$$

and

$$\| (\omega(t), \zeta_\xi(t)) \|_{\Phi_1} \ \text{ is regularizing, as } t \longrightarrow +\infty.$$

It follows from the assumptions of the theorem that $f'(u)\nabla u \in L^2(0, T; L^2(\Omega)^n)$, $\forall T > 0$, and

$$\| f'(u)\nabla u \|_{L^2(0,T;L^2(\Omega)^n)} \leq (T^{\frac{1}{2}} + 1) Q(\| u_0 \|_{H^1}, \| \alpha_0 \|_{H^1}, \| \alpha_1 \|) \tag{2.29}$$

for some function Q. Multiplying (2.25) by $2\partial_t \nu$ and (2.26) by $2\partial \eta$, integrating over Ω and summing the two resulting equations, we have

$$\frac{d}{dt}\Big(\| \nu \|_{H^1}^2 + \| \eta \|_{H^1}^2 + \| \partial_t \eta \|^2 \Big) + 2 \| \partial \nu \|^2 + 2 \| \partial_t \eta \|_{H^1}^2 = 0. \tag{2.30}$$

Multiplying (2.25) by 2ν and (2.26) by 2η, integrating over Ω, we have the two following estimates

$$\frac{d}{dt} \parallel \nu \parallel^2 + \parallel \nu \parallel^2_{H^1} \leq c_0 \parallel \partial\eta \parallel^2, \tag{2.31}$$

$$\frac{d}{dt}\Big(\parallel \eta \parallel^2_{H^1} + 2(\partial_t\eta, \eta)\Big) + \parallel \eta \parallel^2_{H^1} \leq c_0 \parallel \partial_t\nu \parallel^2 + 2 \parallel \partial\eta \parallel^2 . \tag{2.32}$$

Multiply (2.26) by $2(-\Delta)^{-1}\partial_t\eta$ integrate over Ω. We find

$$\frac{d}{dt}\Big(\parallel \partial_t\eta \parallel^2_{H^{-1}} + \parallel \eta \parallel^2 \Big) + \parallel \partial_t\eta \parallel^2 \leq C_1 \parallel \partial_t\nu \parallel^2 . \tag{2.33}$$

Summing (2.30), ϵ_8(2.31), ϵ_9(2.32) and ϵ_{10}(2.33), where ϵ_8, ϵ_9 and $\epsilon_{10} > 0$ are chosen small enough such that

$$1 - c_0^2\epsilon_8 - 2\epsilon_9 c_0 > 0,$$
$$1 - \epsilon_9 c_0 - \epsilon_{10} C_1 > 0,$$

we have an inequality of the form

$$\frac{d}{dt}E_4 + k_2 E_4 \leq 0, \tag{2.34}$$

where

$$\begin{aligned} E_4 = &\parallel \nu \parallel^2_{H^1} + \parallel \eta \parallel^2_{H^1} + \parallel \partial_t\eta \parallel^2 + \epsilon_{10} \parallel \nu \parallel^2 \\ &+ \epsilon_{11}\Big(\parallel \eta \parallel^2_{H^1} + 2(\partial_t\eta, \eta)\Big) + \epsilon_{12}\Big(\parallel \partial_t\eta \parallel^2_{H^{-1}} + \parallel \eta \parallel^2 \Big). \end{aligned}$$

Applying Gronwall's inequality to (2.34) and using the fact that for a sufficiently small $\epsilon_9 > 0$ there exists $k_3 > 0$ such that

$$k_3^{-1} \parallel (\nu(t), \zeta_\eta(t)) \parallel^2_{\Phi_0} \leq E_4(t) \leq k_3 \parallel (\nu(t), \zeta_\eta(t)) \parallel^2_{\Phi_0},$$

we have

$$\parallel (\nu(t), \zeta_\eta(t)) \parallel^2_{\Phi_0} \leq Ce^{-k_2 t} \parallel (u(0), \zeta_\eta(0) \parallel^2_{\Phi_0}, C > 0. \tag{2.35}$$

Then, $\parallel (\nu(t), \zeta_\eta(t)) \parallel_{\Phi_0}$ tends to 0 as $t \longrightarrow +\infty$.

Multiplying (2.27) by $-2\Delta\partial_t\omega$ and (2.28) by $-2\Delta\partial_t\xi$, integrating over Ω, summing the two resulting equations, we have

$$\frac{d}{dt}\Big(\parallel \omega \parallel^2_{H^2} + \parallel \xi \parallel^2_{H^2} + \parallel \partial_t\xi \parallel^2_{H^1} \Big) + \parallel \partial_t\omega \parallel^2_{H^1} + 2 \parallel \partial_t\xi \parallel^2_{H^2} \leq C_1 \parallel f'(u)\nabla u \parallel^2,$$
$$C_1 > 0. \tag{2.36}$$

Multiply (2.28) by $2\partial_t\xi$ and integrate over Ω. We obtain

$$\frac{d}{dt}\Big(\parallel \xi \parallel^2_{H^1} + \parallel \partial_t\xi \parallel^2 \Big) + \parallel \partial_t\xi \parallel^2_{H^1} \leq c_0 \parallel \partial_t\omega \parallel^2, C_2 > 0. \tag{2.37}$$

Summing (2.36) and ϵ_{11}(2.37), where $\epsilon_{11} > 0$ such that

$$1 - c_0^2\epsilon_{11} > 0,$$

we have

$$\frac{d}{dt}E_5 \leq C_1 \parallel f'(u)\nabla u \parallel^2, \tag{2.38}$$

where

$$E_5 = \parallel \omega \parallel_{H^2}^2 + \parallel \xi \parallel_{H^2}^2 + \parallel \partial_t \xi \parallel_{H^1}^2 + \epsilon_{12}\Big(\parallel \xi \parallel_{H^1}^2 + \parallel \partial_t \xi \parallel^2 \Big)$$

satisfies

$$C \parallel (\omega(t), \zeta_\xi(t)) \parallel_{\Phi_1}^2 \leq E_5(t). \tag{2.39}$$

We thus deduce from (2.29), (2.38) and (2.39) that

$$\parallel (\omega(t), \zeta_\xi(t)) \parallel_{\Phi_1}^2 \leq C(T^2+1)Q^2(\parallel u_0 \parallel_{H^1}, \parallel \alpha_0 \parallel_{H^1}, \parallel \alpha_1 \parallel), \quad t \geq 0, \tag{2.40}$$

Then, $\parallel (\omega(t), \zeta_\xi(t)) \parallel_{\Phi_1}$ is regularizing, as $t \longrightarrow +\infty$. □

In order to end this section, we give some lemmas which allow to obtain a uniform estimate of $\parallel \partial_t^2 u \parallel_{H^{-1}}$ that we need in the third section.

Lemma 2.1. *Let the assumptions of Theorem 2.3 hold and (u, α) be a solution of (2.1)-(2.4) such that $(u(0), \zeta_\alpha(0)) \in \Phi_1$. Then, the following estimate*

$$\begin{aligned}
&\|u(t)\|_{H^1}^2 + \parallel \zeta_\alpha(t) \parallel_{\varepsilon(1)}^2 \\
&+ \int_0^t \Big(\|u(s)\|_{H^1}^2 + \parallel \partial_t u(s) \parallel_{H^{-1}}^2 + \|\partial_t \alpha\|_{H^{-1}}^2 + \parallel \partial_t^2 \alpha(s) \parallel_{H^{-1}}^2 \Big) e^{-\beta(t-s)} ds \\
&\leq Q(\parallel (u(0), \zeta_\alpha(0)) \parallel_{\Phi_1}) e^{-\beta t} + C,
\end{aligned} \tag{2.41}$$

where Q is a monotonic function, is valid.

Proof. Multiply (2.1) by $2(-\Delta)^{-1}\partial_t u$ and (2.2) by $2(-\Delta)^{-1}\partial_t \alpha$, integrate over Ω and sum the two resulting equations. We have

$$\frac{d}{dt}\Big(\|u\|^2 + \|\alpha\|^2 + \|\partial_t \alpha\|_{H^{-1}}^2 \Big) + \|\partial_t u\|_{H^{-1}}^2 + 2\|\partial_t \alpha\|^2 \leq C. \tag{2.42}$$

Multiplying (2.1) by $2u$ and (2.2) by 2α and integrating over Ω, we obtain

$$\frac{d}{dt}\|u\|_{H^1}^2 + \|u\|_{H^1}^2 \leq C + c_0 \|\partial_t \alpha\|^2, \tag{2.43}$$

$$\frac{d}{dt}\Big(\|\alpha\|_{H^1}^2 + 2(\alpha, \partial_t \alpha) \Big) + \|\alpha\|_{H^1}^2 \leq \|\partial_t u\|_{H^{-1}}^2 + 2\|\partial_t \alpha\|^2. \tag{2.44}$$

Multiply (2.2) by $2(-\Delta)^{-1}\partial_t^2 \alpha$ and integrate over Ω. We obtain

$$\frac{d}{dt}\Big(\|\partial_t \alpha\|^2 + 2(\alpha, \partial_t \alpha) \Big) + \|\partial_t^2 \alpha\|_{H^{-1}}^2 \leq \|\partial_t u\|_{H^{-1}}^2 + 2\|\partial_t \alpha\|^2. \tag{2.45}$$

Summing (2.42), ϵ_{12}(2.43), ϵ_{13}(2.44) and ϵ_{14}(2.45) where $\epsilon_{12}, \epsilon_{13}$ and $\epsilon_{14} > 0$ are chosen small enough such that

$$\begin{aligned}
&1 - \epsilon_{13} - \epsilon_{14} > 0. \\
&1 - 2\epsilon_{13} - \epsilon_{12} c_0 - 2\epsilon_{14} > 0,
\end{aligned}$$

we have

$$\frac{d}{dt}E_6 + \beta E_6 + \frac{\epsilon_{12}}{2}\|u(t)\|_{H^1}^2 + C_2 \parallel \partial_t u \parallel_{H^{-1}}^2 + \epsilon_{14} \parallel \partial_t^2 \alpha \parallel_{H^{-1}}^2 \leq C, \tag{2.46}$$

where

$$E_6 = \|u\|^2 + \|\alpha\|^2 + \frac{1}{2}\|\partial_t\alpha\|^2_{H^{-1}} + \epsilon_{12}\|u\|^2_{H^1}$$
$$+ \epsilon_{13}(\|\alpha\|^2_{H^1} + 2(\alpha, \partial_t\alpha)) + \epsilon_{14}(\|\partial_t\alpha\|^2 + 2(\alpha, \partial_t\alpha)).$$

Moreover, for a sufficiently small ϵ_{13} and ϵ_{14}, we have

$$C^{-1}(\|u(t)\|^2_{H^1} + \| \alpha(t) \|^2_{H^1} + \| \partial_t\alpha(t) \|^2)$$
$$\leq E_6(t) \leq C(\|u(t)\|^2_{H^1} + \| \alpha(t) \|^2_{H^1} + \| \partial_t\alpha(t) \|^2). \tag{2.47}$$

Applying Gronwall's inequality, we have

$$E_6(t) + \int_0^t \Big(\|u(t)\|^2_{H^1} + \| \partial_t u \|^2_{H^{-1}} + \| \partial_t^2\alpha \|^2_{H^{-1}} \Big)e^{-\beta(t-s)}ds$$
$$\leq E_6(0)e^{-\beta t} + C. \tag{2.48}$$

Thanks to estimate (2.47), we have

$$\|u(t)\|^2_{H^1} + \| \alpha(t) \|^2_{H^1} + \| \partial_t\alpha(t) \|^2$$
$$+ \int_0^t \Big(\|u(s)\|^2_{H^1} + \| \partial_t u(s) \|^2_{H^{-1}} + \|\partial_t\alpha\|^2_{H^{-1}} + \| \partial_t^2\alpha(s) \|^2_{H^{-1}} \Big)e^{-\beta(t-s)}ds$$
$$\leq C + Q(\|(u(0), \zeta_\alpha(0))\|_{\Phi_1})e^{-\beta t} + C.$$

□

Lemma 2.2. *Let the assumptions of Theorem 2.3 hold and (u, α) be a solution of (2.1)-(2.4) with initial data $(u(0), \zeta_\alpha(0)) \in \Phi_1$. Then the following estimate*

$$\| \alpha(t) \|^2_{H^1} + \| \partial_t\alpha(t) \|^2_{H^1} + \int_0^t \|\partial_t^2\alpha(t)\|^2 e^{-\beta(t-s)}ds$$
$$\leq Q(\| (u(0), \zeta_\alpha(0)) \|_{\Phi_1})e^{-\beta t} + C, \tag{2.49}$$

where Q is a monotonic function, is valid.

Proof. Multiply (2.1) by $2\partial_t\alpha$ and integrate over Ω. We have

$$\frac{d}{dt}\Big(\|\alpha\|^2_{H^1} + \|\partial_t\alpha\|^2\Big) + \|\partial_t\alpha\|^2_{H^1} \leq c_0\|\partial_t u\|^2. \tag{2.50}$$

Multiplying (2.1) by $2\partial_t^2\alpha$ and integrate over Ω. We have

$$\frac{d}{dt}\Big(\|\partial_t\alpha\|^2_{H^1} - 2(\partial_t\alpha, \Delta\alpha)\Big) + \|\partial_t^2\alpha\|^2 \leq \|\partial_t u\|^2 + 2\|\partial_t\alpha\|^2_{H^1}. \tag{2.51}$$

Summing (2.50), ϵ_{15}(2.14), ϵ_{16}(2.51) where ϵ_{15} and ϵ_{16} are small enough, we get

$$\frac{d}{dt}E_7 + \beta E_7 + C_3\|\partial_t^2\alpha\|^2 \leq C_4\|\partial_t u\|^2. \tag{2.52}$$

Applying Gronwall's inequality, we have

$$E_7(t) + \int_0^t \|\partial_t^2\alpha(s)\|^2 e^{-\beta(t-s)}ds \leq E_7(0) + \int_0^t \|\partial_t u(s)\|^2 e^{-\beta(t-s)}ds. \tag{2.53}$$

There exists $C>0$ such that

$$C^{-1}(\|\partial_t\alpha(t)\|_{H^1}^2+\|\alpha(t)\|_{H^1}^2)\leq E_7(t)\leq C(\|\partial_t\alpha(t)\|_{H^1}^2+\|\alpha(t)\|_{H^1}^2). \tag{2.54}$$

Thanks to above estimate and estimate (2.12), we obtain estimate (2.49). $\square$

We differentiate the two equations of system (2.1)-(2.4) with respect to t and set $p(t)=\partial_t u$ and $q=\partial_t\alpha$. Then, (p,q) satisfies the following system

$$\partial_t p-\Delta p+f'(u)p=\partial_t q, \tag{2.55}$$

$$\partial_t^2 q-\Delta\partial_t q-\Delta q=-\partial_t p. \tag{2.56}$$

Lemma 2.3. *Let the assumptions of Theorem 2.3 hold and (u,α) be a solution of (2.1)-(2.4) with initial data $(u(0),\zeta_\alpha(0))\in\Phi_1$. Then the following estimate*

$$\begin{aligned}&\|\partial_t u(t)\|_{H^{-1}}^2+\|\partial_t\alpha(t)\|_{H^{-1}}^2+\int_t^{t+1}\|\partial_t u(s)\|_{H^{-1}}^2e^{-\beta(t-s)}ds\\ \leq&Q(\|\ (u(0),\zeta_\alpha(0))\ \|_{\Phi_1})e^{-\beta t}+C\end{aligned} \tag{2.57}$$

is valid for $t\geq 1$.

Proof. Multiply (2.1) by $2t(\partial_t u+\epsilon_{17}(-\Delta)^{-1}\partial_t^2 u)$ and integrate over Ω. We have

$$\begin{aligned}\frac{d}{dt}(tE_8)+\beta tE_8+Ct\|\partial_t u\|^2\leq&C_1t\|u(t)\|_{H^1}^2+C_2t\|\partial_t u(t)\|_{H^{-1}}^2+C_3t\|\partial_t\alpha(t)\|_{H^{-1}}^2\\ &+C_4t\|\partial_t^2\alpha(t)\|_{H^{-1}}^2+C5t+E_8,\end{aligned}$$

where

$$\begin{aligned}E_8(t)=&\|u(t)\|_{H^1}^2+\epsilon_{16}\Big(\|\partial_t u(t)\|_{H^{-1}}^2+2(u,\partial_t u)+2(f(u)-\partial_t\alpha,(-\Delta)^{-1}\partial_u)\Big)\\ &+4\|f(u)\|_{H^{-1}}^2+4\|\partial_t\alpha(t)\|_{H^{-1}}^2.\end{aligned} \tag{2.58}$$

Applying Gronwall's inequality, we have

$$\begin{aligned}&tE_8(t)+\int_0^t s\|\partial_t u(s)\|^2e^{-\beta(t-s)}ds\\ \leq&C(t+1)\int_0^t\Big(1+\|\partial_t u(s)\|_{H^{-1}}^2+C_3\|\partial_t\alpha(s)\|_{H^{-1}}^2\\ &+C_4\|\partial_t^2\alpha(s)\|_{H^{-1}}^2\Big)e^{-\beta(t-s)}ds.\end{aligned} \tag{2.59}$$

Thanks to (2.41) and the fact that there exists $C>0$ such that

$$C^{-1}(\|\partial_t u(t)\|_{H^{-1}}^2+\|\partial_t\alpha(t)\|_{H^{-1}}^2)\leq E_8(t), \tag{2.60}$$

(2.59) implies estimate (2.57). $\square$

Lemma 2.4. *Let the assumptions of Theorem 2.3 hold and (u,α) be a solution of (2.1)-(2.4) with initial data $(u(0),\zeta_\alpha(0))\in\Phi_1$. Then the following estimate*

$$\begin{aligned}&\|\ \partial_t u(t)\ \|^2+\|\ \partial_t\alpha(t)\ \|^2+\|\partial_t^2\alpha(t)\|^2+\|\partial_t^2\alpha(t)\|_{H^{-1}}^2\\ &+\int_t^{t+1}\Big(\ \|\ \partial_t^2 u(s)\ \|_{H^{-1}}^2+\|\partial_t^3\alpha(t)\|_{H^{-1}}^2\Big)e^{-\beta(t-s)}ds\\ \leq&Q(\|\ (u(0),\zeta_\alpha(0))\ \|_{\Phi_1})e^{-\beta t}+C\end{aligned} \tag{2.61}$$

is valid for $t\geq 1$.

Proof. Multiply (2.55) by $2t(-\Delta)^{-1}\partial_t p$ and (2.56) by $2t(-\Delta)^{-1}\partial_t q$, integrate over Ω and sum the two resulting equations. We have

$$t\frac{d}{dt}\Big(\| p \|^2 + \|\partial_t q\|^2_{H^{-1}} + \|q\|^2\Big) + t \| \partial_t p \|^2_{H^{-1}} + 2t\|\partial_t q\|^2 \le C_1 t \| p \|^2_{H^{-1}} . \tag{2.62}$$

Owing to (2.55), we have

$$-\Delta p = -\partial_t p - f'(u)p + \partial_t q. \tag{2.63}$$

Multiply (2.63) par $2(-\Delta)^{-1}p$ and integrate over Ω. We have

$$\|p\|^2 \le C_2\|\partial_t p\|^2_{H^{-1}} + C_3\|p\|^2_{H^{-1}} + C_4\|\partial_t q\|^2_{H^{-1}}. \tag{2.64}$$

Multiply (2.56) by $2(-\Delta)^{-1}\partial_t^2 q$, and integrate over Ω. We obtain

$$\frac{d}{dt}\Big(\|\partial_t q\|^2 + 2(q, \partial_t q)\Big) + \|\partial_t^2 q\|^2_{H^{-1}} \le \|\partial_t p\|^2_{H^{-1}} + 2\|\partial_t q\|^2 \tag{2.65}$$

Multiply (2.56) by $2(-\Delta)^{-1}q$ and integrate over Ω. We have

$$\frac{d}{dt}\Big(\|q\|^2 + 2(-\Delta)^{-1}q, \partial_t q)\Big) + \|q\|^2 \le C_2\|\partial_t p\|^2_{H^{-1}} + 2\|\partial_t q\|^2_{H^{-1}}. \tag{2.66}$$

Summing (2.62), $\epsilon_{18}t(2.65)$, and $\epsilon_{19}t(2.66)$ where ϵ_{18} and $\epsilon_{19} > 0$ are small enough, we have an inequality on the form

$$\begin{aligned}&\frac{d}{dt}(tE_9) + \beta t E_9 + C_1 t \| \partial_t p \|^2_{H^{-1}} + C_3 t \| \partial_t^2 q \|^2_{H^{-1}}\\ \le& C_5 t(\| p(s) \|^2_{H^{-1}} + \| \partial_t q(s) \|^2_{H^{-1}}) + E_9,\end{aligned}$$

where

$$\begin{aligned}E_9 =& \|p\|^2 + \|\partial_t q\|^2_{H^{-1}} + \|q\|^2\\ &+ \epsilon_{18}\Big(\|\partial_t q\|^2 + 2(q, \partial_t q)\Big) + \epsilon_{19}\Big(\|q\|^2 + 2((-\Delta)^{-1}q, \partial_t q)\Big).\end{aligned}$$

Applying Gronwall's inequality, we get

$$\begin{aligned}&tE_9(t) + \int_0^t s(\| \partial_t p(s) \|^2_{H^{-1}} + t \| \partial_t^2 q(s) \|^2_{H^{-1}})e^{-\beta(t-s)}ds\\ \le& \int_0^t s \| p(s) \|^2_{H^{-1}} e^{-\beta(t-s)}ds + \int_0^t E_9(s)e^{-\beta(t-s)}ds.\end{aligned} \tag{2.67}$$

There exists $C > 0$ such that

$$\begin{aligned}&C^{-1}\Big(\|p\|^2 + \|\partial_t q\|^2 + \|\partial_t q\|^2_{H^{-1}} + \|q\|^2\Big)\\ \le& E_9(t) \le C\Big(\|p\|^2 + \|\partial_t q\|^2 + \|\partial_t q\|^2_{H^{-1}} + \|q\|^2\Big).\end{aligned}$$

Thanks to the above estimates, (2.67) can be written as follows

$$\begin{aligned}&t\Big(\|p(t)\|^2 + \|\partial_t q(t)\|^2 + \|\partial_t q(t)\|^2_{H^{-1}} + \|q(t)\|^2\Big)\\ &+ \int_0^t s(\| \partial_t p(s) \|^2_{H^{-1}} + t \| \partial_t^2 q(s) \|^2_{H^{-1}})e^{-\beta(t-s)}ds\\ \le& \int_0^t s(\| p(s) \|^2_{H^{-1}} + \| \partial_t q(s) \|^2_{H^{-1}})e^{-\beta(t-s)}ds\\ &+ \int_0^t (\|p(s)\|^2 + \|\partial_t q(s)\|^2 + \|\partial_t q(s)\|^2_{H^{-1}} + \|q(s)\|^2)e^{-\beta(t-s)}ds,\end{aligned}$$

which implies

$$
\begin{aligned}
&\|p(t)\|^2 + \|\partial_t q(t)\|^2 + \|\partial_t q(t)\|^2_{H^{-1}} + \|q(t)\|^2 \\
&+ \int_t^{t+1} (\| \partial_t p(s) \|^2_{H^{-1}} + t \| \partial_t^2 q(s) \|^2_{H^{-1}}) e^{-\beta(t-s)} ds \\
\le & C \frac{t+1}{t} \int_0^t (\|p(s)\|^2 + \|\partial_t q(s)\|^2 + \|\partial_t q(s)\|^2_{H^{-1}} + \|q(s)\|^2) e^{-\beta(t-s)} ds. \qquad (2.68)
\end{aligned}
$$

Thanks to estimates (2.12), (2.41) and (2.49), we obtain (2.61). □

Lemma 2.5. *Let the assumptions of Theorem 2.3 hold and (u, α) be a solution of (2.1)-(2.4) with initial data $(u(0), \zeta_\alpha(0)) \in \Phi_1$. Then the following estimate*

$$
\begin{aligned}
&\| \partial_t u(t) \|^2_{H^1} + \| \partial_t u(t) \|^2 + \int_t^{t+1} \| \partial_t^2 u(s) \|^2 \; e^{-\beta(t-s)} ds \\
\le & Q(\| (u(0), \zeta_\alpha(0)) \|_{\Phi_1} \; e^{-\beta t} + C \qquad (2.69)
\end{aligned}
$$

is valid for $t \ge 1$.

Proof. Multiplying equation (2.55) by $2\partial_t p$, integrating over Ω, we obtain

$$
\frac{d}{dt}\Big(t(\|p\|^2_{H^1} + \|p\|^2) \Big) + \beta t (\|p\|^2_{H^1} + \|p\|^2) \le C_1 t \|p\|^2_{H^1} + 2t\|\partial_t q\|^2 + \|p\|^2_{H^1} + \|p\|^2.
$$

Applying Gronwall's inequality, owing to estimates (2.18) and (2.49), we have the required estimate. □

Proposition 2.1. *Let the assumptions of Theorem 2.3 hold and (u, α) be a solution of the system (2.1)-(2.4) with initial data $(u(0), \zeta_\alpha(0)) \in \Phi_1$. Then the following estimate*

$$
\begin{aligned}
&\| \partial_t^2 u \|^2_{H^{-1}} + \| \partial_t^2 \alpha \|^2_{H^{-1}} + \int_0^t \| (\partial_t^3 u(s) \|^2_{H^{-1}} \; e^{-\beta(t-s)} ds \\
\le & C + Q(\| (u(0), \zeta_\alpha(0)) \|_{\Phi_1}) e^{-\beta t}, \qquad (2.70)
\end{aligned}
$$

where Q is a monotone increasing function and $t \ge 2$, is valid.

Proof. We differentiate equation (2.55) with respect to t and set $w(t) = \partial_t p = \partial_t^2 u$. We obtain

$$
\partial_t w - \Delta w - f''(u) p^2 + f'(u) w = \partial_t^2 q.
$$

Multiplying the above equation by $2(t-1)(-\Delta)^{-1} w$ and integrating over Ω, we find

$$
\begin{aligned}
&\frac{d}{dt}\Big((t-1) E_{10} \Big) + \beta (t-1) E_{10} + \frac{t-1}{2} \|\partial_t w\|^2_{H^{-1}} \\
\le \; & C_1 (t-1) \|p\|^4_{H^1} + C_3 (t-1) \|w\|^2_{H^{-1}} \\
& + C_2 (t-1) \|\partial_t^2 q\|^2_{H^{-1}} + \|w\|^2 + \|w\|^2_{H^{-1}}, \qquad (2.71)
\end{aligned}
$$

where

$$
E_{10} = \|w\|^2 + \|w\|^2_{H^{-1}}.
$$

Thanks to estimate (2.69) which gives a uniform estimate of $\|\partial_t u\|_{H^1}$, we have

$$
(t-1)\|p\|^4_{H^1} \le (t+1) \Big(Q(\|(u(0), \zeta_\alpha(0))\|_{\Phi_1}) e^{-\beta t} + C \Big), \qquad (2.72)
$$

where Q is an appropriate function and $t \geq 1$. Inserting above estimate into (2.71) and applying Gronwall's inequality, we have

$$
\begin{aligned}
&(t-1)\Big(E_{10}(t) + \int_t^{t+1} \|\partial_t w\|^2_{H^{-1}} e^{-\beta(t-s)} ds\Big) \\
\leq & C(t+1) \int_0^t (\|w\|^2 + \|\partial_t w\|^2_{H^{-1}} + \|\partial_t^2 q\|_{H^{-1}} + \|(u(0), \zeta_\alpha(0))\|_{\Phi_1}) e^{-\beta t} \\
& + C) e^{-\beta(t-s)} ds.
\end{aligned}
$$

Thanks to estimates (2.18) and (2.49), we find the required estimate. This finishes the proof. □

3. The perturbed hyperbolic Caginalp system

In this section, we consider the hyperbolic system associated with the system (1.1)-(1.5) for $\epsilon > 0$. As for the study of the unperturbed system, we give some results that are needed on order to study the continuity of exponential attractors.

Theorem 3.1 (Existence)**.** *We assume that* $(u_0, u_1, \alpha_0, \alpha_1) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$. *Then, the system (1.1)-(1.5) possesses at least one solution* (u, α) *such that* $u, \alpha \in L^\infty(\mathbb{R}_+; H_0^1(\Omega))$, $\partial_t u \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$, $\partial_t \alpha \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, $\forall T > 0$.

Proof. Multiply (1.1) by $2\partial_t u$ and (1.2) by $2\partial_t \alpha$, integrate over Ω and sum the two resulting equations. We obtain

$$\frac{d}{dt} E_{11} + 2 \parallel \partial_t u \parallel^2 + 2 \parallel \partial_t \alpha \parallel^2_{H^1} = 0, \tag{3.1}$$

where

$$E_{11} = \parallel u \parallel^2_{H^1} + \epsilon \parallel \partial_t u \parallel^2 + \parallel \alpha \parallel^2_{H^1} + \parallel \partial_t \alpha \parallel^2 + 2(F(u), 1).$$

The existence of the solution is based on the estimate (3.1) and a standard Galerkin scheme. □

Theorem 3.2 (Uniqueness)**.** *Under the assumptions of the Theorem 3.1, the system (1.1)-(1.5) possesses a unique solution* (u, α) *such that* $u, \alpha \in L^\infty(\mathbb{R}_+; H_0^1(\Omega))$, $\partial_t u \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$ *and* $\partial_t \alpha \in L^\infty(\mathbb{R}_+; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$.

Proof. Let $(u^{(1)}, \alpha^{(1)})$ and $(u^{(2)}, \alpha^{(2)})$ be two solutions of $(1.1) - (1.5)$ with initial data $(u_0^{(1)}(0), u_1^{(1)}(0), \alpha_0^{(1)}(0), \alpha_1^{(1)}(0))$ and $(u_0^{(2)}(0), u_1^{(2)}(0), \alpha_0^{(2)}(0), \alpha_1^{(2)}(0))$, respectively, and set $u = u^{(1)} - u^{(2)}$ and $\alpha = \alpha^{(1)} - \alpha^{(2)}$. Then (u, α) satisfies

$$\epsilon \partial_t^2 u + \partial_t u - \Delta u + f(u^{(1)}) - f(u^{(2)}) = \partial_t \alpha \tag{3.2}$$

$$\partial_t^2 \alpha - \Delta \partial_t \alpha - \Delta \alpha = -\partial_t u. \tag{3.3}$$

Multiplying (3.2) by $2\partial_t u$ and (3.3) by $2\partial_t \alpha$, integrating over Ω, summing the two resulting equations, we have

$$\frac{d}{dt}\Big(\parallel u \parallel^2_{H^1} + \epsilon \parallel \partial_t u \parallel^2 + \parallel \partial_t \alpha \parallel^2 + \parallel \alpha \parallel^2_{H^1} \Big) \leq C_3'' \parallel u \parallel^2_{H^1} . \tag{3.4}$$

We have the continuous dependence of the solution on the initial data, which implies the uniqueness of the solution. □

Theorem 3.3. *We assume that* $(u_0, u_1, \alpha_0, \alpha_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. *Then, the system (1.1)-(1.5) possesses the unique solution* (u, α) *such that* $u, \alpha \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t u \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, $\partial_t \alpha \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ *and* $\partial_t^2 u, \partial_t^2 \alpha \in L^2(0, T; L^2(\Omega))$, $\forall T > 0$.

Proof. It follows from Theorem 3.1 that the perturbed system possesses a solution (u, α) such that $u \in L^\infty(R_+; H_0^1(\Omega))$.

Multiply (1.1) by $-2\Delta\partial_t u$ and (1.2) by $-2\Delta\partial_t \alpha$, integrate over Ω, and sum the two resulting equations. We find

$$\frac{d}{dt}E_{12} + 2 \parallel \partial_t u \parallel_{H^1}^2 + 2 \parallel \partial_t \alpha \parallel_{H^2}^2 \leq C(\parallel u \parallel_{H^1}^4 + 1) \parallel u \parallel_{H^2}^2 + \parallel \partial_t u \parallel_{H^1}^2, \quad (3.5)$$

where

$$E_{12} = \epsilon \|\partial_t u\|_{H^1}^2 + \|u\|_{H^2}^2 + \|\alpha\|_{H^2}^2 + \|\partial_t \alpha\|_{H^1}^2. \quad (3.6)$$

We remark that for $n = 2, 3$, we have $H^1 \subset L^6$, so that by Hölder's inequality,

$$\int_\Omega (|u|^2 + 1)|\nabla u||\nabla \partial_t u| dx \leq C(\parallel u \parallel_{H^1}^2 + 1) \parallel u \parallel_{H^2} \parallel \partial_t u \parallel_{H^1}. \quad (3.7)$$

Thanks to above estimate, (3.5) implies

$$\frac{d}{dt}E_{12} + 2 \parallel \partial_t u \parallel_{H^1}^2 + 2 \parallel \partial_t \alpha \parallel_{H^2}^2 \leq C(\parallel u \parallel_{H^1}^4 + 1) \parallel u \parallel_{H^2}^2 + \parallel \partial_t u \parallel_{H^1}^2, \quad (3.8)$$

which yields, owing to $u \in L^\infty(R_+; H_0^1(\Omega))$,

$$\frac{d}{dt}E_{12} + \parallel \partial_t u \parallel_{H^1}^2 + 2 \parallel \partial_t \alpha \parallel_{H^2}^2 \leq C \parallel u \parallel_{H^2}^2 + C. \quad (3.9)$$

Applying Gronwall's inequality, we have $u, \alpha \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t u \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, $\partial_t \alpha \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ and $\partial_t^2 \alpha \in L^\infty(0, T; L^2(\Omega))$, $\forall T > 0$.

Multiplying (1.1) by $2\partial_t^2 u$, integrating over Ω, we obtain an estimate of the form

$$\frac{d}{dt} \parallel \partial_t u \parallel^2 + \epsilon \parallel \partial_t^2 u \parallel^2 \leq c_9 \parallel u \parallel_{H^2}^2 + c_{10} \parallel \partial_t \alpha \parallel^2 + c_{11} \parallel f(u) \parallel^2, \quad (3.10)$$

which yields that $\partial_t^2 u \in L^2(0, T; L^2(\Omega))$, $\forall T > 0$. □

The phase spaces have the form $\Phi_{2+\kappa} = \varepsilon^\kappa(\epsilon) \times \varepsilon^\kappa(1)$ with $\kappa = 0, 1$. The standard energy norms for the perturbed system are

$$\parallel (\zeta_u(t), \zeta_\alpha(t)) \parallel_{\Phi_{2+\kappa}}^2 = \parallel \zeta_u(t) \parallel_{\varepsilon^\kappa(\epsilon)}^2 + \parallel \zeta_\alpha(t) \parallel_{\varepsilon^\kappa(1)}^2 .$$

Thanks to Theorem 3.1 and 3.3, we define the solving semigroup $S_\epsilon(t)$ associated with system (1.1)-(1.5) by

$$\begin{aligned} &S_t(\epsilon) : \Phi_{2+\kappa} \longrightarrow \Phi_{2+\kappa} \\ &(\zeta_{u_0}, \zeta_{\alpha_0}) \longmapsto (\zeta_u(t), \zeta_\alpha(t)), \end{aligned}$$

where $(\zeta_u(t), \zeta_\alpha(t))$ is such that (u, α) is the unique solution of $(1.1) - (1.5)$ with initial data $(\zeta_u(0), \zeta_\alpha(0)) \in \Phi_{2+\kappa}$ for $\kappa = 0, 1$. The following lemma allows to give uniform estimate for $\parallel u \parallel_{H^1}$, $\parallel \partial_t u \parallel$ and $\parallel \partial_t \alpha \parallel$.

Theorem 3.4. *Let the assumptions of Theorem 3.1 hold and (u,α) be the solution of the system (1.1)-(1.5) with initial data $(\zeta_u(0),\zeta_\alpha(0))\in\Phi_2$. Then, the following estimate is valid*

$$\begin{aligned}&\epsilon \parallel \partial_t u \parallel^2 + \parallel u \parallel_{H^1}^2 + \parallel \partial_t\alpha \parallel^2 + \parallel \alpha \parallel_{H^1}^2\\ &+\int_0^t e^{-\beta(t-s)}\Big(\parallel \partial_t u(s) \parallel^2 + \parallel \partial_t\alpha(s) \parallel_{H^1}^2 \Big)ds\\ \leq &Q(\parallel (\zeta_u(0),\zeta_\alpha(0)) \parallel_{\Phi_2})e^{-\beta t}+C,\end{aligned} \tag{3.11}$$

where C and β are independent of ϵ, and Q is a monotonic function.

Proof. Multiplying (1.1) by $2u$ and integrating over Ω, we obtain

$$\frac{d}{dt}\Big(\parallel u \parallel^2 +2\epsilon(\partial_t u,u)\Big)+ \parallel u \parallel_{H^1}^2\leq C''+c_0 \parallel \partial_t\alpha \parallel^2 +2\epsilon \parallel \partial_t u \parallel^2 . \tag{3.12}$$

Multiplying (1.2) by 2α and integrating over Ω, we obtain the following estimate

$$\frac{d}{dt}\Big(\parallel \alpha \parallel_{H^1}^2 +2(\alpha,\partial_t\alpha)\Big)+ \parallel \alpha \parallel_{H^1}^2 \quad\leq\quad c_0 \parallel \partial_t u \parallel^2 +2 \parallel \partial_t\alpha \parallel^2 . \tag{3.13}$$

We sum (3.1), ϵ_{20}(3.12) and ϵ_{21}(3.13) where ϵ_{20} and $\epsilon_{21}>0$ are chosen small enough, we have an inequality of the form

$$\frac{d}{dt}E_{13}+\beta E_{13}+ \parallel \partial_t u \parallel^2 + \parallel \partial_t\alpha \parallel_{H^1}^2\leq C',\quad \beta, C'>0, \tag{3.14}$$

where β is independent of ϵ and

$$\begin{aligned}E_{13}(t) =&E_{11}(t)+\epsilon_{20}\Big(\parallel u(t) \parallel^2 +2\epsilon(\partial_t u(t),u(t))\Big)\\ &+\epsilon_{21}\Big(\parallel \alpha \parallel_{H^1}^2 +2(\alpha(t),\partial_t\alpha(t))\Big).\end{aligned}$$

Moreover, for ϵ_{20} and $\epsilon_{21}>0$ sufficiently small, we have, obviously

$$\begin{aligned}&C_1^{-1}(\epsilon\|\partial_t u(t)\|^2+\|u(t)\|_{H^1}^2+\|\partial_t\alpha(t)\|^2+\|\alpha(t)\|_{H^1}^2)\\ \leq\ & E_{13}(t)\leq C_1(\epsilon\|\partial_t u(t)\|^2+\|u(t)\|_{H^1}^2+\|\partial_t\alpha(t)\|^2+\|\alpha(t)\|_{H^1}^2),\end{aligned}$$

where the constant C_1 is independent of ϵ.

Applying Gronwall's inequality to (3.14), we obtain (3.11). □

Theorem 3.5. *We assume that the assumptions of Theorem 3.1 hold and (u,α) is the solution of the system (1.1)-(1.5) such that $(\zeta_u(0),\zeta_\alpha(0))\in\Phi_2$. Then, $(\zeta_u(t),\zeta_\alpha(t))$ verifies the following estimate*

$$\begin{aligned}&\parallel (\zeta_u(t),\zeta_\alpha(t)) \parallel_{\Phi_2}^2+\int_0^t e^{-\beta(t-s)}\Big(\parallel \partial_t u \parallel^2 + \parallel \partial_t\alpha(s) \parallel_{H^1}^2 \Big)ds\\ \leq\ & Q(\parallel (\zeta_u(0),\zeta_\alpha(0)) \parallel_{\Phi_2})e^{-\beta t}+C,\end{aligned} \tag{3.15}$$

where the positive constants C and β are independent of ϵ and Q is monotonic a function.

Proof. Equation (1.1) is the initial and boundary value problem for the singularly perturbed damped hyperbolic equation and can be written on the form

$$\epsilon\partial_t^2 u(t)+\partial_t u(t)-\Delta u(t)=-f(u(t))+\partial_t\alpha(t)=h_{u,\alpha}(t),\quad h_{u,\alpha}(t)|_{\partial\Omega}=u(t)|_{\partial\Omega}=0. \tag{3.16}$$

In order to deduce the uniform energy estimate for the initial and boundary value problem for a singularly perturbed damped hyperbolic equation (3.16), we apply estimate (5.37) in the appendix and have an estimate of the form

$$\begin{aligned}&\| \zeta_u(t) \|^2_{\varepsilon(\epsilon)}+\int_0^t e^{-\beta(t-s)}\| \partial_t u(s) \|^2\,ds\\ \leq& Ce^{-\beta t}(\| \zeta_u(0) \|^2_{\varepsilon(\epsilon)}+\| h_{u,\alpha}(0) \|^2_{H^{-1}})\\ &+C\Big(\int_0^t e^{-\beta(t-s)}(\| \partial_t h(s) \|_{H^{-1}}+\| h(s) \|_{H^{-1}})ds\Big)^2,\end{aligned} \tag{3.17}$$

where the positive constants β and C are independent of ϵ. In order to estimate the last term in the right-hand side of (3.17), we first estimate $\| h_{u,\alpha}(s) \|_{H^{-1}}+\| \partial_t h_{u,\alpha}(s) \|_{H^{-1}}$. Thus, we have

$$\begin{aligned}\| h_{u,\alpha}(s) \|_{H^{-1}}+\| \partial_t h_{u,\alpha}(s) \|_{H^{-1}}\leq& \| f(u) \|_{H^{-1}}+\| \partial_t\alpha \|_{H^{-1}}+\| f'(u)\partial_t u \|_{H^{-1}}\\ &+\| \partial_t^2\alpha \|_{H^{-1}}.\end{aligned} \tag{3.18}$$

Thanks to estimate (3.11), we have uniform estimates for $\| u \|_{H^1}$, $\| \partial_t u \|$, $\| \alpha \|_{H^1}$ and $\| \partial_t\alpha \|$. Then, for $w\in H_0^1(\Omega)$, we have

$$\begin{aligned}|(f(u),w)|\leq&\| u \|_{L^4}(\| u \|^2_{L^4}+1)\| w \|_{L^4}\\ \leq&\| u \|_{H^1}(\| u \|^2_{H^1}+1)\| w \|_{H^1}\leq C\| w \|_{H^1},\end{aligned}$$

and, thus

$$\| f(u) \|_{H^{-1}}\leq C, \tag{3.19}$$

where the constant C is independent of ϵ. For the second term of right-hand side (3.18), we have

$$\| \partial_t\alpha \|_{H^{-1}}\leq C. \tag{3.20}$$

For the third term of right-hand side (3.18), we have

$$\begin{aligned}|(f'(u)\partial_t u,w)|\leq&(\| u \|^2_{L^6}+1)\| \partial_t u \|\| w \|_{L^6}\\ \leq&(\| u \|^2_{H^1}+1)\| \partial_t u \|\| w \|_{H^1}\leq C\| w \|_{H^1},\end{aligned}$$

which yields

$$\| f'(u)\partial_t u \|_{H^{-1}}\leq C. \tag{3.21}$$

From equation (1.2), we have

$$\| \partial_t^2\alpha \|_{H^{-1}}\leq\| \partial_t\alpha \|_{H^1}+\| \alpha \|_{H^1}+\| \partial_t u \|_{H^{-1}}\leq C \tag{3.22}$$

Thanks to estimates (3.19) – (3.22), we have

$$\| h_{u,\alpha}(s) \|_{H^{-1}}+\| \partial_t h_{u,\alpha}(s) \|_{H^{-1}}\leq C+\| \partial_t\alpha \|_{H^1}, \tag{3.23}$$

where C is independent of ϵ.

Inserting (3.23) into the right-hand side of (3.17), we have, owing to (3.11),

$$\| \zeta_u(t) \|^2_{\varepsilon(\epsilon)} + \int_0^t e^{-\beta(t-s)} \| \partial_t u(s) \|^2 \, ds \quad \leq \quad Ce^{-\beta t} + C, \tag{3.24}$$

where the positive constants C is independent of ϵ.

In order to obtain the desired estimate for $\| \zeta_\alpha(t) \|_{\varepsilon(1)}$, we multiply (1.2) by $2(I + (-\Delta)^{-1})\partial_t \alpha$ and integrate over Ω. We have

$$\frac{d}{dt}(\| \zeta_\alpha \|^2_{\varepsilon(1)} + \| \alpha \|^2) + \| \partial_t \alpha \|^2_{H^1} + \| \partial_t \alpha \|^2 \leq C \| \partial_t u \|^2 . \tag{3.25}$$

Summing (3.25) and ϵ_{22}(3.13) where $\epsilon_{22} > 0$ is chosen small enough, we have an inequality of the form

$$\frac{d}{dt} E_{14} + \beta E_{14} + \| \partial_t \alpha \|^2_{H^1} \leq C \| \partial_t u \|^2, \tag{3.26}$$

where

$$E_{14} = \| \zeta_\alpha \|^2_{\varepsilon(1)} + \| \alpha \|^2 + \epsilon_{22} \Big(\|\alpha\|^2_{H^1} + 2(\alpha, \partial \alpha) \Big).$$

Applying Gronwall's inequality, we obtain

$$E_{14}(t) + \int_0^t \| \partial_t \alpha(s) \|^2_{H^1} e^{-\beta(t-s)} ds \leq E_{14}(0) + C \int_0^t \| \partial_t u(s) \|^2 e^{-\beta(t-s)} ds.$$

Moreover, for $\epsilon_{22} > 0$ sufficiently small, we have, obviously

$$C_1^{-1} \|\zeta_\alpha(t)\|^2_{\varepsilon(1)} \leq E_{14}(t) \leq C_1 \|\zeta_\alpha(t)\|^2_{\varepsilon(1)}.$$

Thanks to the above inequality and estimate (3.11), we have

$$\|\zeta_\alpha(t)\|^2_{\varepsilon(1)} + \int_0^t \| \partial_t \alpha(s) \|^2_{H^1} e^{-\beta(t-s)} ds \leq Q(\| (\zeta_u(0), \zeta_\alpha(0)) \|_{\Phi_2}) e^{-\beta t} + C, \tag{3.27}$$

where the positive constants C,C_1 and β are independent of ϵ. Combining (3.24) and (3.27) we obtain the desired estimate. This finishes the proof. □

Theorem 3.6. *The semigroup associated with system (2.1)-(2.4) is dissipative in Φ_2, i.e., it possesses a bounded absorbing set $\mathcal{B}^1_{R_0}(\epsilon)$ in Φ_2.*

In order to prove this Theorem, use estimate (3.15) and make as in the proof of Theorem (2.6).

Note

$$B^1_{R_0}(\epsilon) = \{ (\zeta_u, \zeta_\alpha) \in \Phi_2 : \| (\zeta_u, \zeta_\alpha) \|_{\Phi_2} \leq R_0 \}, \tag{3.28}$$

the bounded absorbing set for $S_t(\epsilon)$ in phases space $\varepsilon(\epsilon) \times \varepsilon(1)$, where R_0 is large enough.

Lemma 3.1. *Let the assumptions of Theorem 3.1 hold and (u, α) be the solution of system (1.1)-(1.5) with initial data $(\zeta_u(0), \zeta_\alpha(0)) \in B^1_{R_0} \cap \Phi_3$. Then, $(u(t), \alpha(t))$ verifies the following estimate*

$$\begin{aligned}
&\| u(t) \|^2_{H^2} + \epsilon \| \partial_t u(t) \|^2_{H^1} + \| \alpha(t) \|^2_{H^2} + \| \partial_t \alpha(t) \|_{H^1} \\
&+ \int_0^t (\| \partial_t u(s) \|^2_{H^1} + \| \partial_t \alpha(s) \|^2_{H^2}) e^{-\beta(t-s)} ds \\
&\leq Q(\| (\zeta_u(0), \zeta_\alpha(0)) \|_{\Phi_3}) e^{-\beta t} + C,
\end{aligned} \tag{3.29}$$

where the positive constants β and C are independent of ϵ, and Q is a monotonic function.

Proof. Multiply (1.1) by $-2\Delta u$ and (1.2) by $-2\Delta\alpha$ and integrate over Ω. We have

$$\frac{d}{dt}\Big(\parallel u \parallel_{H^1}^2 + 2\epsilon(\nabla u, \nabla\partial_t u)\Big) + \parallel u \parallel_{H^2} \leq C_1 + c_0 \parallel \partial_t\alpha \parallel_{H^1}^2 + 2\epsilon \parallel \partial_t u \parallel_{H^1}^2, \tag{3.30}$$

$$\frac{d}{dt}\Big(\parallel \alpha \parallel_{H^2}^2 + 2(\nabla\alpha, \nabla\partial_t\alpha)\Big) + \parallel \alpha \parallel_{H^2}^2 \leq 2 \parallel \partial_t\alpha \parallel_{H^1}^2 + \parallel \partial_t u \parallel^2 . \tag{3.31}$$

Summing (3.14), ϵ_{23}(3.9), ϵ_{24}(3.30) and ϵ_{25}(3.31) where $\epsilon_{23}, \epsilon_{24}$, and $\epsilon_{25} > 0$ are chosen small enough such that

$$1 - c_0\epsilon_{25} > 0, \qquad \frac{\epsilon_{23}}{2} - 2\epsilon c_0\epsilon_{24} > 0,$$
$$\frac{1}{2} - \epsilon_{24}c_0 - 2\epsilon_{25} > 0, \qquad \epsilon_{24} - \epsilon_{23}C > 0,$$

we have

$$\frac{d}{dt}E_{15}(t) + \beta E_{15} + \frac{\epsilon_{23}}{2} \parallel \partial_t u \parallel_{H^1}^2 + \frac{1}{2} \parallel \partial_t\alpha \parallel_{H^2}^2 \leq C, \tag{3.32}$$

where the positive constants C is independent of ϵ, and

$$\begin{aligned} E_{15} =& \epsilon_{23}E_{12} + E_{13} + \epsilon_{24}\Big(\parallel u \parallel^2 + 2\epsilon(u, \partial_t u)\Big) \\ &+ \epsilon_{25}\Big(\parallel \alpha \parallel_{H^2}^2 + 2(\nabla\alpha, \nabla\partial_t\alpha)\Big), \end{aligned}$$

is such that for some $C > 0$, we have

$$\begin{aligned} &C^{-1}(\epsilon \parallel \partial_t u(t) \parallel_{H^1}^2 + \parallel u(t) \parallel_{H^2}^2 + \parallel \partial_t\alpha(t) \parallel_{H^1}^2 + \parallel \alpha(t) \parallel_{H^2}^2)(t) \\ \leq& E_{15} \leq C(\epsilon \parallel \partial_t u(t) \parallel_{H^1}^2 + \parallel u(t) \parallel_{H^2}^2 + \parallel \partial_t\alpha(t) \parallel_{H^1}^2 + \parallel \alpha(t) \parallel_{H^2}^2). \end{aligned} \tag{3.33}$$

Applying Gronwall's inequality to (3.32), owing to (3.33), we obtain the desired estimate. $\square$

Theorem 3.7. *Let the assumptions of Theorem 3.1 hold and (u, α) be the solution of system (1.1)-(1.5) with initial data $(\zeta_u(0), \zeta_\alpha(0)) \in B^1_{R_0}(\epsilon) \cap B^2_R(\epsilon)$. Then, $(\zeta_u(t), \zeta_\alpha(t))$ verifies the following estimate*

$$\begin{aligned} &\parallel (\zeta_u(t), \zeta_\alpha(t)) \parallel_{\Phi_3}^2 + \int_0^t (\parallel \partial_t u(s) \parallel_{H^1}^2 + \parallel \partial_t\alpha(s) \parallel_{H^2}^2)e^{-\beta(t-s)}ds \\ \leq& Q(\parallel (\zeta_u(0), \zeta_\alpha(0)) \parallel_{\Phi_3})e^{-\beta t} + C, \end{aligned} \tag{3.34}$$

where the positive constants β and C are independent of ϵ, and Q is a monotonic function.

Proof. We apply estimate (5.28) to deduce the uniform energy estimate for the initial and boundary value problem for a singularly perturbed damped hyperbolic equation (3.16). We have

$$
\begin{aligned}
&\| \zeta_u(t) \|_{\varepsilon^1(\epsilon)} + \int_0^t e^{-\beta(t-s)} \| \partial_t u(s) \|_{H^1}^2 \, d \\
\leq & C e^{-\beta t} (\| \zeta_u(0) \|_{\varepsilon^1(\epsilon)}^2 + \| h_{u,\alpha}(0) \|^2) \\
&+ C \Big(\int_0^t e^{-\beta(t-s)} (\| h_{u,\alpha}(s) \|_{H^1}^2 + \| \partial_t h_{u,\alpha}(s) \|_{H^{-1}}^2) ds \Big),
\end{aligned} \tag{3.35}
$$

where the positive constants C and β are independent of ϵ. In order to estimate the last term in the right-hand side of (3.35), we first estimate $\| h_{u,\alpha}(s) \|_{H^1}^2 + \| \partial_t h_{u,\alpha}(s) \|_{H^{-1}}^2$. We have

$$
\begin{aligned}
\| h_{u,\alpha}(s) \|_{H^1}^2 + \| \partial_t h_{u,\alpha}(s) \|_{H^{-1}}^2 \leq \| f(u) \|_{H^1}^2 + \| \partial_t \alpha \|_{H^1}^2 + \| f'(u) \partial_t u \|_{H^{-1}}^2 \\
+ \| \partial_t^2 \alpha \|_{H^{-1}}^2 .
\end{aligned} \tag{3.36}
$$

Thanks to estimate (3.29), we have uniform estimates for $\| u \|_{H^2}$, $\| \alpha \|_{H^2}$, $\| \partial_t u \|_{H^1}$ and $\| \partial_t \alpha \|_{H^1}$. Then, for $w \in H_0^1(\Omega)$, we have

$$
\begin{aligned}
\| f(u) \|_{H^1} &\leq \| u \|_{L^6} (\| u \|_{L^6}^2 + 1) \\
&\leq C \| u \|_{H^1} (\| u \|_{H^1}^2 + 1) \leq C \| w \|_{H^1} .
\end{aligned}
$$

Hence

$$
\| f(u) \|_{H^1} \leq C, \tag{3.37}
$$

where C depends on R, but is independent of ϵ.

For the third term of right-hand side of (3.35), we have

$$
\begin{aligned}
|(f'(u) \partial_t u, w)| &\leq (\| u \|_{L^4}^2 + 1) \| \partial_t u \|_{L^4} \| w \|_{L^4} \\
&\leq (\| u \|_{H^1}^2 + 1) \| \partial_t u \|_{H^1} \| w \|_{H^1} \leq C \| w \|_{H^1},
\end{aligned}
$$

hence

$$
\| f'(u) \partial_t u \|_{H^{-1}} \leq C, \tag{3.38}
$$

where C depends on R, but is independent of ϵ. For the last term of right-hand side of (3.36), we have

$$
\| \partial_t^2 \alpha \|_{H^{-1}} \leq \| \partial_t \alpha \|_{H^1} + \| \alpha \|_{H^1} + \| \partial_t u \|_{H^{-1}} \leq C, \tag{3.39}
$$

where C depends on R, but is independent of ϵ.

Inserting estimates (3.22) and (3.37) – (3.39) into the right-hand side of (3.36), we have

$$
\| h_{u,\alpha}(s) \|_{H^1}^2 + \| \partial_t h_{u,\alpha}(s) \|_{H^{-1}}^2 \leq C, \tag{3.40}
$$

where C depends on R, but is independent of ϵ. Inserting estimate (3.40) into the right-hand side of (3.35), we have

$$
\| \zeta_u(t) \|_{\varepsilon^1(\epsilon)} + \int_0^t e^{-\beta(t-s)} \| \partial_t u(s) \|_{H^1}^2 \, ds \leq C e^{-\beta t} + C. \tag{3.41}
$$

In order to obtain the desired estimate for $\| \zeta_\alpha(t) \|_{\varepsilon^1(1)}$, we multiply (1.2) by $2(I - \Delta) \partial_t \alpha$ and integrate over Ω. We have

$$
\frac{d}{dt} \Big(\| \zeta_\alpha(t) \|_{\varepsilon^1(1)}^2 + \| \alpha(t) \|_{H^1}^2 \Big) + \| \partial_t \alpha \|_{H^1}^2 + \| \partial_t \alpha \|_{H^2}^2 \leq C \| \partial_t u \|^2 . \tag{3.42}
$$

Summing (3.42) and ϵ_{26}(3.31) where $\epsilon_{26} > 0$ is small enough such that

$$1 - 2\epsilon_{26} > 0,$$

we have

$$\frac{d}{dt}E_{16}(t) + \beta E_{16}(t) + \| \partial_t \alpha(t) \|_{H^2}^2 \leq C \| \partial_t u(t) \|^2,$$

where $C_1 > 0$ and

$$E_{16} = \| \zeta_\alpha(t) \|_{\varepsilon^1(1)}^2 + \| \alpha(t) \|_{H^1}^2 + \epsilon_{26}\Big(\| \alpha \|_{H^2}^2 + 2(\nabla\alpha, \nabla\partial_t\alpha)\Big). \tag{3.43}$$

Applying Gronwall's inequality to above estimate, we obtain

$$E_{16}(t) + \int_0^t \| \partial_t \alpha(s) \|_{H^2}^2 \, e^{-\beta(t-s)} ds \leq C \int_0^t \| \partial_t u(s) \|^2 \, e^{-\beta(t-s)} ds. \tag{3.44}$$

There exists also $C_2 > 0$ such that

$$C_2^{-1} \|\zeta_\alpha(t)\|_{\varepsilon(1)}^2 \leq E_{16}(t) \leq C_2 \| \zeta_\alpha(t) \|_{\varepsilon^1(1)}^2, \tag{3.45}$$

where the constant C_2 is independent of ϵ. Thanks to estimates (3.45) and (3.11), we have

$$\begin{aligned} &\| \zeta_\alpha(t) \|_{\varepsilon^1(1)}^1 + \int_0^t \| \partial_t \alpha(s) \|_{H^2}^2 \, e^{-\beta(t-s)} ds \\ \leq &Q(\| (\zeta_u(0), \zeta_\alpha(0)) \|_{\Phi_3}) e^{-\beta t} + C, \end{aligned} \tag{3.46}$$

where the positive constants C and C_1 are independent of ϵ and Q is a monotonic function. Combining (3.46) and (3.41) we obtain the desired estimate. This finishes the proof. □

Theorem 3.8. *The semigroup $S_t(\epsilon)$ associated with system (1.1)-(1.5) possesses a bounded absorbing set in Φ_3.*

Proof. Let B be a bounded subset of Φ_3 and R be such that $\| (\zeta_{u_0}, \zeta_{\alpha_0}) \|_{\Phi_3} \leq R$, $\forall (\zeta_{u_0}, \zeta_{\alpha_0}) \in B$. Owing to estimate (3.34), we have $\forall t \geq 0$

$$\| (\zeta_u, \zeta_\alpha) \|_{\Phi_3}^2 \leq C', \tag{3.47}$$

which implies that $S_t(\epsilon)$ possesses a bounded absorbing set in Φ_3. □

In the sequel, we will also need more regular solution of system (1.1)-(1.5). To this end, we introduce the set $B_R^2(\epsilon)$ as follows

$$B_R^2(\epsilon) = \{(\zeta_u, \zeta_\alpha) \in \Phi_3 : \| (\zeta_u(t), \zeta_\alpha(t)) \|_{\Phi_3} \leq R\}.$$

Theorem 3.9. *Under the assumptions of Theorem 3.1, the semigroup $S_\epsilon(t)$ associated to (1.1)-(1.5) possesses the global attractor $\mathcal{A}_\epsilon$ which is bounded in Φ_2.*

Proof. To prove this Theorem, we proceed as in the proof of the Theorem 2.8. We decompose the solution $(u, \alpha) \in B_{R_0}^1(\epsilon)$ in the form

$$(u, \alpha) = (\nu, \eta) + (\omega, \xi),$$

where (ν, η) solves

$$\epsilon\partial_t^2\nu + \partial_t\nu - \Delta\nu = \partial_t\eta \tag{3.48}$$
$$\partial_t^2\eta - \partial_t\Delta\eta - \Delta\eta = -\partial_t\nu \tag{3.49}$$
$$\nu = \eta = 0 \quad \text{ on } \Omega$$
$$\zeta_\nu|_{t=0} = [u_0, u_1], \quad \zeta_\eta|_{t=0} = [\alpha_0, \alpha_1],$$

and (ω, ξ) solves

$$\epsilon\partial_t^2\omega + \partial_t\omega - \Delta\omega + f(u) = \partial_t\xi \tag{3.50}$$
$$\partial_t^2\xi - \partial_t\Delta\xi - \Delta\xi = -\partial_t\omega \tag{3.51}$$
$$\omega = \xi = 0 \quad \text{ on } \Omega$$
$$\zeta_\omega|_{t=0} = \zeta_\xi|_{t=0} = 0,$$

and we shall show that

$$\| (\zeta_\nu(t), \zeta_\eta(t)) \|_{\Phi_2} \text{ tends to 0 as } t \longrightarrow +\infty,$$

and

$$\| (\zeta_\omega(t), \zeta_\xi(t)) \|_{\Phi_3} \text{ is regularizing, as } t \longrightarrow +\infty.$$

Multiply (3.48) to $2\partial_t\nu$ and (3.49) by $2\partial_t\eta$, integrate over Ω, and sum the two resulting equations. We have

$$\frac{d}{dt}\Big(\epsilon\|\partial_t\nu\|^2 + \|\nu\|^2_{H^1} + \|\partial_t\eta\|^2 + \|\eta\|^2_{H^1}\Big) + 2\|\partial_t\nu\|^2 + 2\|\partial_t\eta\|^2_{H^1} = 0. \tag{3.52}$$

Multiplying (3.48) to $2(-\Delta)^{-1}\partial_t^2\nu$ and (3.49) by $2(-\Delta)^{-1}\partial_t\eta$ and integrating over Ω, we have

$$\frac{d}{dt}\Big(\|\partial_t\nu\|^2_{H^{-1}} + 2(\nu, \partial_t\nu)\Big) + \epsilon\|\partial_t^2\nu\|^2_{H^{-1}} \leq 2\|\partial_t\nu\|^2 + C_1\|\partial_t\eta\|^2, \tag{3.53}$$
$$\frac{d}{dt}\Big(\|\partial_t\eta\|^2_{H^{-1}} + \|\eta\|^2\Big) + \|\partial_t\eta\|^2_{H^{-1}} \leq C_2\|\partial_t\nu\|^2. \tag{3.54}$$

Multiply (3.48) to 2ν and (3.49) by 2η and integrate over Ω. We have

$$\frac{d}{dt}\Big(\|\nu\|^2 + 2\epsilon(\nu, \partial_t\nu)\Big) + \|\nu\|^2_{H^1} \leq 2\epsilon\|\partial_t\nu\|^2 + C_3\|\partial_t\eta\|^2, \tag{3.55}$$
$$\frac{\partial}{\partial t}\Big(\|\eta\|^2_{H^1} + 2(\eta, \partial_t\eta)\Big) + \|\eta\|^2_{H^1} \leq C_4\|\partial_t\nu\|^2 + 2\|\partial_t\eta\|^2. \tag{3.56}$$

Multiply (3.48) to $2(\Delta)^{-1}\partial_t\nu$ and integrate over Ω. We have

$$\frac{d}{dt}\Big(\epsilon\|\partial_t\nu\|^2_{H^{-1}} + \|\nu\|^2\Big) + \|\partial_t\nu\|^2_{H^{-1}} \leq C_5\|\partial_t\eta\|^2. \tag{3.57}$$

Summing (3.52), ϵ_{27}(3.53), ϵ_{28}(3.54), ϵ_{29}(3.55), ϵ_{30}(3.56) and ϵ_{31}(3.57) where ϵ_{27}, ϵ_{28}, ϵ_{29}, ϵ_{30} and $\epsilon_{31} > 0$ such that

$$\frac{2}{c_0} - \epsilon_{27}C_1 - \epsilon_{29}C_3 - 2\epsilon_{30} - C_5\epsilon_{31} > 0, \tag{3.58}$$
$$2 - 2\epsilon_{27} - \epsilon_{28} - 2\epsilon\epsilon_{29} - C_4\epsilon_{30} > 0, \tag{3.59}$$

we obtain

$$\frac{d}{dt}E_{17}(t)+kE_{17}(t)\leq 0, \tag{3.60}$$

where

$$\begin{aligned} E_{17}(t)=&\epsilon\|\partial_t\nu\|^2+\|\nu\|^2_{H^1}+\|\partial_t\eta\|^2+\|\eta\|^2_{H^1}+\epsilon_{27}\Big(\|\partial_t\nu\|^2_{H^{-1}}+2(\nu,\partial_t\nu)\Big)\\ &+\epsilon_{28}\Big(\|\partial_t\eta\|^2_{H^{-1}}+\|\eta\|^2)\Big)+\epsilon_{29}\Big(\|\nu\|^2+2\epsilon(\nu,\partial_t\nu)\Big)\\ &+\epsilon_{30}\Big(\|\eta\|^2_{H^1}+2(\eta,\partial_t\eta)\Big)+\epsilon_{31}\Big(\epsilon\|\partial_t\nu\|^2_{H^{-1}}+\|\nu\|^2\Big)+\|\partial_t\nu\|^2_{H^{-1}}. \end{aligned}$$

Choosing ϵ_{27}, ϵ_{29} and ϵ_{30} small enough, we have C_5 and $C_6>0$, and there exist k and $C>0$ such that

$$C^{-1}\|(\zeta_\nu(t),\zeta_\eta(t))\|^2_{\Phi_2}\leq E_{17}(t)\leq C\|(\nu(t),\eta(t))\|^2_{\Phi_2}. \tag{3.61}$$

Applying Gronwall inequality to (3.60), we have

$$E_{17}(t)\leq E_{17}(0)e^{-kt},$$

which implies, owing to (3.61)

$$\|(\zeta_\nu(t),\zeta_\eta(t))\|^2_{\Phi_2}\leq C^2\|(\zeta_\nu(0),\zeta_\eta(0))\|^2_{\Phi_2}e^{-kt}.$$

Then, $\|(\zeta_\nu(t),\zeta_\eta(t))\|_{\Phi_2}$ tends to 0, as $t\longrightarrow+\infty$.

It remains to prove that

$$\|\ (\zeta_\omega(t),\zeta_\xi(t))\ \|_{\Phi_3}\text{ is regularizing, as }t\longrightarrow+\infty.$$

Multiplying (3.52) by $-2\Delta\partial_t\omega$ and (3.53) by $-2\Delta\partial_t\xi$, integrating over Ω, and summing the two resulting equalities, we obtain

$$\frac{d}{dt}\Big(\epsilon\|\partial_t\omega\|^2_{H^1}+\|\omega\|^2_{H^2}+2\|\partial_t\xi\|^2_{H^1}+\|\xi\|^2_{H^2}\Big)+\|\partial_t\omega\|^2_{H^1}+\|\partial_t\xi\|^2_{H^2}\leq\|f'(u)\nabla u\|^2. \tag{3.62}$$

Multiply (3.52) by $2\partial_t^2\omega$ and (3.53) by $2\partial_t\xi$ and integrate over Ω. We obtain

$$\frac{d}{dt}\Big(\|\partial_t\omega\|^2+2(\nabla\omega,\nabla\partial_t\omega)\Big)+\epsilon\|\partial_t^2\omega\|^2\leq 2\|\partial_t\omega\|^2_{H^1}+\|\partial_t\xi\|^2+C_5, \tag{3.63}$$

$$\frac{d}{dt}\Big(\|\partial_t\xi\|^2+\|\xi\|^2_{H^1}\Big)+\|\partial_t\xi\|^2_{H^1}\leq C_6\|\partial_t\omega\|^2_{H^1}. \tag{3.64}$$

Summing (3.62), ϵ_{32}(3.63) and ϵ_{33}(3.65) where ϵ_{31} and $\epsilon_{32}>0$ such that

$$\begin{aligned}&1-2\epsilon_{32}-C_6\epsilon_{33}>0,\\ &\epsilon_{32}-2c_0\epsilon_{31}>0,\end{aligned}$$

we have

$$\frac{d}{dt}E_{18}+C_1\|\partial_t\omega\|^2_{H^1}+C_2\|\partial_t\xi\|^2_{H^1}+\epsilon_{32}\|\partial_t\xi\|^2_{H^2}\leq\|f'(u)\nabla u\|^2+C_7. \tag{3.65}$$

where

$$\begin{aligned} E_{18} =& \epsilon\|\partial_t\omega\|^2_{H^1} + \|\omega\|^2_{H^2} + \|\partial_t\xi\|^2_{H^1} + \|\xi\|^2_{H^2} \\ &+ \epsilon_{32}\Big(\|\partial_t\omega\|^2 + 2(\nabla\omega, \nabla\partial_t\omega)\Big) + \epsilon_{33}\Big(\|\partial_t\xi\|^2 + \|\xi\|^2_{H^1}\Big) \end{aligned}$$

Choosing $\epsilon_{32} > 0$ small enough, there exists $C > 0$ such that

$$C^{-1} \parallel (\zeta_\omega(t), \zeta_\xi(t)) \parallel^2_{\Phi_3} \leq E_{18}(t) \leq C \parallel (\zeta_\omega(t), \zeta_\xi(t)) \parallel^2_{\Phi_3} . \tag{3.66}$$

Thanks to (3.66), integrate (3.65), we have

$$\parallel (\zeta_\omega(t), \zeta_\xi(t)) \parallel^2_{\Phi_3} \leq C(T^2 + 1)Q(\|u_0\|_{H^2}, \|u_1\|_{H^1}, \|\alpha_0\|_{H^2}, \|\alpha_1\|_{H^1}).$$

Then, $\parallel (\zeta_\omega(t), \zeta_\xi(t)) \parallel_{\Phi_3}$ is regularizing, as $t \longrightarrow +\infty$. $\square$

4. Estimates on the difference of solutions

In this section, we first establish estimates of the difference between two solutions of the hyperbolic system $(1.1)-(1.5)$, before giving estimates of the difference between the solution $(u^\epsilon, \alpha^\epsilon)$ of the hyperbolic system $(1.1) - (1.5)$ and the solution (u^0, α^0) of the limit parabolic-hyperbolic system $(2.1) - (2.4)$.

Theorem 4.1. *Let the assumptions of Theorem 3.3 hold, $\epsilon \leq 1$ and (u^1, α^1) and (u^2, α^2) be two solutions of the system (1.1)-(1.5) with initial data belonging to $B^1_R(\epsilon)$. Then, the following estimate is valid*

$$\begin{aligned} &\parallel (\zeta_{u^1}(t) - \zeta_{u^2}(t), \zeta_{\alpha^1}(t) - \zeta_{\alpha^2}(t)) \parallel^2_{\Phi_2} \\ \leq &+ \int_0^t \Big(\parallel u(s) \parallel^2_{H^1} + \parallel \partial_t u(s) \parallel^2 + \epsilon \parallel \partial_t^2 u(s) \parallel^2_{H^{-1}} \\ &+ \parallel \alpha(s) \parallel^2_{H^1} + \parallel \partial_t\alpha(s) \parallel^2_{H^1} + \parallel \partial_t^2\alpha(s) \parallel^2_{H^{-1}} \Big) ds \\ &C \parallel (\zeta_{u^1}(0) - \zeta_{u^2}(0), \zeta_{\alpha^1}(0) - \zeta_{\alpha^2}(0)) \parallel^2_{\Phi_2} e^{Kt}, \end{aligned} \tag{4.1}$$

where the positive constants C and K depend on R, but they are independent of ϵ.

Proof. We set $u = u^1 - u^2$ and $\alpha = \alpha^1 - \alpha^2$. Then, (u, α) verifies the following system

$$\epsilon\partial_t^2 u(t) + \partial_t u(t) - \Delta u(t) = -f(u^1(t)) + f(u^2(t)) + \partial_t\alpha(t), \tag{4.2}$$

$$\partial_t^2\alpha(t) - \partial_t\Delta\alpha(t) - \Delta\alpha(t) = -\partial_t u(t), \tag{4.3}$$

where the first equation is the initial and boundary value problem for the singularly perturbed damped hyperbolic equation.

Multiplying (4.2) by $2\partial_t u$ and (4.3) by $2\partial_t\alpha$, integrating over Ω and summing the two resulting equations, we have

$$\frac{d}{dt}E_{19}(t) + \parallel \partial_t u(t) \parallel^2 + \parallel \partial_t\alpha(t) \parallel^2_{H^1} \leq C \parallel u(t) \parallel^2_{H^1}, \tag{4.4}$$

where

$$E_{19}(t) = \epsilon \parallel \partial_t u(t) \parallel^2 + \parallel u(t) \parallel^2_{H^1} + \parallel \partial_t\alpha(t) \parallel^2 + \parallel \alpha(t) \parallel^2_{H^1} .$$

Multiply (4.2) by $2u$ and (4.3) by 2α and integrate over Ω, we find

$$\frac{d}{dt}\Big(\| u \|^2 + 2\epsilon(u, \partial_t u)\Big) + \| u \|^2_{H^1} \leq 2 \| u \|^2 + c_0 \| \partial_t \alpha \|^2 + 2\epsilon \| \partial_t u \|^2, \tag{4.5}$$

$$\frac{d}{dt}\Big(\| \alpha \|^2_{H^1} + 2(\alpha, \partial_t \alpha)\Big) + \| \alpha \|^2_{H^1} \leq c_0 \| \partial_t u \|^2 + 2 \| \partial_t \alpha \|^2 . \tag{4.6}$$

Adding (4.4), ϵ_{34}(4.5) and ϵ_{35}(4.6) where ϵ_{34} and $\epsilon_{35} > 0$ are such that

$$1 - 2\epsilon_{34}\epsilon - \epsilon_{35} c_0 > 0,$$

we have

$$\frac{d}{dt} E_{20} + C_1 \| \partial_t u \|^2 + C_2 \| u \|^2_{H^1} + C_3 \| \alpha \|^2_{H^1} + C_4 \| \partial_t \alpha \|^2_{H^1} \leq K E_{20}, \tag{4.7}$$

where C_i and K are independent of ϵ and

$$\begin{aligned} E_{20} =& \epsilon \| \partial_t u \|^2 + \| u \|^2_{H^1} + \| \partial_t \alpha \|^2 + \| \alpha \|^2_{H^1} \\ &+ \epsilon_{34}\Big(\| u \|^2 + 2\epsilon(u, \partial_t u)\Big) + \epsilon_{35}\Big(\| \alpha \|^2_{H^1} + 2(\alpha, \partial_t \alpha)\Big). \end{aligned}$$

Moreover, for sufficient small values of ϵ_{34} et $\epsilon_{35} > 0$, there exist $C > 0$ such that

$$\begin{aligned} &C^{-1}(\epsilon \| \partial_t u(t) \|^2 + \| u(t) \|^2_{H^1} + \| \partial_t \alpha(t) \|^2 + \| \alpha(t) \|^2_{H^1}) \\ \leq & E_{20}(t) \leq C(\epsilon \| \partial_t u(t) \|^2 + \| u(t) \|^2_{H^1} + \| \partial_t \alpha(t) \|^2 + \| \alpha(t) \|^2_{H^1}). \end{aligned}$$

Applying Gronwall's inequality (4.7), owing to the above estimate, we have

$$\begin{aligned} &\epsilon \| \partial_t u(t) \|^2 + \| u(t) \|^2_{H^1} + \| \partial_t \alpha(t) \|^2 + \| \alpha(t) \|^2_{H^1} \\ &+ \int_0^t (\| u(s) \|^2_{H^1} + \| \partial_t u(s) \|^2 + \| \alpha(s) \|^2_{H^1} + \| \partial_t \alpha(s) \|^2_{H^1}) ds \\ \leq & \| (\zeta_{u^1 - u^2}(0), \zeta_{\alpha^1 - \alpha^2}(0)) \|^2_{\Phi_2} e^{Kt} + C. \end{aligned} \tag{4.8}$$

Multiplying (4.2) by $2(-\Delta)^{-1}\partial_t^2 u$ and integrating over Ω, we obtain

$$\frac{d}{dt} E_{21} + 2\epsilon \|\partial_t^2 u\|^2_{H^{-1}} \leq C_1 \|\partial_t u\|^2 + C_2 \|u\|^2_{H^1} + \|\partial_t^2 \alpha\|^2_{H^{-1}}, \tag{4.9}$$

where

$$E_{21} = \|\partial_t u\|^2_{H^{-1}} + 2(u, \partial_t u) + 2(f(u^1) - f(u^2) - \partial_t \alpha, (-\Delta)^{-1}\partial_t u).$$

Multiply (4.2) by $2(-\Delta)^{-1}\partial_t^2 \alpha$ and integrate over Ω. We obtain

$$\frac{d}{dt} \|\partial_t \alpha\|^2 + \|\partial_t^2 \alpha\|^2_{H^{-1}} \leq C_3 \|\alpha\|^2_{H^1} + C_4 \|\partial_t u\|^2. \tag{4.10}$$

Add (4.9) and ϵ_{36}(4.10) where $\epsilon_{36} > 0$ is such that

$$1 - \epsilon_{36} C_3 > 0,$$

we obtain

$$\frac{d}{dt} E_{22} + 2\epsilon_{36}\epsilon \|\partial_t^2 u\|^2_{H^{-1}} + C_4 \|\partial_t^2 \alpha\|^2_{H^{-1}} \leq C_5(\|\partial_t u\|^2 + \|u\|^2_{H^1} + \|\alpha\|^2_{H^1}), \tag{4.11}$$

where

$$E_{22} = \|\partial_t \alpha\|^2 + \epsilon_{36} E_{21}.$$

Integrate (4.11) from 0 to t, we obtain, thanks to estimate (4.8)

$$\begin{aligned}
&\|\partial_t \alpha(t)\|^2 \\
&+ \epsilon_{36}\Big(\|\partial_t u(t)\|^2_{H^{-1}} + 2(u, \partial_t u(t)) + 2(f(u^1(t)) - f(u^2(t)) - \partial_t \alpha(t), (-\Delta)^{-1}\partial_t u(t))\Big) \\
&+ C\int_0^t (\epsilon\|\partial_t^2 u(s)\|^2_{H^{-1}} + \|\partial_t^2 \alpha(s)\|^2_{H^{-1}})ds \leq C \parallel (\zeta_{u^1-u^2}(0), \zeta_{\alpha^1-\alpha^2}(0)) \parallel^2_{\Phi_2} e^{Kt}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|\partial_t u(t)\|^2_{H^{-1}} + \int_0^t (\epsilon\|\partial_t^2 u(s)\|^2_{H^{-1}} + \|\partial_t^2 \alpha(s)\|^2_{H^{-1}})ds \\
\leq& C \parallel (\zeta_{u^1-u^2}(0), \zeta_{\alpha^1-\alpha^2}(0)) \parallel^2_{\Phi_2} e^{Kt}.
\end{aligned}$$

Combining the above estimate and estimate (4.8) , we have

$$\begin{aligned}
&\|\zeta_u(t)\|^2_{\varepsilon(\epsilon)} + \int_0^t \Big(\epsilon\|\partial_t^2 u(s)\|^2_{H^{-1}} + \|\partial_t u(s)\|^2 \\
&+ \|u\|^2_{H^1} + \|\partial_t^2 \alpha(s)\|^2_{H^{-1}} + \|\partial_t \alpha(s)\|^2_{H^1} + \|\alpha\|^2_{H^1}\Big)ds \\
\leq& C \parallel (\zeta_{u^1-u^2}(0), \zeta_{\alpha^1-\alpha^2}(0))) \parallel^2_{\Phi_2} e^{Kt}.
\end{aligned} \tag{4.12}$$

Multiply (4.3) by $2(-\Delta)^{-1}\partial_t \alpha$, integrate over Ω, we have

$$\frac{d}{dt}\Big(\parallel \alpha \parallel^2 + \parallel \partial_t \alpha \parallel^2_{H^{-1}} \Big) + \parallel \partial_t \alpha \parallel^2 \quad \leq \quad C_1 \parallel \partial_t u \parallel^2 . \tag{4.13}$$

Integrate (4.13) from 0 to t, we find, owing to estimate (4.8)

$$\begin{aligned}
&\parallel \alpha(t) \parallel^2 + \parallel \partial_t \alpha(t) \parallel^2_{H^{-1}} + \int_0^t \parallel \partial_t \alpha(s) \parallel^2 ds \\
\leq& C \parallel (\zeta_{u^1-u^2}(0), \zeta_{\alpha^1-\alpha^2}(0)) \parallel^2_{\Phi_2}) e^{Kt}.
\end{aligned}$$

Combining the above estimate and estimate (4.8), we obtain

$$\begin{aligned}
&\|\zeta_\alpha(t)\|^2_{\varepsilon(1)} + \int_0^t \Big(\epsilon\|\partial_t^2 u(s)\|^2_{H^{-1}} + \|\partial_t u(s)\|^2 + \|u\|^2_{H^1} \\
&+ \|\partial_t^2 \alpha(s)\|^2_{H^{-1}} + \|\partial_t \alpha(s)\|^2_{H^1} + \|\alpha\|^2_{H^1}\Big)ds \\
\leq& C \parallel (\zeta_{u^1-u^2}(0), \zeta_{\alpha^1-\alpha^2}(0)) \parallel^2_{\Phi_2} e^{Kt}.
\end{aligned} \tag{4.14}$$

Combining estimates (4.12) and (4.14), we obtain the result. □

We now show an asymptotic smoothing property for the difference of solutions of system (1.1) – (1.5). To this end, we split the solution (u, α) of system (4.2) – (4.3) as follows

$$(u, \alpha) = (v^1, w^1) + (v^2, w^2),$$

where (v^1, w^1) solves

$$\epsilon\partial_t^2 v^1 + \partial_t v^1 - \Delta v^1 = \partial_t w^1, \tag{4.15}$$

$$\partial_t^2 w^1 - \Delta\partial_t w^1 - \Delta w^1 = -\partial_t v^1, \tag{4.16}$$

$$\zeta_{v^1} = w^1 = 0 \quad \text{on } \partial\Omega$$

$$\zeta_{v^1}|_{t=0} = \zeta_u(0), \qquad \zeta_{w^1}|_{t=0} = \zeta_\alpha(0),$$

and (v^2, w^2) solves

$$\epsilon\partial_t^2 v^2 + \partial_t v^2 - \Delta v^2 + f(u^1) - f(u^2)u = \partial_t w^2, \tag{4.17}$$

$$\partial_t^2 w^2 - \Delta\partial_t w^2 - \Delta w^2 = -\partial_t v^2, \tag{4.18}$$

$$v^2 = w^2 = 0 \quad \text{on } \partial\Omega,$$

$$\zeta_{v^2}|_{t=0} = \zeta_{w^2}|_{t=0} = 0.$$

Theorem 4.2. *Let the assumptions of Theorem 3.1 hold and let (v^1, w^1) and (v^2, w^2) be two solutions of systems (4.15)-(4.16) and (4.17)-(4.18), respectively with initial data belonging to $\mathbb{B}^1_R(\epsilon)$. Then, the solutions (v^1, w^1) and (v^2, w^2) satisfy the following estimates*

$$\| (\zeta_{v^1}(t), \zeta_{w^1}(t)) \|^2_{\Phi_2} \leq K_1 e^{-\beta t} \| (\zeta_{v^1}(0), \zeta_{w^1}(0)) \|^2_{\Phi_2}, \tag{4.19}$$

$$\| (\zeta_{v^2}(t), w^2(t)) \|^2_{\Phi_3} \leq C e^{Kt} \| (\zeta_{v^1}(0), \zeta_{w^1}(0)) \|^2_{\Phi_2} . \tag{4.20}$$

where the positive constants K_1, β, K and C depend on R, but are independent of ϵ.

Proof. Multiplying (4.15) by $2\partial_t v^1$ and (4.16) by $2\partial_t w^1$, integrating over Ω, and summing the two resulting equations, we have the following estimate

$$\frac{d}{dt}\Big(\epsilon \| \partial_t v^1 \|^2 + \| v^1 \|^2_{H^1} + \| \partial_t w^1 \|^2 + \| w^1 \|^2_{H^1}\Big) + 2 \| \partial_t v^1 \|^2 + 2 \| \partial_t w^1 \|^2_{H^1} = 0. \tag{4.21}$$

We multiply (4.15) by $2v^1$ and (4.16) by $2w^1$ et integrate over Ω. We obtain

$$\frac{d}{dt}\Big(\| v^1 \|^2 + 2\epsilon(\partial_t v^1, v^1)\Big) + \| v^1 \|^2_{H^1} \leq 2\epsilon \| \partial_t v^1 \|^2 + c_0 \| \partial_t w^1 \|^2, \tag{4.22}$$

$$\frac{d}{dt}\Big(\| w^1 \|^2_{H^1} + 2(w^1, \partial_t w^1)\Big) + \| w^1 \|^2_{H^1} \leq c_0 \| \partial_t v^1 \|^2 + 2 \| \partial_t w^1 \|^2 . \tag{4.23}$$

Summing (4.21), ϵ_{37}(4.22) and ϵ_{38}(4.23) where ϵ_{37}, and $\epsilon_{38} > 0$ are chosen small enough such that

$$1 - 2\epsilon\epsilon_{37} - \epsilon_{38}c_0 > 0,$$

$$1 - \epsilon_{37}c_0^2 - 2c_0\epsilon_{38} > 0,$$

we have an inequality of the form

$$\frac{d}{dt}E_{23} + \beta E_{23} \leq 0, \tag{4.24}$$

where the positive constant β is independent of ϵ, and

$$E_{23} = \epsilon \| \partial_t v^1 \|^2 + \| v^1 \|^2_{H^1} + \| \partial_t w^1 \|^2 + \| w^1 \|^2_{H^1} + \epsilon_{37}\Big(\| v^1 \|^2 + 2\epsilon(\partial_t v^1, v^1)\Big) + \epsilon_{38}\Big(\| w^1 \|^2_{H^1} + 2(w^1, \partial_t w^1)\Big).$$

Moreover, for a sufficiently $\epsilon_{38} > 0$, there exists $C > 0$ such that

$$\begin{aligned} &C^{-1}(\epsilon \parallel \partial_t v^1(t) \parallel^2 + \parallel v^1(t) \parallel^2_{H^1} + \parallel \partial_t w^1(t) \parallel^2 + \parallel w^1(t) \parallel^2_{H^1}) \\ \le& E_{23}(t) \le C(\epsilon \parallel \partial_t v^1(t) \parallel^2 + \parallel v^1(t) \parallel^2_{H^1} + \parallel \partial_t w^1(t) \parallel^2 + \parallel w^1(t) \parallel^2_{H^1}). \end{aligned} \tag{4.25}$$

Applying Gronwall's inequality to (4.24), owing to (4.25), we obtain the following estimate

$$\begin{aligned} &\epsilon \parallel \partial_t v^1(t) \parallel^2 + \parallel v^1(t) \parallel^2_{H^1} + \parallel \partial_t w^1(t) \parallel^2 + \parallel w^1(t) \parallel^2_{H^1} \\ \le& \parallel (\zeta_{v^1}(0), \zeta_{w^1}(0)) \parallel^2_{\Phi_2} e^{-\beta t}, \end{aligned} \tag{4.26}$$

where β is independent of ϵ. In order to prove estimate (4.19), we first deduce the required estimate of $\parallel \zeta_{v^1}(t) \parallel_{\varepsilon(\epsilon)}$. Equation (4.15) can be written as follows

$$\epsilon \partial_t^2 v^1 + \partial_t v^1 - \Delta v^1 = \partial_t w^1 = h_{w^1}(t), \qquad h_{w^1}|_{\partial\Omega} = 0. \tag{4.27}$$

Applying estimate (5.37) to the initial and boundary value problem for the singularly perturbed damped hyperbolic equation (4.27), we have

$$\begin{aligned} &\parallel \zeta_{v^1}(t) \parallel^2_{\varepsilon(\epsilon)} + \int_0^t e^{-\beta(t-s)} \parallel \partial_t v^1(s) \parallel^2 ds \\ \le& C e^{-\beta t}\Big(\parallel \zeta_{v^1}(0) \parallel^2_{\varepsilon(\epsilon)} + \parallel \partial_t w^1(0) \parallel^2_{H^{-1}} \Big) \\ &+ \int_0^t e^{-\beta(t-s)}(\parallel \partial_t w^1(s) \parallel^2_{H^{-1}} + \parallel \partial_t^2 w^1(s) \parallel^2_{H^{-1}}) ds, \end{aligned} \tag{4.28}$$

where β and C are independent of ϵ. In order to estimate $\parallel \partial_t^2 w^1 \parallel_{H^{-1}}$, we use equation (4.16) which implies

$$\parallel \partial_t^2 w^1 \parallel_{H^{-1}} \le \parallel \partial_t w^1 \parallel_{H^1} + \parallel w^1 \parallel_{H^1} + \parallel \partial_t v^1 \parallel_{H^{-1}} . \tag{4.29}$$

Inserting (4.29) into the right-hand side of (4.28), owing to (4.26), we have

$$\parallel \zeta_{v^1}(t) \parallel^2_{\varepsilon(\epsilon)} + \int_0^t e^{-\beta(t-s)} \parallel \partial_t v^1(s) \parallel^2 ds \le C \parallel (\zeta_{v^1}(0), \zeta_{w^1}(0)) \parallel^2_{\Phi_2} e^{-\beta t}, \tag{4.30}$$

where the positive constant C is independent of ϵ. In order to deduce the required estimate of $\parallel \zeta_{w^1}(t) \parallel_{\varepsilon(1)}$, we multiply (4.28) by $2(I + (-\Delta)^{-1})\partial_t w^1$ and integrate over Ω. We obtain

$$\frac{d}{dt}\Big(\parallel \zeta_{w^1} \parallel^2_{\varepsilon(1)} + \parallel w^1 \parallel^2 \Big) + \parallel \partial_t w^1 \parallel^2_{H^1} + \parallel \partial_t w^1 \parallel^2 \le C \parallel \partial_t v^1 \parallel^2 . \tag{4.31}$$

Summing (4.31) and ϵ_{39}(4.23) where $\epsilon_{39} > 0$ is small enough such that

$$1 - 2\epsilon_{39} > 0,$$

we have

$$\frac{d}{dt} E_{24}(t) + \beta E_{24}(t) + \parallel \partial_t w^1(t) \parallel^2_{H^1} \le C \parallel \partial_t v^1(t) \parallel^2, \tag{4.32}$$

where β and C are independent of ϵ, and

$$E_{24}(t) = \parallel \zeta_{w^1}(t) \parallel^2_{\varepsilon(1)} + \parallel w^1(t) \parallel^2 + \epsilon_{39}\Big(\parallel w^1(t) \parallel^2_{H^1} + 2(w^1(t), \partial_t w^1(t))\Big). \tag{4.33}$$

Applying Gronwall's inequality, we have

$$E_{24}(t)+\int_0^t \| \partial_t w^1(s) \|_{H^1}^2 \, e^{-\beta(t-s)}ds \leq E_{24}(0)+C\int_0^t \| \partial_t v^1(s) \|^2 \, e^{-\beta(t-s)}ds. \tag{4.34}$$

Using estimate (4.30) and the fact that for $\epsilon_{39}>0$ small enough there exists $C_2>0$ such that

$$C_2^{-1}\|\zeta_\alpha(t)\|_{\varepsilon(1)}^2 \leq E_{23}(t) \leq C_2\|\zeta_\alpha(t)\|_{\varepsilon(1)}^2, \tag{4.35}$$

estimate (4.34) implies

$$\|\zeta_\alpha(t)\|_{\varepsilon(1)}^2+\int_0^t \| \partial_t w^1(s) \|_{H^1}^2 \, e^{-\beta(t-s)}ds \leq C \, \| \, (\zeta_{v^1}(0),\zeta_{w^1}(0)) \, \|_{\Phi_2}^2 \, e^{-\beta t}. \tag{4.36}$$

Combining (4.28) and (4.36), we obtain estimate (4.19).

Multiply (4.17) by $-2\Delta\partial_t v^2$ and (4.18) by $-2\Delta\partial_t w^2$ and integrate over Ω, sum the two resulting equations. We have

$$\frac{d}{dt}E_{25}+ \| \partial_t v^2 \|_{H^1}^2 +2 \| \partial_t w^2 \|_{H^2}^2 \leq C_1(\| v^2 \|_{H^1}^2 + \| v^1 \|_{H^1}^2), \tag{4.37}$$

where the positive constant C_1 is independent of ϵ and

$$E_{25}=\epsilon \| \partial_t v^2 \|_{H^1}^2 + \| v^2 \|_{H^2}^2 + \| w^2 \|_{H^2}^2 + \| \partial_t w^2 \|_{H^1}^2 \, .$$

Multiply (4.18) by $2\partial_t w^2$ and integrate over Ω. We have

$$\frac{d}{dt}\Big(\| v^2 \|_{H^1}^2 +2(w^2,\partial_t w^2)\Big)+ \| w^2 \|_{H^1}^2 \leq c_0 \| \partial_t v^2 \|^2 +2 \| \partial_t w^2 \|^2 \, . \tag{4.38}$$

Summing (4.37) and ϵ_{40}(4.38) where $\epsilon_{40}>0$, we have

$$\frac{d}{dt}E_{26}+ \| \partial_t v^2 \|_{H^1}^2 +2 \| \partial_t w^2 \|_{H^2}^2 +\epsilon_{40} \| w^2 \|_{H^1}^2 \quad \leq \quad KE_{23}+C_1 \| v^1 \|_{H^1}^2,$$

where the positive constants K and C_1 are independent of ϵ and

$$\begin{aligned} E_{26} =& \epsilon \| \partial_t v^2 \|_{H^1}^2 + \| v^2 \|_{H^2}^2 + \| w^2 \|_{H^2}^2 + \| \partial_t w^2 \|_{H^1}^2 \\ &+\epsilon_{40}\Big(\| v^2 \|_{H^1}^2 +2(w^2,\partial_t w^2)\Big). \end{aligned}$$

Applying Gronwall's inequality, thanks to estimate (4.19), we have

$$\begin{aligned} &\epsilon \| \partial_t v^2(t) \|_{H^1}^2 + \| v^2(t) \|_{H^2}^2 + \| w^2(t) \|_{H^2}^2 + \| \partial_t w^2(t) \|_{H^1}^2 \\ &+\int_0^t \Big(\| \partial_t v^2(s) \|_{H^1}^2 + \| w^2(s) \|_{H^1}^2 +2 \| \partial_t w^2(s) \|_{H^2}^2 \Big)ds \\ \leq& C \, \| \, (\zeta_{v^1}(0),\zeta_{w^1}(0)) \, \|_{\Phi_2}^2 \, e^{Kt}, \end{aligned} \tag{4.39}$$

where the positive constant C is independent of ϵ.

In order to deduce the estimate of $\| \zeta_{v^2}(t) \|_{\varepsilon(\epsilon)}$, equation (4.17) can be written as follows

$$\epsilon\partial_t^2 v^2+\partial_t v^2-\Delta v^2=-l(t)u+\partial_t w^2=h_{v^1,v^2,w^2}(t), \qquad h_{v^1,v^2,w^2}|_{\partial\Omega}=0. \tag{4.40}$$

Applying estimate (5.28) to the initial and boundary value problem for singularly perturbed damped hyperbolic equation (4.40) in order to deduce the uniform energy estimate, we find

$$\begin{aligned}&\| \zeta_{v^2}(t) \|^2_{\varepsilon^1(\epsilon)} + \int_0^t \| \partial_t v^2(s) \|^2_{H^1} \, ds \\ \leq & Ce^{-\beta t}(\| h_{v^1,v^2,w^2}(0) \|^2) + \int_0^t e^{-\beta(t-s)} \Big(\| l(t)u \|^2_{H^1} + \| \partial_t w^2 \|^2_{H^1} \\ &+ \| \partial_t [l(t)u] \|^2_{H^{-1}} + \| \partial_t^2 w^2 \|^2_{H^{-1}} \Big) ds. \end{aligned} \tag{4.41}$$

Following estimate (4.37), v^2, w^2 and $\partial_t w^2 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\partial_t v^2 \in H^1_0(\Omega)$, which implies

$$\| h_{v^1,v^2,w^2}(0) \|^2 \leq C' e^{Kt}(\| \zeta_{v^1}(0) \|^2_{\varepsilon(\epsilon)} + \| \zeta_{w^1}(0) \|^2_{\varepsilon(1)}), \tag{4.42}$$

$$\| l(t)u \|_{H^1} \leq C \| u \|_{H^1} \leq C_1, \tag{4.43}$$

$$\| \partial_t [l(t)u(t)] \|_{H^{-1}} \leq C_2. \tag{4.44}$$

In order to estimate $\| \partial_t w^2 \|_{H^{-1}}$, we use equation (4.18) which implies

$$\| \partial_t^2 w^2 \|_{H^{-1}} \leq \| \partial_t w^2 \|^2_{H^1} + \| w^2 \|^2_{H^1} + \| \partial_t v^2 \|_{H^{-1}} \, . \tag{4.45}$$

Inserting estimates (4.42) – (4.45) into the right-hand side of (4.41), using estimate (4.39), we find

$$\| \zeta_{v^2}(t) \|^2_{\varepsilon^1(\epsilon)} + \int_0^t \| \partial_t v^2(s) \|^2_{H^1} \, ds \leq Ce^{Kt} \| (\zeta_{v^2}(0), \zeta_{w^2}(0)) \|^2_{\Phi_3} \, . \tag{4.46}$$

In order to deduce the desired estimate of $\| \zeta_{w^2}(t) \|_{\varepsilon^1(1)}$, we multiply (4.18) by $2(I - \Delta)\partial_t w^2$ and integrate over Ω. We have

$$\frac{d}{dt}\Big(\| \zeta_{w^2} \|^2_{\varepsilon^1(1)} + \| w^2 \|^2 \Big) + \| \partial_t w^2 \|^2_{H^2} + \| \partial_t w^2 \|^2_{H^1} \leq C \| \partial_t v^2 \|^2_{H^1} \, . \tag{4.47}$$

Integrating over $[0, t]$, owing to estimate (4.39), we have

$$\begin{aligned}&\| \zeta_{w^2}(t) \|^2_{\varepsilon^1(1)} + \int_0^t \Big(\| \partial_t w^2(s) \|^2_{H^2} + \| \partial_t w^2(s) \|^2_{H^1} \Big) ds \\ \leq & C_1 \| (\zeta_{v^1}(0) \, \zeta_{w^1}(0)) \|^2_{\Phi_2} \, e^{Kt}, \end{aligned} \tag{4.48}$$

where the positive constant C_1 is independent of ϵ. Combining estimates (4.46) and (4.48), we obtain the result. This finishes the proof. □

In order to find the estimate between the difference of the solution of the perturbed system (1.1) – (1.5) and the solution of the to unperturbed system (2.1) – (2.4), we need the first term of asymptotic expansions of $(u^\epsilon, \alpha^\epsilon)$ near $t = 0$ with respect ϵ. Following the general scheme (see Lyusternik & Vishik [10], Babin & Vishik [1], Fabrie & Galusinski [8] and Grasselli & Miranville [9]), we seek for asymptotic expansions of the form

$$u^\epsilon(t) = u^0(t) + \epsilon \tilde{u}^1(\frac{t}{\epsilon}) + \epsilon \mathcal{R}(t), \qquad \alpha^\epsilon(t) = \alpha^0(t) + \epsilon \mathcal{P}(t), \tag{4.49}$$

where (u^0,α^0) solves the limit system parabolic-hyperbolic system $(2.1)-(2.4)$ with initial data $(u^0(0),\alpha^0(0)) = (u^\epsilon(0),\alpha^\epsilon(0))$, the boundary layer term $\tilde{u}^1$ satisfies the following equation:

$$\partial_\tau^2\tilde{u}^1 + \partial_\tau\tilde{u}^1 = 0, \quad \partial_\tau\tilde{u}^1(0) = \partial_t u^0(0) - \partial_t u^\epsilon(0) \text{ and } \lim_{\tau\to+\infty}\tilde{u}^1(\tau) = 0, \quad (4.50)$$

and $(\mathcal{R}(t),\mathcal{P}(t))$ is the remainder. Solving (4.50), we have

$$\tilde{u}^1(\tau) = e^{-\tau}\theta_{u^\epsilon,\alpha^\epsilon}(0), \text{ where } \theta_{u^\epsilon,\alpha^\epsilon}(t) = -\partial_t u^\epsilon(t)+\Delta u^\epsilon(t)-f(u^\epsilon(t))+\alpha^\epsilon(t). \quad (4.51)$$

Following the construction of the asymptotic expansion, the remainder $(\mathcal{R}(t),\mathcal{P}(t))$ verifies the following equations

$$\epsilon\partial_t^2\mathcal{R}(t) + \partial_t\mathcal{R}(t) - \Delta\mathcal{R}(t) = h_{\mathcal{R},\mathcal{P}}(t) \quad \zeta_{\mathcal{R}}|_{t=0} = (-\theta_{u,\alpha}(0),0), \quad (4.52)$$

$$\partial_t^2\mathcal{P}(t) - \partial_t\Delta\mathcal{P}(t) - \Delta\mathcal{P}(t) = -\partial_t\mathcal{R}(t) - \partial_t\tilde{u}^1(\frac{t}{\epsilon}), \quad \zeta_{\mathcal{P}}|_{t=0} = 0, \quad (4.53)$$

where

$$h_{\mathcal{R},\mathcal{P}}(t) = \frac{1}{\epsilon}[f(u^0(t)) - f(u^\epsilon(t))] + \partial_t\mathcal{P}(t) + \Delta\tilde{u}^1(\frac{t}{\epsilon}) - \partial_t^2 u^0(t). \quad (4.54)$$

The next theorem gives an estimate of $\|\ (\zeta_{\mathcal{R}}(t),\zeta_{\mathcal{P}}(t))\ \|_{\Phi_2}^2$.

Theorem 4.3. *Let the assumptions of Theorem 3.3 hold, $\epsilon < 1$ and $(u^\epsilon,\alpha^\epsilon)$ be the solution of the hyperbolic system (1.1)-(1.5) with initial data belonging to $B^1_{R_0}(\epsilon)\cap B^2_R(\epsilon)$. Then, the remainder $(\mathcal{R}(t),\mathcal{P}(t))$ in the asymptotic expressions (4.49) satisfies the following estimate*

$$\|\ (\zeta_{\mathcal{R}}(t),\zeta_{\mathcal{P}}(t))\ \|_{\Phi_2}^2 \quad \le \quad Ce^{Kt}, \quad (4.55)$$

where the constants C and K depend on $\|\ \zeta_{\mathcal{R}}(0)\ \|_{\varepsilon(\epsilon)}$, but are independent of ϵ.

Proof. According to the explicit expression (4.50) of the boundary layer term, we have the following estimates

$$\|\ \tilde{u}^1(\frac{t}{\epsilon})\ \|_{H^1} \le Ce^{\frac{-t}{\epsilon}}, \quad (4.56)$$

$$\|\ \partial_t\tilde{u}^1(\frac{t}{\epsilon})\ \|_{H^1} \le C\epsilon^{-1}e^{\frac{-t}{\epsilon}}, \quad (4.57)$$

$$\|\ \tilde{u}^1(\frac{t}{\epsilon})\ \|_{H^{-1}} \le Ce^{\frac{-t}{\epsilon}}, \quad (4.58)$$

where the positice constant C is independent of ϵ.

Applying estimate (5.37) to the initial and boundary value problem for the singularly perturbed damped hyperbolic equation (4.53), we deduce

$$\begin{aligned}&\|\ \zeta_{\mathcal{R}}(t)\ \|_{\varepsilon(\epsilon)}^2 + \int_0^t e^{-\beta(t-s)}\ \|\ \partial_t\mathcal{R}(s)\ \|^2\ ds\\ \le\ &Ce^{-\beta t}(\|\ \zeta_{\mathcal{R}}(0)\ \|_{\varepsilon(\epsilon)}^2 + \|\ h_{\mathcal{R},\mathcal{P}}(0)\ \|_{H^{-1}}^2) \quad (4.59)\\ &+ C_1(\int_0^t e^{-\beta(t-s)}(\|\ h_{\mathcal{R},\mathcal{P}}(s)\ \|_{H^{-1}} + \|\ \partial_t h_{\mathcal{R},\mathcal{P}}(s)\ \|_{H^{-1}})ds)^2, \quad (4.60)\end{aligned}$$

where the positive constants β, C and C_1 are independent of ϵ.

In order to estimate the last term of the right-hand side of (4.60), we first estimate $\| h_{\mathcal{R},\mathcal{P}}(s) \|_{H^{-1}}$ and $\| \partial_t h_{\mathcal{R},\mathcal{P}}(s) \|_{H^{-1}}$.Thanks to (4.54), we have

$$\begin{aligned}
\| h_{\mathcal{R},\mathcal{P}}(s) \|_{H^{-1}} \leq & \quad \frac{1}{\epsilon}\|f(u^0)-f(u^\epsilon)\|_{H^{-1}} + \|\partial_t\mathcal{P}\|_{H^{-1}} + \|\tilde{u}^1\|_{H^1} + \|\partial_t^2 u^0\|_{H^{-1}},\\
\| \partial_t h_{\mathcal{R},\mathcal{P}}(s) \|_{H^{-1}} \leq & \quad \frac{1}{\epsilon}\|f'(u^0)\partial_t u\|_{H^{-1}} + \frac{1}{\epsilon}\|[f'(u^0)-f'(u^\epsilon)]\partial_t u^\epsilon\|_{H^{-1}} + \|\partial_t^2\mathcal{P}\|_{H^{-1}}\\
& + \|\partial_t\tilde{u}^1\|_{H^1} + \|\partial_t^3 u^0\|_{H^{-1}}.
\end{aligned}$$

We have for all $w \in H^1(\Omega)$,

$$\begin{aligned}
|(f(u^0)-f(u^\epsilon),w)| \leq & (\|u^0\|_{L^4}^2 + \|u^\epsilon\|_{L^4}^2 + 1)\|u\|_{L^4}\|w\|_{L^4}\\
\leq & C(\|u^0\|_{H^1}^2 + \|u^\epsilon\|_{H^1}^2 + 1)\epsilon(\|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1})\|w\|_{H^1}\\
\leq & C\epsilon(\|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1})\|w\|_{H^1},
\end{aligned}$$

which implies

$$\|f(u^0)-f(u^\epsilon)\|_{H^{-1}} \leq C\epsilon(\|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1}), \tag{4.61}$$

and, owing to (2.70),

$$\| h_{\mathcal{R},\mathcal{P}}(s) \|_{H^{-1}} \leq C(\|\mathcal{R}\|_{H^1} + \|\partial_t\mathcal{P}\|_{H^{-1}} + \|\tilde{u}^1\|_{H^1} + 1), \tag{4.62}$$

where the positive constant C is independent of ϵ. We have also for all $w \in H^1(\Omega)$,

$$\begin{aligned}
\|f'(u^0)\partial_t u, w)\|_{H^{-1}} \leq & \|\partial_t u\|_{H^{-1}}\|((u^0)^2+1)w\|_{H^1}\\
\leq & \|\partial_t u\|_{H^{-1}}\Big((\|u^0\|_{L^\infty}^2 + 1)\|\nabla w\| + (\|u^0\|_{L^6}\|\nabla u^0\|_{L^6} + 1)\|w\|_{L^6}\Big)\\
\leq & C\|\partial_t u\|_{H^{-1}}\|w\|_{H^1},
\end{aligned}$$

which implies

$$\|f'(u^0)\partial_t u\|_{H^{-1}} \leq C\epsilon(\|\partial_t\mathcal{R}\|_{H^{-1}} + \| \partial_t\tilde{u}^1 \|_{H^{-1}}), \tag{4.63}$$

where the positive constant is independent of ϵ, and

$$\begin{aligned}
|([f'(u^0)-f'(u^\epsilon)]\partial_t u^\epsilon, w)| \leq & (\|u^0\|_{L^6} + \|u^\epsilon\|_{L^6} + 1)\|u\|_{L^6}\|\partial_t u^\epsilon\|\|w\|_{L^6}\\
\leq & C\epsilon(\|u^0\|_{H^1} + \|u^\epsilon\|_{H^1} + 1)(\|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1})\\
& \times \|\partial_t u^\epsilon\|\|w\|_{H^1}\\
\leq & C\epsilon(\|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1})\|w\|_{H^1},
\end{aligned}$$

which implies

$$\|[f'(u^0)-f'(u^\epsilon)]\partial_t u^\epsilon\|_{H^{-1}} \leq C\epsilon(\|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1}). \tag{4.64}$$

We have, owing to (4.53),

$$\| \partial_t^2\mathcal{P} \|_{H^{-1}} \quad \leq \quad \| \partial_t\mathcal{P} \|_{H^1} + \| \mathcal{P} \|_{H^1} + \| \partial_t\mathcal{R} \|_{H^{-1}} + \| \partial_t\tilde{u}^1 \|_{H^{-1}}\ . \tag{4.65}$$

Thanks to estimates (4.63) – (4.65), we have

$$\begin{aligned}
\| \partial_t h_{\mathcal{R},\mathcal{P}} \|_{H^{-1}} \leq & C(\| \mathcal{R} \|_{H^1} + \| \partial_t\mathcal{R} \|_{H^{-1}} + \| \partial_t\mathcal{P} \|_{H^1} + \| \mathcal{P} \|_{H^1} + \|\tilde{u}^1\|_{H^1}\\
& + \|\partial_t\tilde{u}^1\|_{H^1} + \|\partial_t^3 u^0\|_{H^{-1}}),
\end{aligned} \tag{4.66}$$

where the positive constant C is independent of ϵ. Inserting estimate (4.62) and (4.66) into the right-hand side of (4.60), we have

$$\begin{aligned}&\| \zeta_{\mathcal{R}}(t) \|^2_{\varepsilon(\epsilon)} + \int_0^t e^{-\beta(t-s)} \| \partial_t \mathcal{R}(s) \|^2 \, ds \\ \leq &Ce^{-\beta t} + C\Big(\int_0^t (\| \zeta_{\mathcal{R}}(s) \|^2_{\varepsilon(\epsilon)} + \int_0^t (\| \mathcal{P}(s) \|^2_{H^1} + \| \partial_t \mathcal{P}(s) \|^2_{H^1}) ds \\ &+ (\int_0^t [\|\tilde{u}^1\|_{H^1} + \|\partial_t \tilde{u}^1\|_{H^1}] ds)^2 + \int_0^t ds + \int_0^t e^{-\beta(t-s)} \|\partial_t^3 u^0\|^2_{H^{-1}} ds \Big).\end{aligned} \tag{4.67}$$

It remains to estimate $\|\mathcal{P}\|^2_{H^1}$ and the integral of $\|\partial_t \mathcal{P}\|^2_{H^1}$. Multiply (4.53) by $2\partial_t \mathcal{P}$ and (4.52) by $2\partial_t \mathcal{R}$, integrate over Ω and sum the two resulting equations. After transformation, using the fact that

$$(\tilde{u}^1, \partial_t^2 \mathcal{P}) = (\tilde{u}^1, \Delta \partial_t \mathcal{P}) + (\tilde{u}^1, \Delta \mathcal{P}) - (\tilde{u}^1, \partial_t \mathcal{R}) - (\tilde{u}^1, \partial_t \tilde{u}^1),$$

and owing to (2.70), we obtain

$$\frac{d}{dt}\Gamma_1 + \| \partial_t \mathcal{R} \|^2 + \| \partial_t \mathcal{P} \|^2_{H^1} \leq K(\| \mathcal{R} \|^2_{H^1} + \| \mathcal{P} \|^2_{H^1}) + C(1 + e^{\frac{-2t}{\epsilon}}), \tag{4.68}$$

where

$$\Gamma_1 = \| \mathcal{R} \|^2_{H^1} + \epsilon \| \partial_t \mathcal{R} \|^2 + \| \mathcal{P} \|^2_{H^1} + \| \partial_t \mathcal{P} \|^2 + 2(\tilde{u}^1, \partial_t \mathcal{P}) + \|\tilde{u}^1\|^2.$$

Applying Gronwall's inequality to (4.68), we have

$$\begin{aligned}&\epsilon \| \partial_t \mathcal{R}(t) \|^2 + \| \mathcal{R}(t) \|^2_{H^1} + \| \partial_t \mathcal{P}(t) \|^2 + \| \mathcal{P}(t) \|^2_{H^1} \\ &+ \int_0^t (\| \partial_t \mathcal{R}(s) \|^2 + \| \partial_t \mathcal{P}(s) \|^2_{H^1}) ds \leq Ce^{Kt}.\end{aligned} \tag{4.69}$$

Using (4.69) in the right-hand side of (4.67), we have

$$\begin{aligned}\| \zeta_{\mathcal{R}}(t) \|^2_{\varepsilon(\epsilon)} \leq &Ce^{-\beta t} + C\Big(\int_0^t \| \zeta_{\mathcal{R}}(s) \|^2_{\varepsilon(\epsilon)} \, ds + e^{Kt} + (\int_0^t [\|\tilde{u}^1\|_{H^1} + \|\partial_t \tilde{u}^1\|_{H^1}] ds)^2 \\ &+ \int_0^t ds + \int_0^t e^{-\beta(t-s)} \|\partial_t^3 u^0(s)\|^2_{H^{-1}} ds \Big),\end{aligned}$$

which implies, owing to estimates (2.70), (4.56) et (4.57),

$$\| \zeta_{\mathcal{R}}(t) \|^2_{\varepsilon(\epsilon)} \leq C\Big(e^{Kt} + 1 + t + (1+\epsilon)^2 + \int_0^t \| \zeta_{\mathcal{R}}(s) \|^2_{\varepsilon(\epsilon)} \, ds \Big). \tag{4.70}$$

Applying Gronwall's inequality to (4.70), we obtain

$$\| \zeta_{\mathcal{R}}(t) \|^2_{\varepsilon(\epsilon)} \leq Ce^{Kt}, \tag{4.71}$$

where the positive constants C and K depend on R, but they are independent of ϵ. Thus, the $\mathcal{R}$-part (4.55) is proven. In order to obtain the $\mathcal{P}$-part, we multiply

(4.53) by $2(-\Delta)^{-1}\partial_t\mathcal{P}$ and integrate over Ω. After transformation and using the fact that

$$(\tilde{u}^1, (-\Delta)^{-1}\partial_t^2\mathcal{P}) = (\tilde{u}^1, \partial_t\mathcal{P}) + (\tilde{u}^1, \mathcal{P}) - (\tilde{u}^1, (-\Delta)^{-1}\partial_t\mathcal{R}) - (\tilde{u}^1, (-\Delta)^{-1}\partial_t\tilde{u}^1),$$

thanks to estimates (4.58), (4.69) and (4.71), we have

$$\begin{aligned}&\frac{d}{dt}\Big(\|\partial_t\mathcal{P}\|^2_{H^{-1}} + \|\mathcal{P}\|^2 + 2(\tilde{u}^1, (-\Delta)^{-1}\partial_t\mathcal{P} + \|\tilde{u}^1\|^2_{H^{-1}}) + 2 \parallel \partial_t\mathcal{P} \parallel^2)\\ \leq &C\Big(e^{Kt} + e^{\frac{-t}{\epsilon}}).\end{aligned}$$

This implies, by integration over $[0, t]$,

$$\|\partial_t\mathcal{P}(t)\|^2_{H^{-1}} \leq Ce^{Kt}, \tag{4.72}$$

where the positive constants C and K depend on R, but they are independent of ϵ. Combining estimates (4.69) and (4.72), we have

$$\parallel \zeta_{\mathcal{P}}(t) \parallel^2_{\varepsilon(1)} \leq Ce^{Kt}, \tag{4.73}$$

where the positive constants C and K depend on R, but they are independent of ϵ. Combining estimates (4.71) and estimate (4.73), we obtain the desired estimate. This finishes the proof. □

Corollary 4.1. *Let the assumptions of Theorem 4.3 hold and let $(u^\epsilon, \alpha^\epsilon)$ and (u^0, α^0) be solutions of (1.1)-(1.5) and (2.1)-(2.4), respectively such that $u^\epsilon(0) = u^0(0) = u_0$, $\alpha^\epsilon(0) = \alpha^0(0) = \alpha_0$ and $\partial_t\alpha^\epsilon(0) = \partial_t\alpha^0(0) = \alpha_1$. Then*

$$\begin{aligned}&\parallel (\zeta_{u^\epsilon - u^0}(t), \zeta_{\alpha^\epsilon - \alpha^0}(t)) \parallel^2_{\Phi_2}\\ \leq &C(1 + e^{-\beta t}) + C\epsilon^2\Big(e^{Kt} + t + (\epsilon \parallel \theta_{u^\epsilon,\alpha^\epsilon}(0) \parallel_{H^1} + \parallel \theta_{u^\epsilon,\alpha^\epsilon}(0) \parallel_{H^{-1}})^2\\ &+ \frac{1}{\epsilon} \parallel \theta_{u^\epsilon,\alpha^\epsilon}(0) \parallel^2_{H^{-1}}\Big),\end{aligned} \tag{4.74}$$

where $\theta_{u^\epsilon,\alpha^\epsilon}(t) = \partial_t u^\epsilon(t) - \Delta u^\epsilon(t) + f(u^\epsilon(t)) - \partial\alpha^\epsilon(t)$ and the constants C_1, C_2 and K depend on $\parallel \zeta_{u^\epsilon}(0) \parallel_{\varepsilon(\epsilon)}$, but are independent of ϵ.

Proof. Setting $u = u^\epsilon - u^0$ and $\alpha = \alpha^\epsilon - \alpha^0$, then, (u, α) verifies the system

$$\epsilon\partial_t^2 u + \partial_t u - \Delta u = f(u^0) - f(u^\epsilon) + \partial_t\alpha - \epsilon\partial_t^2 u^0 = h_{u,\alpha}(t), \tag{4.75}$$

$$\partial_t^2\alpha - \partial_t\Delta\alpha - \Delta\alpha = -\partial_t u. \tag{4.76}$$

Applying estimate (5.37) to the initial and boundary value problem for the singularly perturbed damped hyperbolic equation(4.75), we find

$$\begin{aligned}&\parallel \zeta_u(t) \parallel^2_{\varepsilon(\epsilon)} + \int_0^t e^{-\beta(t-s)} \parallel \partial_t u(s) \parallel^2 ds\\ \leq &Ce^{-\beta t} + C\epsilon^2(\int_0^t e^{-\beta(t-s)}(\parallel h_{u^2,\alpha}(s) \parallel_{H^{-1}} + \parallel \partial_t h_{u,\alpha}(s) \parallel_{H^{-1}})ds)^2,\end{aligned}$$

where

$$h_{u,\alpha}(t) = f(u^0) - f(u^\epsilon) + \partial_t\alpha.$$

We have, owing to (2.70) and (4.61),

$$\begin{aligned}\| h_{u,\alpha}(t) \|_{H^{-1}} \leq& \|f(u^0)-f(u^\epsilon)\|_{H^{-1}} + \|\partial_t\alpha\|_{H^{-1}} + \epsilon\|\partial_t^2 u^0\|_{H^{-1}}\\ \leq& C\epsilon(\| \mathcal{R}(t) \|_{H^1} + \| \tilde{u}^1 \|_{H^1} + \|\partial_t\mathcal{P}\|_{H^{-1}} + 1).\end{aligned}$$

From (4.76), we deduce

$$\begin{aligned}\|\partial_t^2\alpha\|_{H^{-1}} \leq& \|\partial_t\alpha\|_{H^1} + \|\alpha\|_{H^1} + \|\partial_t u\|_{H^{-1}}\\ \leq& \epsilon(\|\partial_t\mathcal{P}\|_{H^1} + \|\mathcal{P}\|_{H^1} + \|\partial_t\mathcal{R}\|_{H^{-1}} + \|\partial_t\tilde{u}^1\|_{H^{-1}}),\end{aligned}$$

which implies, thanks to (4.63) and (4.64),

$$\begin{aligned}&\| \partial_t h_{u,\alpha}(t) \|_{H^{-1}}\\ \leq& \|f'(u^0)\partial_t u\|_{H^{-1}} + \|[f'(u^0)-f'(u^\epsilon)]\partial_t u^\epsilon\|_{H^{-1}} + \|\partial_t^2\alpha\|_{H^{-1}} + \epsilon\|\partial_t^3 u^0\|_{H^{-1}}\\ \leq& C\epsilon(\| \partial_t\mathcal{R} \|_{H^{-1}} + \| \partial_t\tilde{u}^1 \|_{H^{-1}} + \| \mathcal{R} \|_{H^1} + \| \tilde{u}^1 \|_{H^1}) + \|\partial_t^2\alpha\|_{H^{-1}}\\ &+ \|\partial_t^3 u^0\|_{H^{-1}})\\ \leq& C\epsilon(\| \partial_t\mathcal{R} \|_{H^{-1}} + \| \mathcal{R} \|_{H^1} + \|\partial_t\mathcal{P}\|_{H^1} + \|\mathcal{P}\|_{H^1} + \| \tilde{u}^1 \|_{H^1} + \|\partial_t\tilde{u}^1\|_{H^{-1}}\\ &+ \|\partial_t^3 u^0\|_{H^{-1}}).\end{aligned}$$

Thanks to (4.69), (4.15), (4.56) and (4.57), we have

$$\begin{aligned}&\| \zeta_u(t) \|^2_{\varepsilon(\epsilon)} + \int_0^t e^{-\beta(t-s)} \| \partial_t u(s) \|^2\\ \leq& Ce^{-\beta t} + C\epsilon^2\Big(\int_0^t e^{Ks}ds + t + \int_0^t e^{-\beta(t-s)}\|\partial_t^3 u^0(s)\|^2_{H^{-1}}ds\\ &+ \int_0^t (\|\partial_t\mathcal{P}(s)\|^2_{H^1} + \|\mathcal{P}(s)\|^2_{H^1})ds\\ &+ ([\|\theta_{u^\epsilon,\alpha^\epsilon}(0) \|_{H^1} + \frac{1}{\epsilon}\|\theta_{u^\epsilon,\alpha^\epsilon}(0))\|_{H^{-1}}]\int_0^t e^{\frac{-s}{\epsilon}}ds)^2\Big),\end{aligned}$$

which implies

$$\| \zeta_u(t) \|^2_{\varepsilon(\epsilon)} \leq Ce^{-\beta t} + C\epsilon^2\Big(e^{Kt} + 1 + t + (\epsilon\|\theta_{u^\epsilon,\alpha^\epsilon}(0) \|_{H^1} + \|\theta_{u^\epsilon,\alpha^\epsilon}(0))\|_{H^{-1}})^2\Big),$$

where the positive constants C and K are independent of ϵ.

Multiply (4.76) by $2(I+(-\Delta)^{-1})\partial_t\alpha$ and integrate over Ω. We obtain, owing to (4.71) and (4.51),

$$\begin{aligned}\frac{d}{dt}\|\zeta_\alpha\|^2_{\varepsilon(1)} + \|\partial_t\alpha\|^2 \leq& C(\|\partial_t u\|^2_{H^{-1}} + \|\alpha\|^2)\\ \leq& C\epsilon^2(\|\partial_t\mathcal{R}\|^2_{H^{-1}} + \|\mathcal{P}\|^2_{H^1} + \|\partial_t\tilde{u}^1\|^2_{H^{-1}})\\ \leq& C\epsilon^2\Big(e^{Kt} + \frac{1}{\epsilon^2}\|\theta_{u^\epsilon,\alpha^\epsilon}(0))\|^2_{H^{-1}}e^{\frac{-2t}{\epsilon}}\Big).\end{aligned}$$

Integrating over $[0,t]$, we have,

$$\|\zeta_\alpha\|^2_{\varepsilon(1)} \leq C\epsilon^2\Big(e^{Kt} + \frac{1}{\epsilon}\|\theta_{u^\epsilon,\alpha^\epsilon}(0)\|^2_{H^{-1}}\Big). \tag{4.77}$$

We obtain the result by combining (4.77) and (4.77). □

We now generalize estimate (4.74) to the case where the solutions $(u^\epsilon,\alpha^\epsilon)$ and (u^0,α^0) have different initial data.

Corollary 4.2. *Let the assumptions of Theorem 4.3 hold and (u,α) be solution of the system (1.1)-(1.5) with initial data in $B^2_R(0)$. Then, the following estimate*

$$\begin{aligned}\| (\zeta_{u^\epsilon - u^0}(t), \zeta_{\alpha^\epsilon-\alpha^0}(t)) \|^2_{\Phi_0} \leq & C(1+e^{-\beta t}) + Ce^{Kt} + C\epsilon^2\Big(e^{Kt} + t \\ & + (\epsilon \| \theta_{u^\epsilon,\alpha^\epsilon}(0) \|_{H^1} + \| \theta_{u^\epsilon,\alpha^\epsilon}(0) \|_{H^{-1}})^2 \\ & + \frac{1}{\epsilon} \| \theta_{u^\epsilon,\alpha^\epsilon}(0) \|^2_{H^{-1}} \Big), \end{aligned} \tag{4.78}$$

is valid, where the positive constants C and K depend on R and $\| u^0(0) \|_{H^3}$, but are independent of ϵ.

Proof. Let (u^0,α^0) be the same as in Theorem 4.3. Then, the difference $u^\epsilon - u^0$ satisfies estimate (4.74) and, since (u^0,α^0) and (u,α) solve the parabolic-hyperbolic system $(2.1)-(2.4)$, thanks to estimate (4.1) with $\epsilon = 0$, we have

$$\| (u(t) - u^0(t), \zeta_\alpha(t) - \zeta_{\alpha^0}(t)) \|^2_{\Phi_0} \leq Ce^{Kt} \| (u(0) - u^0(0), \zeta_\alpha(0) - \zeta_{\alpha^0}(0)) \|^2_{\Phi_0} . \tag{4.79}$$

Combining (4.74) and (4.79), we obtain the estimate (4.78). □

The following corollary allows to control the evolution of the quantity

$$\theta_{u^\epsilon,\alpha^\epsilon}(t) = \epsilon\partial_t^2 u^\epsilon(t). \tag{4.80}$$

Corollary 4.3. *Let the assumptions of Theorem 4.3 hold. Then, we have the following estimate*

$$\| \theta_{u^\epsilon,\alpha^\epsilon}(t) \|_{H^{-1}} \leq C\epsilon\Big(e^{Kt} + (\| \theta_{u^\epsilon,\alpha^\epsilon}(0) \|_{H^1} + \frac{1}{\epsilon} \| \theta_{u^\epsilon,\alpha^\epsilon}(0) \|)e^{\frac{-t}{\epsilon}}\Big), \tag{4.81}$$

where the positive constant C depends on R, but is independent of ϵ.

Proof. Inserting the asymptotic expansion (4.49) in (4.51), we have

$$\begin{aligned} & \theta_{u^\epsilon,\alpha^\epsilon}(t) \\ =& -\partial_t(u^\epsilon - u^0) + \Delta(u^\epsilon - u^0) + f(u^0) - f(u^\epsilon) + \partial_t(\alpha^\epsilon - \alpha^0) \\ =& -\epsilon\partial_t\Big(\mathcal{R}(t) + \tilde{u}^1(\frac{t}{\epsilon})\Big) + \epsilon\Delta\Big(\mathcal{R}(t) + \tilde{u}^1(\frac{t}{\epsilon})\Big) + [f(u^0(t)) - f(u^\epsilon(t))] + \epsilon\partial_t\mathcal{P}(t) \\ =& \epsilon h_{\mathcal{R},\mathcal{P}}(t) - \epsilon\Big(\partial_t\mathcal{R}(t) - \Delta\mathcal{R}(t) + \partial_t\tilde{u}^1(\frac{t}{\epsilon})\Big), \end{aligned} \tag{4.82}$$

where $h_{\mathcal{R},\mathcal{P}}(t)$ is defined by (4.54). Moreover, without loss generality, we can assume that $t \leq 1$. Then, (4.82) implies

$$\| \theta_{u^\epsilon,\alpha^\epsilon}(t) \|_{H^{-1}} \leq \epsilon\|h_{\mathcal{R},\mathcal{P}}(t)\|_{H^{-1}} + \epsilon\Big(\| \partial_t\mathcal{R} \|_{H^{-1}} + \| \mathcal{R}(t) \|_{H^1} + \| \partial_t\tilde{u}^1 \|_{H^{-1}} \Big), \tag{4.83}$$

where

$$\|h_{\mathcal{R},\mathcal{P}}(t)\|_{H^{-1}} \leq \|\partial_t\mathcal{P}\|_{H^{-1}} + \frac{1}{\epsilon}\|f(u^0) - f(u^\epsilon)\|_{H^{-1}} + \|\tilde{u}^1\|_{H^1}.$$

In order to estimate $\|f(u^0) - f(u^\epsilon)\|_{H^{-1}}$, we take $w \in H^1(\Omega)$, then we have

$$\begin{aligned} |(f(u^0) - f(u^\epsilon), w)| \leq & 3(\|u^0\|^2_{L^4} + \|u^\epsilon\|^2_{L^4} + 1)\epsilon(\|\mathcal{R}\|_{L^4} + \|\tilde{u}^1\|_{L^4})\|w\|_{L^4} \\ \leq & C\epsilon(\|u^0\|^2_{H^1} + \|u^\epsilon\|^2_{H^1} + 1)(\|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1})\|w\|_{H^1} \\ \leq & C\epsilon(\|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1})\|w\|_{H^1}, \end{aligned}$$

which implies

$$\|f(u^0) - f(u^\epsilon)\|_{H^{-1}} \leq C\epsilon(\|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1}). \tag{4.84}$$

Thus

$$\|h_{\mathcal{R},\mathcal{P}}(t)\|_{H^{-1}} \leq C(\|\partial_t \mathcal{P}\|_{H^{-1}} + \|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^2}). \tag{4.85}$$

Inserting (4.85) into (4.83), we have

$$\begin{aligned}\| \theta_{u^\epsilon,\alpha^\epsilon}(t) \|_{H^{-1}} \leq & C\epsilon(\|\partial_t \mathcal{P}\|_{H^{-1}} + \|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1}) \\ & + \epsilon\Big(\| \partial_t \mathcal{R} \|_{H^{-1}} + \| \mathcal{R}(t) \|_{H^1} + \| \partial_t \tilde{u}^1 \|_{H^{-1}} \Big) \\ \leq & C\epsilon(\|\partial_t \mathcal{P}\|_{H^{-1}} + \| \partial_t \mathcal{R} \|_{H^{-1}} + \|\mathcal{R}\|_{H^1} + \|\tilde{u}^1\|_{H^1} + \| \partial_t \tilde{u}^1 \|_{H^{-1}}),\end{aligned}$$

which implies, owing to estimates (4.55), (4.56) and (4.57),

$$\| \theta_{u^\epsilon,\alpha^\epsilon}(t) \|_{H^{-1}} \leq C\epsilon\Big(1 + (\| \theta_{u^\epsilon,\alpha^\epsilon}(0) \|_{H^1} + \frac{1}{\epsilon} \| \theta_{u^\epsilon,\alpha^\epsilon}(0) \|_{H^{-1}})e^{\frac{-t}{\epsilon}}\Big), \tag{4.86}$$

where the constant C depends on R, but is independent of ϵ. □

Remark 4.1. According (4.86), we have

$$\| \partial_t^2 u^\epsilon(t) \|_{H^{-1}} \leq C\Big(1 + (\| \theta_{u^\epsilon,\alpha^\epsilon}(0) \|_{H^1} + \frac{1}{\epsilon} \| \theta_{u^\epsilon,\alpha^\epsilon}(0) \|_{H^{-1}})e^{\frac{-t}{\epsilon}}\Big), \tag{4.87}$$

where the constant C depends on R, but is independent of ϵ.

5. Robust exponential attractors

In this section, we construct a robust family of exponential attractors $\mathcal{M}_\epsilon$ for system (1.1) − (1.5) as $\epsilon \to 0$. We know that the semigroup $S_t(\epsilon)$ generated by the system (1.1) − (1.5) with $\epsilon > 0$ and the semigroup S_t associated with the limit system (2.1) − (2.4) are defined on different phase spaces (since the limit system (2.1) − (2.4) does not require to have an initial data for the derivative $\partial_t u|_{t=0}$). In order to overcome this difficulty, following the standard procedure (for the theory of singularly perturbed hyperbolic equations,(see Lyusternik & Vishik [10], Babin & Vishik [1], Fabrie & Galusinski [8] and Grasselli & Miranville [9]), we define the infinite dimensional submanifold $\mathcal{N}_0$ of the space Φ_0 as follows:

$$\mathcal{N}_0 = \{([u,v],\zeta_\alpha) \in \Phi_0, v = \mathcal{N}(u,\alpha) = \partial_t \alpha + \Delta u - f(u)\}, \tag{5.1}$$

and define the semigroup $S_t(0) : \mathcal{N}_0 \longrightarrow \mathcal{N}_0$ by the following expression

$$S_t(0)(u_0,u_1,\zeta_{\alpha_0}) = (S_t(u_0,\zeta_{\alpha_0}),\ \ \mathcal{N}(u_0,\zeta_{\alpha_0})), (u_0,u_1,\zeta_{\alpha_0}) \in \mathcal{N}_0. \tag{5.2}$$

In order to construct a robust family of exponential attractors we need the following theorem.

Theorem 5.1. *Let the assumptions of Theorem 3.3 hold. Then, there exist a positive number R_0 and a family of exponential attractors $\mathcal{M}_\epsilon$, $\epsilon \in [0,\epsilon_0]$, of the semigroups $S_t(\epsilon)$ such that*

1) the following inclusions hold

$$\mathcal{M}_\epsilon \subset \mathbb{B}^1_{R_0}(\epsilon), \qquad S_t(\epsilon)\mathcal{M}_\epsilon \subset \mathcal{M}_\epsilon, \tag{5.3}$$

for all $t \in \mathbb{R}_+, \epsilon \in [0,\epsilon_0]$;

2) the fractal dimension of $\mathcal{M}_\epsilon$ is uniform bounded with respect ϵ

$$\dim_F \Big(\mathcal{M}_\epsilon, \Phi_2\Big) \leq C, \quad \epsilon \in [0, \epsilon_0]; \tag{5.4}$$

3) the attractors $\mathcal{M}_\epsilon$ converge to the limit attractor $\mathcal{M}_0$ in the following sense

$$dist^{sym}_{\Phi_2}(\mathcal{M}_\epsilon, \mathcal{M}_0) \leq C\epsilon^\kappa, \tag{5.5}$$

where $dist^{sym}_V$ denotes the symmetric distance between sets in the space V and the positive constants C and κ are independent of ϵ;

4) the set $\mathcal{M}_\epsilon$ attract exponentially the trajectories of the semigroups $S_t(\epsilon)$, i.e., there exists a positive constant β (which is independent of ϵ) such that, for every $\epsilon \in [0, \epsilon_0]$,

$$dist^{sym}_{\Phi_2}(S_t(\epsilon)\mathbb{B}^2_R(\epsilon), \mathcal{M}_\epsilon) \leq Q(R)e^{-\beta t}, \text{ for all } t \in \mathbb{R}_+, \tag{5.6}$$

where Q is a monotonic function independent of initial data.

Proof. We first construct the exponential attractors $\mathcal{M}_\epsilon$ for more regular initial data belonging to $B^1_{R_0}(\epsilon) \cap B^2_R(\epsilon)$. Thanks to estimates (3.34), it is sufficient to construct the exponential attractors $\mathcal{M}_\epsilon$ for initial data belonging to $B_\epsilon = B^2_{R_1}(\epsilon)$ where R_1 is large enough and

$$B^2_{R_1}(\epsilon) = \{(\zeta_u, \zeta_\alpha) \in \Phi_2, \| (\zeta_u, \zeta_\alpha) \|_{\Phi_2} \leq R_1\}.$$

According to this estimate, there exists $T = T(R_0)$ which is independent of $\epsilon \in (0, \epsilon_0)$, such that

$$S_t(\epsilon)B_\epsilon \subset B_\epsilon, \qquad \text{for all } t \geq T. \tag{5.7}$$

For $\epsilon = 0$, we define the set $\mathbb{B}_0$ by

$$B_0 = \{([u, v], \zeta_\alpha) \in \mathbb{N}^2, \| (u, \zeta_\alpha) \|_{\Phi_1} \leq R_0\}. \tag{5.8}$$

Then, we have $B_0 \subset B^2_{\overline{R}}(0)$, where $\overline{R} = \overline{R}(R_0)$ is large enough, and due to estimate (2.18), B_0 is an absorbing set for the semigroups $S_t(0)$ in $\mathcal{N}^2$ if R_0 is large enough, i.e., we have

$$S_t B_0 \subset B_0, \qquad \text{for all } t \geq T. \tag{5.9}$$

Thus, we define the discrete semigroups $S^{(n)}_\epsilon = S_{nT}(\epsilon)$ acting on the phase spaces $\mathbb{B}_\epsilon$:

$$S^{(n)}_\epsilon : B_\epsilon \to B_\epsilon, \qquad \text{for all } n \in \mathbb{N}, \epsilon \in [0, \epsilon_0]. \tag{5.10}$$

Instead of constructing the exponential attractors $\mathcal{M}_\epsilon$ for the continuous semigroups $S_t(\epsilon)$, we first construct the exponential attractors $\mathcal{M}^d_\epsilon$ for the discrte semigroups $S^{(n)}_\epsilon$.

To this end, we apply the abstract theorem on perturbations of exponential attractors proved by Fabrie & Galusinski in [8]. To do this, we need to verify that there exist two families of Banach spaces $E(\epsilon)$ and $E^1(\epsilon)$, $\epsilon \in [0, \epsilon_0]$, such that:

1) the set B_ϵ is a closed bounded subset of $E(\epsilon)$, $\mathbb{B}_0 \subset E(\epsilon)$ for all $\epsilon \in [0, \epsilon_0]$ and

$$\| b_0 \|_{E(\epsilon)} \leq C_1 \| b_0 \|_{E(0)} + C_2\epsilon^\delta, \text{ for all } b_0 \in \mathbb{B}_0, \tag{5.11}$$

where the positive constants C_1, C_2 and δ are independent of ϵ;

2) the space $E^1(\epsilon)$ is compactly embedded into the space $E(\epsilon)$, for all $\epsilon \in [0, \epsilon_0]$, and this compactness is uniform with respect to ϵ in the following sense:

$$N_\mu(B(1,0,E^1(\epsilon)), E(\epsilon)) \leq \mathbb{M}(\mu), \text{ for all } \mu > 0, \tag{5.12}$$

where $B(1,0,E^1(\epsilon))$ is the unit ball of $E^1(\epsilon)$, $N_\mu(X,Y)$ denotes the minimal number of μ-balls in V which are necessary to cover the subset $X \subset V$ and the monotonic decreasing function $\mathbb{M}$ is independent of ϵ;

3) there exist two maps $\mathbf{C}_\epsilon$ and $\mathbf{K}_\epsilon$ (which map B_ϵ onto $E(\epsilon)$) such that $S_\epsilon = \mathbf{C}_\epsilon + \mathbf{K}_\epsilon$ and, for every $b_1^\epsilon, b_2^\epsilon \in B_\epsilon$, we have

$$\| \mathbf{K}_\epsilon b_1^\epsilon - \mathbf{K}_\epsilon b_2^\epsilon \|_{E^1(\epsilon)} \leq K \| \mathbf{K}_\epsilon b_1^\epsilon - \mathbf{K}_\epsilon b_2^\epsilon \|_{E(\epsilon)}, \tag{5.13}$$

$$\| \mathbf{C}_\epsilon b_1^\epsilon - \mathbf{C}_\epsilon b_2^\epsilon \|_{E(\epsilon)} \leq \delta \| \mathbf{C}_\epsilon b_1^\epsilon - \mathbf{C}_\epsilon b_2^\epsilon \|_{E(\epsilon)}, \tag{5.14}$$

where $\delta < \frac{1}{2}$ and K are independent of ϵ;

4) there exist nonlinear projectors $\Pi_\epsilon : B_\epsilon \to B_0$ such that $\Pi_\epsilon B_\epsilon = B_0$ and

$$\| S_\epsilon^{(n)} b_\epsilon - S_0^{(n)} \Pi_\epsilon b_\epsilon \|_{E(\epsilon)} \leq C \epsilon L^n, \quad n \in \mathbb{N}, \quad b_\epsilon \in \mathbf{B}_\epsilon, \tag{5.15}$$

where the positive constants C and L are also independent of ϵ.

We are going to verify these conditions for the semigroups $S_t(\epsilon)$ generated by the hyperbolic system $(2.1) - (2.4)$. To this end, we set

$$E^i(\epsilon) = \varepsilon^i(\epsilon) \times \varepsilon^i(1), \qquad i = 0, 1. \tag{5.16}$$

Then, the first condition (with $\delta = \frac{1}{2}$ in (5.11)) follows immediately from the definition of the sets $\mathbb{B}_\epsilon$. The second condition is verified for $E(\epsilon) = E^0(\epsilon)$. The third assumption follows from Theorem 4.2 if T is large enough (we recall that $S_\epsilon = S_T(\epsilon)$). To prove the fourth assumption we define the projectors $\Pi_\epsilon = \Pi$ by following expression

$$\Pi([u,v], \zeta_\alpha) = ([u, \mathcal{N}(u,\alpha)], \zeta_\alpha), \tag{5.17}$$

where the map $\mathcal{N}$ is the same as in (5.1). Following the definition of the set B_0 we have $\Pi B_\epsilon \subset B_0$. Moreover, as $([u,0],\alpha,w) \in \Pi^{-1}([u,\mathcal{N}(u,\alpha,w)],\alpha,w)$, then, $\Pi B_\epsilon = B_0$, for every $\epsilon > 0$. We can remark that estimate (5.6) is an immediate consequence of estimate (4.74). Thus, all the assumptions of the abstract theorem on the existence of a robust family of exponential attractors are satisfied and, due to this theorem (see Fabrie & Galusinski [8]), there exists a family $\mathcal{M}_\epsilon^d \subset B_\epsilon$ of exponential attractors for the semigroups $S_\epsilon^{(n)}$ such that

$$dist_{E(\epsilon)}\Big(S_\epsilon^{(n)} B_\epsilon, \mathcal{M}_\epsilon^d\Big) \leq C e^{-Ln}, \tag{5.18}$$

where the constants C and L are independent of t and ϵ,

$$dim_F\Big(\mathcal{M}_\epsilon^d, E(\epsilon)\Big) \leq C_1, \tag{5.19}$$

where the constant C_1 is independent of ϵ, and

$$dist_{E(\epsilon)}^{sym}\Big(\mathcal{M}_\epsilon^d, \mathcal{M}_0^d\Big) \leq C_2 \epsilon^\kappa, \tag{5.20}$$

where the positive constants C_2 and κ are independent of ϵ.

Thus, we have constructed the desired exponential attractors for the discrete semigroups. In order to obtain the exponential attractors for the continuous semigroups $S_t(\epsilon)$, we use the following standard formula:

$$\mathcal{M}_\epsilon = \bigcup_{t\in[T,2T]} S_t(\epsilon)\mathcal{M}_\epsilon^d. \tag{5.21}$$

In order to verify that the exponential attractors $\mathcal{M}_\epsilon$ so constructed satisfy all assumptions of Theorem 5.1, we use estimates (4.1), (3.29), (4.87) and (4.86) which prove that the semigroups $S_t(\epsilon)$ are uniformly Lipschitz continuous on $[T, 2T] \times B_\epsilon$ in the metric of $E(\epsilon)$. Consequently, due to (5.18) and (5.19) , we have

$$dist_{E(\epsilon)}\Big(S_\epsilon\ B_\epsilon, \mathcal{M}_\epsilon\Big) \le C' e^{-L't}, \tag{5.22}$$

$$dim_F\Big(\mathcal{M}_\epsilon, E(\epsilon)\Big) \le C_1', \tag{5.23}$$

where the new constants C', C_1' and L' are independent of ϵ. From the estimates (4.78), (4.81) and (5.11), we have the analogue of (5.20) for the continuous attractors

$$dist^{sym}_{E(\epsilon)}\Big(\mathcal{M}_\epsilon, \mathcal{M}_0\Big) \le C_2' \epsilon^\kappa, \tag{5.24}$$

see Fabrie & Galusinski [8] for the details. We also recall that, due to Theorem 3.5, the trajectories of the semigroups $S_t(\epsilon)$ are uniformly bounded in Φ_1 and we deduce that estimates (5.22) and (5.24) remain valid with the space $E(\epsilon)$ replaced by Φ_0.

Thus, all the assertions of Theorem 5.1, except (5.6), are satisfied and instead of estimate (5.6), we only have

$$dist_{\Phi_0}(S_t(\epsilon)[B_R^1(\epsilon) \cap B_R^2(\epsilon)], \mathcal{M}_\epsilon) \le Q(R)e^{-\beta t}, \text{ for all } t \in \mathbb{R}_+, \tag{5.25}$$

where the positive constant β and the monotonic function Q are independent of ϵ. Using now estimate (5.25), Theorem 3.5 and the transitivity of the exponential attraction (see Fabrie & Galusinski [8]), we derive estimate (5.6) for initial data in $B_R^1(\epsilon)$. This finishes the proof of the theorem. $\square$

Remark 5.1. It follows from Theorem 5.1 that there exists a family of global attractors $\mathcal{A}_\epsilon, \epsilon \in [0, \epsilon_0]$, of the semigroups $S_t(\epsilon)$ such that

1) $\mathcal{A}_\epsilon \subset B^2_{R_0}(\epsilon), S_t(\epsilon)\mathcal{A}_\epsilon = \mathcal{A}_\epsilon, \forall t \in \mathbb{R}_+, \epsilon \in [0, \epsilon_0]$;
2) the fractal dimension of $\mathcal{A}_\epsilon$ is uniformly bounded with respect to ϵ, that is,

$$\dim_F(\mathcal{A}_\epsilon, \Phi_0) \le C, \epsilon \in [0, \epsilon_0]. \tag{5.26}$$

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Appendix

The aim here is to give the proof of Corollary 5.2 and Proposition 5.3 of Grasselli & Miranville (see [9]). Actually we make use of these results to deduce uniform energy estimates for the following initial and boundary value problem for a singularly perturbed damped hyperbolic equation

$$\epsilon\partial_t^2 v + \partial_t v - \Delta v = h(t), \quad \zeta_v|_{t=0} = \zeta^0, \quad v|_{\partial\Omega} = 0. \tag{5.27}$$

The following theorem gives the uniform $\varepsilon^1(\epsilon)$-norm energy estimate of equation (5.27).

Theorem 5.2. *Let v be a solution of (5.27) and the function $h(t)$ be such that $h(t)|_{\partial\Omega} = 0$ for all $t \geq 0$. Then, the following estimate*

$$\begin{aligned}
&\|\zeta_v(t)\|^2_{\varepsilon^1(\epsilon)} + \int_0^t e^{-\beta(t-s)}\|\partial_t v(s)\|^2_{H^1} ds \\
\leq &Ce^{-\beta t}\Big(\|\zeta_v(0)\|^2_{\varepsilon^1(\epsilon)} + \|h(0)\|^2\Big) + \int_0^t e^{-\beta(t-s)}\Big(\|h(s)\|^2_{H^1} + \|\partial_t h(s)\|^2_{H^{-1}}\Big)ds,
\end{aligned} \tag{5.28}$$

is valid, where the positive constants C and β are independent of ϵ.

Proof. Multiplying (5.27) by $(-\Delta)(\partial_t v + \gamma v)$, where $\gamma > 0$ is a sufficiently small (but independent of ϵ) number, integrating over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\epsilon\|\partial_t v\|^2_{H^1} + \|v\|^2_{H^2} + \gamma(\|v\|^2_{H^1} + 2\epsilon(\Delta v, \Delta\partial_t v))\Big) \\
&+ (1 - 2\epsilon\gamma)\|\partial_t v\|^2_{H^1} + \gamma\|v\|^2_{H^2} \leq C\|h(t)\|^2_{H^1},
\end{aligned} \tag{5.29}$$

where the positive constant C is independent of ϵ. Moreover, for a sufficiently small $\gamma > 0$, there exists $C_1 > 0$ such that

$$\begin{aligned}
&C_1^{-1}(\epsilon\|\partial_t v(t)\|^2_{H^1} + \|v(t)\|^2_{H^2}) \\
\leq &\epsilon\|\partial_t v\|^2_{H^1} + \|v\|^2_{H^2} + \gamma(\|v\|^2_{H^1} + 2\epsilon\gamma(\Delta v, \Delta\partial_t v)) \\
\leq &C_1(\epsilon\|\partial_t v(t)\|^2_{H^1} + \|v(t)\|^2_{H^2}).
\end{aligned} \tag{5.30}$$

Thanks to (5.30), (5.29) can be written in the form

$$\begin{aligned}
&\frac{d}{dt}\Big(\epsilon\|\partial_t v\|^2_{H^1} + \|v\|^2_{H^2} + \gamma(\|v\|^2_{H^1} + 2\epsilon(\Delta v, \Delta\partial_t v))\Big) \\
&+ \beta\Big(\epsilon\|\partial_t v\|^2_{H^1} + \|v\|^2_{H^2} + \gamma(\|v\|^2_{H^1} + 2\epsilon(\Delta v, \Delta\partial_t v))\Big) + \beta\|\partial_t v\|^2_{H^1} \leq C\|h(t)\|^2_{H^1},
\end{aligned}$$

where the positive constants C and β are independent of ϵ. Applying Gronwall's inequality, owing to (5.30), we have

$$\begin{aligned}
&\epsilon\|\partial_t v(t)\|^2_{H^1} + \|v(t)\|^2_{H^2} + \int_0^t e^{-\beta(t-s)}\|\partial_t v(s)\|^2_{H^1} ds \\
\leq &Ce^{-\beta t}\Big(\|\partial_t v(0)\|^2_{H^1} + \|v(0)\|^2_{H^2}\Big)ds + C\int_0^t e^{-\beta(t-s)}\|h(s)\|^2_{H^1} ds.
\end{aligned} \tag{5.31}$$

Thus, it only remains to deduce the estimate for L^2-norm of $\partial_t v$. To do that, we multiply (5.27) by $\partial_t^2 v$ and integrate over Ω. We obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\partial_t v\|^2 + 2(v,(-\Delta)\partial_t v) - 2(h,\partial_t v)\Big) + \beta\Big(\|\partial_t v\|^2 + 2(u,(-\Delta)\partial_t v) - 2(h,\partial_t u)\Big)\\
&+ 2\epsilon\|\partial_t^2 v\|^2 = 2\|\partial_t v\|_{H^1}^2 - 2(\partial_t h,\partial_t v) + \Big(\|\partial_t v\|^2 + 2(v,(-\Delta)\partial_t v) - 2(h,\partial_t v)\Big)\\
&= g(t), \qquad (5.32)
\end{aligned}$$

where $\beta > 0$ and the function g is such that

$$|g(t)| \le C\|\partial_t v\|_{H^1}^2 + \beta\|v(t)\|_{H^1}^2 + C(\|h(t)\|_{H^1}^2 + \|\partial_t h(t)\|_{H^{-1}}^2). \qquad (5.33)$$

Applying Gronwall's inequality to (5.32), owing to (5.31) and (5.33), we have

$$\begin{aligned}
&\|\partial_t v(t)\|^2 + 2(v(t),(-\Delta)\partial_t v(t)) - 2(h(t),\partial_t v(t)) + \int_0^t e^{-\beta(t-s)}2\epsilon\|\partial_t^2 v(s)\|^2 ds\\
\le&\Big(\|\partial_t v(0)\|^2 + 2(v(0),(-\Delta)\partial_t v(0)) - 2(h(0),\partial_t v(0))\Big)e^{-\beta t}\\
&+ C\int_0^t e^{-\beta(t-s)}\Big(\|h(s)\|_{H^1}^2 + \|\partial_t h(s)\|_{H^{-1}}^2\Big)ds, \qquad (5.34)
\end{aligned}$$

where the positive costant C is independent of ϵ. We know that

$$\begin{aligned}
\|h(t)\|^2 - e^{-\beta t}\|h(0)\|^2 &= \int_0^t \frac{d}{dt}\Big(e^{-\beta(t-s)}\|h(s)\|^2\Big)ds\\
&= \int_0^t e^{-\beta(t-s)}\Big(\beta\|h(s)\|^2 + 2(h(s),\partial_t h(s))\Big)ds\\
&\le \int_0^t e^{-\beta(t-s)}\Big(\beta\|h(s)\|^2 + 2\|h(s)\|_{H^1}\|\partial_t h(s)\|_{H^{-1}}\Big)ds\\
&\le C\int_0^t e^{-\beta(t-s)}\Big(\|h(s)\|_{H^1}^2 + \|\partial_t h(s)\|_{H^{-1}}^2\Big)ds. \qquad (5.35)
\end{aligned}$$

Adding (5.31) and γ(5.34) where $\gamma > 0$ is sufficiently small, we have, owing to (5.34)

$$\begin{aligned}
\|\partial_t v(t)\|^2 \le& C(\|\zeta_v(0)\|_{\varepsilon(\epsilon)}^2 + \|h(0)\|^2)ds e^{-\beta t}\\
&+ C\int_0^t e^{-\beta(t-s)}\Big(\|h(s)\|_{H^1}^2 + \|\partial_t h(s)\|_{H^{-1}}^2\Big)ds. \qquad (5.36)
\end{aligned}$$

Combining (5.31) and (5.36), we obtain the result. $\square$

The following theorem gives the uniform $\varepsilon(\epsilon)$-norm energy estimate of equation (5.27), in the case where the H^{-1}-norm of the right-hand side h is known or can be obtained.

Theorem 5.3. *Let v be a solution of (5.27) such that $\zeta_v(0) \in \varepsilon(\epsilon)$. Then, the following estimate*

$$\begin{aligned}
&\| \zeta_v(t) \|_{\varepsilon(\epsilon)}^2 + \int_0^t e^{-\beta(t-s)} \| \partial_t v(s) \|^2 ds\\
\le& Ce^{-\beta t}(\| \zeta_v(0) \|_{\varepsilon(\epsilon)}^2 + \| h(0) \|_{H^{-1}}^2)\\
&+ C\Big(\int_0^t e^{-\beta(t-s)}(\| \partial_t h(s) \|_{H^{-1}} + \| h(s) \|_{H^{-1}})ds\Big)^2, \qquad (5.37)
\end{aligned}$$

is valid, where the positive constants β and C are independent of ϵ.

Proof. In order to prove this theorem, we proceed as Grasselli & Miranville (see [9]). We first multiply (5.27) by $2(\partial_t v+\gamma v)$, where $\beta>0$ is small (but independent of ϵ) number, integrate over Ω and have

$$\frac{d}{dt}\Gamma+2(1-\beta\epsilon)\|\partial_t v\|^2+2\beta\|u\|^2_{H^1}=4(h(t),(-\Delta)^{-1}\partial_t h)+2(\beta h-\partial_t h,v), \quad (5.38)$$

where

$$\Gamma=\epsilon\|\partial_t v\|^2+\|v\|^2_{H^1}+\beta\|v\|^2+2\beta\epsilon(v,\partial_t v)-2(h(t),v(t))+2\|h\|^2_{H^{-1}}.$$

Moreover, for a sufficiently small $\beta>0$, there exists a constant $C>0$ such that

$$\begin{aligned} &C\Big(\epsilon\|\partial_t v(t)\|^2+\|v(t)\|^2_{H^1}+\|h(t)\|^2_{H^{-1}}\Big)\le\Gamma(t)\\ \le&C\Big(\epsilon\|\partial_t v(t)\|^2+\|v(t)\|^2_{H^1}+\|h(t)\|^2_{H^{-1}}\Big), \end{aligned} \quad (5.39)$$

where the positive constant C is independent of ϵ. Then, we can write (5.27) in the form

$$\partial_t\Gamma+\beta'\Gamma+\beta\|\partial_t v\|^2\le CH(t)\Gamma^{\frac{1}{2}}, \quad (5.40)$$

where the positive constants C and β' are independent of ϵ, and $H(t)=\|h(t)\|_{H^{-1}}+\|\partial_t h(t)\|_{H^{-1}}$. In order to solve (5.40), we first solve the following inequality

$$\partial_t\Gamma+\beta'\Gamma\le CH(t)\Gamma^{\frac{1}{2}}. \quad (5.41)$$

Here, $\Gamma=0$ is the trivial solution of (5.41). Assume that $\Gamma\neq 0$. We set $U(t)=\sqrt{\Gamma(t)}$ in (5.41) and we obtain

$$\frac{d}{dt}U+\beta'U\le CH(t). \quad (5.42)$$

Applying Gronwall's inequality, we obtain

$$U(t)\le U(0)e^{-\beta' t}+C\int_0^t e^{-\beta'(t-s)}H(s)ds$$

which implies

$$\begin{aligned} U(t)^2\le&\frac{3}{2}U(0)^2e^{-2\beta' t}+C(\int_0^t e^{-\beta'(t-s)}H(s)ds)^2\\ &\le U(0)^2e^{-\beta' t+1}+C(\int_0^t e^{-\beta'(t-s)}H(s)ds)^2, \end{aligned}$$

which we can write in the form

$$\Gamma(t)\le\Gamma(0)e^{\beta' t+1}+C(\int_0^t e^{\beta'(t-s)/2}H(s)ds)^2, \quad (5.43)$$

where the positive constants β' and C are independent of ϵ. Then, using (5.44), we have

$$\begin{aligned} &\epsilon\|\partial_t v(t)\|^2+\|v(t)\|^2_{H^1}+\|u(t)\|^2+\|h(t)\|^2_{H^{-1}}\\ \le&Ce^{-\beta' t}(\epsilon\|\partial_t v(0)\|^2+\|v(0)\|^2_{H^1}+\|v(t)\|^2+\|h(0)\|^2_{H^{-1}}\\ &+C\Big(\int_0^t e^{\beta'(t-s)/2}H(s)ds\Big)^2, \end{aligned} \quad (5.44)$$

where the positive constants β and C are independent of ϵ. In order to obtain the desired estimate for integral of $\|\partial_t u(s)\|^2$, we multiply (5.40) by $e^{-\beta'(T-s)}$, integrate over $[0,T]$ and use the following equality

$$\int_0^T e^{-\beta'(T-t)}[U'(t)+\beta' U(t)]dt = U(t) - U(0)e^{-\beta' T}. \tag{5.45}$$

Thanks to (5.39), (5.40) and (5.45), we have

$$\begin{aligned}
&\beta' \int_0^T e^{-\beta'(T-t)}\|\partial_t v(t)\| dt \\
\leq &\Gamma(0)e^{-\beta' T} - \Gamma(T) + C\int_0^T e^{-\beta'(T-t)} H(t)[\Gamma(t)]^{\frac{1}{2}} dt \\
\leq &\Gamma(0)e^{-\beta' T} + C\int_0^T e^{-\beta'(T-t)} H(t).e^{-\beta' t/2}[\Gamma(0)]^{\frac{1}{2}} dt \\
&+ C\int_0^T e^{-\beta'(T-t)/2} H(t).e^{-\beta'(T-t)}\Big([\Gamma(t)]^{\frac{1}{2}} - e^{\beta' t/2}[\Gamma(0)]^{\frac{1}{2}})\Big) dt \\
\leq &C_1\Gamma(0)e^{-\beta' T} + C\int_0^T e^{-\beta'(T-t)/2} H(t).e^{-\beta'(T-t)}\Big(U(t) - U(0)e^{\beta' t/2})\Big) dt \\
\leq &C_1\Gamma(0)e^{-\beta' T} + C\int_0^T e^{-\beta'(T-t)/2} H(t).e^{-\beta'(T-t)}\int_0^t e^{-\beta'(t-s)/2}[v'(s) + \frac{\beta'}{2}U(s)] ds dt \\
\leq &C_1\Gamma(0)e^{-\beta' T} + C\int_0^T e^{-\beta'(T-t)/2} H(t).e^{-\beta'(T-t)/2}\int_0^t e^{-\beta'(t-s)/2} H(s) ds dt \\
\leq &C_1\Gamma(0)e^{-\beta' T} + C\int_0^T e^{-\beta'(T-t)/2} H(t).\int_0^t e^{-\beta'(T-s)/2} H(s) ds dt \\
\leq &C_1\Gamma(0)e^{-\beta' T} + C\Big(\int_0^T e^{-\beta'(T-t)/2} H(t) dt\Big)^2.
\end{aligned} \tag{5.46}$$

In order to obtain the estimate for H^{-1}-norm of $\partial_t v(t)$, due to estimate (3.11), we can assume $\zeta_v(0) = 0$. We differentiate equation (5.27) with respect t and we obtain

$$\epsilon\partial_t^3 v + \partial_t^2 v - \Delta\partial_t v = \partial_t h. \tag{5.47}$$

Multiply (5.47) by $(-\Delta)^{-1}(2\epsilon\partial_t^2 v + \partial_t v)$ and integrate over Ω. We obtain

$$\frac{d}{dt}\Gamma_v + \epsilon\|\partial_t^2 v\|_{H^{-1}}^2 + \|\partial_t v\|^2 = 2(\partial_t h, (-\Delta)^{-1}, (-\Delta)^{-1}(\epsilon\partial_t^2 v + \partial_v)) + 2\epsilon\|\partial_t v\|^2, \tag{5.48}$$

where

$$\Gamma_v = \epsilon^2\|\partial_t^2 v\|_{H^{-1}}^2 + \epsilon\|\partial_t v\|^2 + \frac{1}{2}\|\partial_t v\|_{H^{-1}}^2 + \epsilon(\partial_t v, \partial_t^2 v).$$

There exists $C > 0$ such that

$$\begin{aligned}
&C^{-1}(\epsilon^2\|\partial_t^2 v(t)\|_{H^{-1}}^2 + \epsilon\|\partial_t v(t)\|^2 + \|\partial_t v(t)\|_{H^{-1}}^2) \\
&\leq \Gamma_v(t) \\
&C(\epsilon^2\|\partial_t^2 v(t)\|_{H^{-1}}^2 + \epsilon\|\partial_t v(t)\|^2 + \|\partial_t v(t)\|_{H^{-1}}^2),
\end{aligned} \tag{5.49}$$

where the constants C is independent of ϵ. Moreover, for sufficiently small $\epsilon > 0$, (5.48) can be written in the form

$$\frac{d}{dt}\Gamma_v + \beta'\Gamma_v = C_1\|\partial_t h\|_{H^{-1}}\Gamma_u^{\frac{1}{2}}, \tag{5.50}$$

where the positive constants β' and C_1 are independent of ϵ. Applying estimate (5.43) for the above equation, we have

$$\begin{aligned}&\epsilon^2\|\partial_t^2 v(t)\|^2_{H^{-1}} + \epsilon\|\partial_t v(t)\|^2 + \|\partial_t v(t)\|^2_{H^{-1}})\\ \leq& Ce^{-\beta' t}\epsilon^2\|\partial_t^2 v(0)\|^2_{H^{-1}} + C\Big(\int_0^t e^{-\beta'(t-s)}C_1\|\partial_t h(s)\|_{H^{-1}}ds\Big)^2.\end{aligned} \tag{5.51}$$

Here, we have also used the fact that $\zeta_u(0) = 0$, which yields, owing to (5.27), $\epsilon\partial_t^2 v = h(0)$ and the desired estimate for the H^{-1} of $\partial_t u$ is an immediate consequence of (5.51). □

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