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PHASE PORTRAITS OF Z_7-EQUIVARIANT SEXTIC HAMILTONIAN SYSTEM*

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Abstract In this paper, a sextic Hamiltonian system with Z_7-equivariant property is considered. Using the methods of qualitative analysis of differential equations, bifurcations of the above system are analyzed, the phase portraits of the system are classified and the corresponding representative orbits are shown by Maple software.

Keywords Sextic Hamiltonian system, Z_7-equivariant, qualitative analysis, phase portraits, bifurcations.

MSC(2000) 34C07, 34C23, 34C37, 37G15.

1. Introduction

Hamiltonian system has a quite important position in the study of mathematical and physical problems. Studying the phase portraits of Hamiltonian system has important meanings in analyzing its perturbation system and it is also helpful in studying the number of limit cycles of plane polynomial differential system [5].

In 2001, Li Yanmei [6] studied a class of quintic Hamiltonian system with Z_2-symmetry. She used the methods of qualitative analysis and gave the global phase portraits. Huang Yi and Wu Yuhai [4] studied a class of quartic Hamiltonian system with Z_5-symmetry and the phase portraits of the system are also given. Longwei Chen and Jianguo Ning [1] studied phase portraits of a Z_6-equivariant quintic Hamiltonian system and gave sixteen types phase portraits. In 2004, Guowei Chen and Xinan Yang [2] studied topological classifications of phase portraits of quintic Hamiltonian system which are symmetric with respect to both x-axis and y-axis, and gave 33 types phase portraits. In 2012, Fang Xinyu, Huang Wentao and Chen Aiyong [3] studied the number of limit cycles for a class of quartic Hamiltonian system. They used the accurate method to calculate the number of zeros of Abelian integrals of the system and obtained that the system had at least 14 limit cycles(For more results of limit cycles bifurcated from near Hamiltonian system see [8, 9]).

In this article, we study the Z_7-equivariant sextic Hamiltonian system. First we give the general form of Z_7-equivariant sextic Hamiltonian system.

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Consider a real planar system

$$\begin{cases} \dfrac{dx}{dt} = P(x,y), \\ \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are real sextic polynomials.

Let $z = x + yi$, $\bar{z} = x - yi$, $i^2 = -1$. Then system (1.1) is changed into the following complex form

$$\begin{cases} \dfrac{dz}{dt} = F(z,\bar{z}), \\ \dfrac{d\bar{z}}{dt} = \overline{F(z,\bar{z})}, \end{cases} \tag{1.2}$$

where $F(z,\bar{z}) = P(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}) + iQ(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i})$.

By noticing that system (1.1) is Z_7-equivariant real planar and applying the conclusion in [7], we have

$$F(Z,\bar{Z}) = \sum_{k=1} g_k(r^2)\bar{Z}^{kq-1} + \sum_{l=0} h_l(r^2)\bar{Z}^{lq+1}, \tag{1.3}$$

where $q = 7$, $k = 1$, $l = 0$.

That means the function $F(z,\bar{z})$ in (1.3) has the following form

$$F(Z,\bar{Z}) = A_0\bar{Z}^6 + A_1 Z + A_2 Z^2\bar{Z} + A_3 Z^3\bar{Z}^2, \tag{1.4}$$

where A_i, $i = 0,1,2,3$ are complex numbers.

Further, if the system (1.2) is also Hamiltonian, that is $\frac{\partial F}{\partial Z} + \frac{\partial \bar{F}}{\partial \bar{Z}} = 0$, then we have $(A_1 + \bar{A}_1) + 2r^2(A_2 + \bar{A}_2) + 3r^4(A_3 + \bar{A}_3) = 0$, where $r^2 = Z\bar{Z}$. So A_1, A_2, A_3 are pure imaginary numbers.

In the following, we assume $A_0 = a_0 + b_0 i$, $A_1 = b_1 i$, $A_2 = b_2 i$, $A_3 = b_3 i$, $Z = x + yi$, where a_0, b_0, b_1, b_2, b_3 are real parameters and $a_0 b_0 \neq 0$.

Rewriting $F(Z,\bar{Z})$ into an algebraic form, then we have

$$F(Z,\bar{Z}) = (a_0+b_0 i)(x-yi)^6 + b_1 i(x+yi) + b_2 i(x+yi)^2(x-yi) + b_3 i(x+yi)^3(x-yi)^2. \tag{1.5}$$

Then we have the following general real form of Z_7-invariant sextic Hamiltonian system

$$\begin{cases} \dfrac{dx}{dt} = & -15a_0x^4y^2 - b_2y^3 - a_0y^6 - b_2x^2y + 6b_0x^5y + 6b_0xy^2 - b_3x^4y \\ & -20b_0x^3y^3 + 15a_0x^2y^4 - 2b_3x^2y^3 + a_0x^6 - b_1y - b_3y^5, \\ \dfrac{dy}{dt} = & b_2x^3 + b_0x^6 + b_3x^5 + b_1x - b_0y^6 - ba_0x^5y - 15b_0x^4y^2 + 20a_0x^3y^3 \\ & +b_2xy^2 + 2b_3x^3y^2 + 15b_0x^2y^4 - 6a_0xy^5 + b_3xy^4, \end{cases} \tag{1.6}$$

where a_0, b_0, b_1, b_2, b_3 are five real parameters and $a_0 b_0 \neq 0$.

2. The properties of finite singular points of system (1.6)

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then system (1.6) is changed into

$$\begin{cases} \dot{r} = kr^6 \sin 7(\theta+\varphi), \\ \dot{\theta} = kr^5 \cos 7(\theta+\varphi) + b_3r^4 + b_2r^2 + b_1, \end{cases} \tag{2.1}$$

where $\varphi = \frac{1}{7}\arccos\frac{b_0}{\sqrt{a_0^2+b_0^2}}$, $k = \sqrt{a_0^2+b_0^2}$. Clearly the system (2.1) is invariant under the rotation over $2\pi/7$. By direct computation, we get that the first integral of system (2.1) is

$$H(r,\theta) = -(\frac{1}{7}kr^7\cos 7(\theta+\varphi) + \frac{1}{6}b_3r^6 + \frac{1}{4}b_2r^4 + \frac{1}{2}b_1r^2). \tag{2.2}$$

And Jacobian matrix of system (2.1) at the singular point (r^*, θ^*) is

$$J(r^*,\theta^*) = \begin{pmatrix} 6kr^5\sin 7(\theta+\varphi) & 7kr^6\cos 7(\theta+\varphi) \\ 5kr^4\cos 7(\theta+\varphi) + 4b_3r^3 + 2b_2r & -7kr^5\sin 7(\theta+\varphi) \end{pmatrix}_{(r^*,\theta^*)}.$$

To study number of singular points of system (2.1), we solve the following equations

$$\begin{cases} kr^6 \sin 7(\theta+\varphi) = 0, \\ kr^5 \cos 7(\theta+\varphi) + b_3r^4 + b_2r^2 + b_1 = 0. \end{cases} \tag{2.3}$$

From the first equation of (2.3), we get that

$$\begin{cases} \theta = -\varphi, \\ kr^5 + b_3r^4 + b_2r^2 + b_1 = 0, \end{cases} \quad \text{or} \quad \begin{cases} \theta = \frac{\pi}{7} - \varphi, \\ -kr^5 + b_3r^4 + b_2r^2 + b_1 = 0, \end{cases}$$

where $g_1(r) = kr^5 + b_3r^4 + b_2r^2 + b_1$ and $g_2(r) = -kr^5 + b_3r^4 + b_2r^2 + b_1$.

Let $g_1'(r) = 0$. Then we have $r = 0$ or $5kr^3 + 4b_3r^2 + 2b_2 = 0$. Since $5kr^3 + 4b_3r^2 + 2b_2 = 0$ is a cubic equation with real coefficients, it has at least one real solution $r = r^*$. As to the sign of $r = r^*$, there are the following three cases.

(I) $5kr^3 + 4b_3r^2 + 2b_2 = 0$ has only real root and a triple zero.

(II) $5kr^3 + 4b_3r^2 + 2b_2 = 0$ at least has $r^* > 0$.

(III) $5kr^3 + 4b_3r^2 + 2b_2 = 0$ has a negative root and the maximal real root is zero or negative.

In the following, we give the number and types of singular points of system (2.1) in different cases.

Case I: When both b_2 and b_3 are zeros, that is $r = 0$ is the unique singular point of system (2.1) and the origin is a degenerated saddle. The corresponding phase portraits of system (2.1) are plotted in Fig.1.

Case II: By scale transformation of variables, it is not loosing generality to suppose that $r^* = 1$. Then $b_3 = -\frac{5k+2b_2}{4}$ and we can rewrite $g_m'(r) = 0$ into the following forms $g_m'(r) = r(r + (-1)^m)(r^2 - \frac{2b_2r}{5k} - \frac{2b_2}{5k}) = 0$, where $m = 1, 2$.

According to the sign of $\Delta = \frac{4b_2^2}{25k^2} + \frac{8b^2}{5k}$, we have following categories about the number of positive real solutions of $g_m(r) = 0$, where $m = 1, 2$.

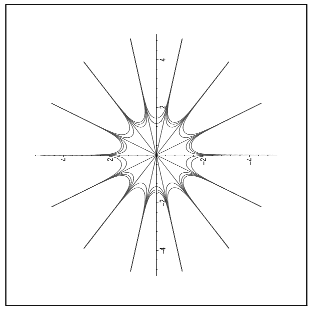

Figure 1.

(1) When $-10k < b_2 < 0$, $g_1(r) = 0$ has at most three solutions.

(2) When $b_2 = 0$ or $b_2 = -10k$, $g_1(r) = 0$ has at most four solutions.

(3) When $b_2 > 0$ or $b_2 < -10k$, $g_1(r) = 0$ has at most five solutions.

Case III: If $r^* < 0$, then we find that $-r^* > 0$ is the root of $g_2'(r) = 0$. By applying similar process to Case (II) to analyze positive solutions of equation $g_m(r) = 0$, $m = 1, 2$, we get similar results about the number and types of singularities of system (2.1).

When $\Delta = \frac{4b_2^2}{25k^2} + \frac{8b^2}{5k} < 0$, which corresponds to (1) in case (II), as to number and types of singularities of system (2.1), we have the following conclusions.

Theorem 2.1. *(1) When $-10k < b_2 < 0$, $b_1 \leq 0$ or $b_1 > \frac{1}{4}k - \frac{1}{2}b_2$, system (2.1) has seven saddle points and one center point $(0,0)$, and the phase portraits are plotted in Fig.2.*

(2) When $-10k < b_2 < 0$, $0 < b_1 < \frac{1}{4}k - \frac{1}{2}b_2$, system (2.1) has fourteen saddle points and eight centers (including point $(0,0)$), and the phase portraits are plotted in Fig.3.

(3) When $-10k < b_2 < 0$, $b_1 = \frac{1}{4}k - \frac{1}{2}b_2$, system (2.1) has seven saddle points, seven degenerate saddle points and one center point $(0,0)$, and the phase portraits are plotted in Fig.4.

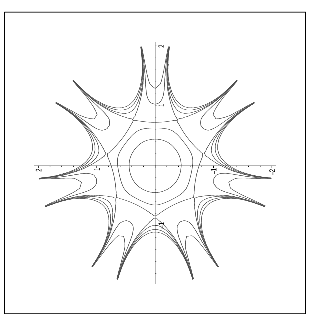

Figure 2.

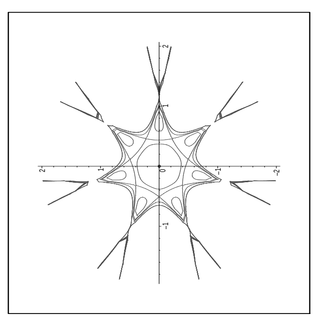

Figure 3.

When $\Delta = \frac{4b_2^2}{25k^2} + \frac{8b^2}{5k} = 0$, which corresponds to (2) in case (II), as to number and types of singularities of system (2.1), we have the following conclusions.

Theorem 2.2. *(1) When $b_2 = 0$, $b_1 \le 0$ or $b_1 > \frac{1}{4}k$, or $b_2 = -10k$, $b_1 \le 0$ or $b_1 > 12k$ or $\frac{21}{4} < b_1 < 12k$, system* (2.1) *has seven saddle points and one center(point $(0,0)$), and the phase portraits are plotted in Fig.2.*

(2) When $b_2 = 0$, $0 < b_1 < \frac{1}{4}k$, or $b_2 = -10k$, $0 < b_1 < \frac{21}{4}k$, system (2.1) *has fourteen saddle points and eight centers (including point $(0,0)$), and the phase portraits are plotted in Fig.3.*

(3) When $b_2 = 0$, $b_1 = \frac{1}{4}k$, or $b_2 = -10k$, $b_1 = \frac{21}{4}k$, system (2.1) *has seven saddle points, seven degenerate saddle points and one center point $(0,0)$, and the phase portraits are plotted in Fig.4.*

(4) When $b_2 = 0$, $b_1 = 12k$, system (2.1) *has seven degenerate saddle points and one center point $(0,0)$, and the phase portraits are plotted in Fig.5.*

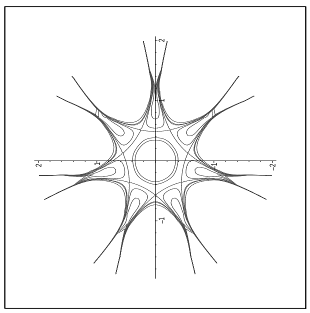

Figure 4.

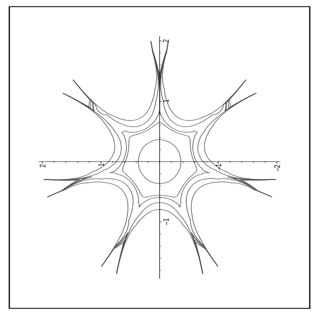

Figure 5.

When $\Delta = \frac{4b_2^2}{25k^2} + \frac{8b^2}{5k} > 0$, which corresponds to (3) in case (II). We study the number and types of singularities of system (2.1) in details. For hardness of symbol computations of equation, we choose proper value of parameter in different situations about extremum of $h_1(r) = kr^5 + b_3r^4 + b_2r^2$ and $h_2(r) = -kr^5 + b_3r^4 + b_2r^2$, and get the following results about phase portraits of system (2.1) for $k > 0$, particularly we choose $k = 1$ in this sub-case.

Theorem 2.3. *(1) When $0 < b_2 < \frac{1}{2}$, then we have $r_4 < r_1 < r_3 < r_2$, where* $r_1 = 0$, $r_2 = 1$, $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$, $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When $b_1 < -h_1(r_3)$ or $b_1 > -h_1(r_2)$, system* (2.1) *has seven saddle points and one center point $(0,0)$, and the phase portraits are plotted in Fig.2.*

② *When $-h_1(r_3) < b_1 < -h_1(r_4)$ or $0 \le b_1 < -h_1(r_2)$, system* (2.1) *has fourteen saddle points and eight centers (including point $(0,0)$), and the phase portraits are plotted in Fig.3.*

③ *When $b_1 = -h_1(r_3)$ or $b_1 = -h_1(r_2)$, system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point $(0,0)$, and the phase portraits are plotted in Fig.4.*

④ *When $b_1 = -h_1(r_4)$, system* (2.1) *has fourteen saddle points, seven degenerate saddle points and eight centers (including point $(0,0)$), and the phase portraits are plotted in Fig.6.*

⑤ *When $-h_1(r_4) < b_1 < 0$, system* (2.1) *has twenty-one saddle points and fifteen centers (including point $(0,0)$), and the phase portraits are plotted in Fig.7.*

(2) When $b_2 = \frac{1}{2}$*, then we have* $r_4 < r_1 < r_3 < r_2$*, where* $r_1 = 0$*,* $r_2 = 1$*,* $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$*,* $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When* $b_1 < -h_1(r_3)$ *or* $b_1 > -h_1(r_2)$*, system* (2.1) *has seven saddle points and one center point* $(0,0)$*, and the phase portraits are plotted in Fig.2.*

② *When* $-h_1(r_3) < b_1 < -h_1(r_4)$*, system* (2.1) *has fourteen saddle points and eight centers (including point* $(0,0)$*), and the the phase portraits are plotted in Fig.3.*

③ *When* $b_1 = -h_1(r_3)$ *or* $b_1 = -h_1(r_2)$*, system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point* $(0,0)$*, and the phase portraits are plotted in Fig.4.*

④ *When* $b_1 = -h_1(r_4)$*, system* (2.1) *has fourteen saddle points, seven degenerate saddle points and eight centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.6.*

⑤ *When* $-h_1(r_4) < b_1 < 0$*, system* (2.1) *has twenty-one saddle points and fifteen centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.7.*

(3) When $\frac{1}{2} < b_1 < \frac{5}{4}$*, then we have* $r_4 < r_1 < r_3 < r_2$*, where* $r_1 = 0$*,* $r_2 = 1$*,* $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$*,* $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When* $b_1 < -h_1(r_3)$ *or* $-h_1(r_2) < b_1 < -h_1(r_4)$ *or* $b_1 \geq 0$*, system* (2.1) *has seven saddle points and one center point* $(0,0)$*, and the phase portraits are plotted in Fig.2.*

② *When* $-h_1(r_3) < b_1 < -h_1(r_2)$ *or* $-h_1(r_4) < b_1 < 0$*, system* (2.1) *has fourteen saddle points and eight centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.3.*

③ *When* $b_1 = -h_1(r_3)$ *or* $b_1 = -h_1(r_2)$ *or* $b_1 = -h_1(r_4)$*, system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point* $(0,0)$*, and the phase portraits are plotted in Fig.4.*

(4) When $b_1 = \frac{5}{4}$*, then we have* $r_4 < r_1 < r_3 = r_2$*, where* $r_1 = 0$*,* $r_2 = 1$*,* $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$*,* $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When* $b_1 < -h_1(r_3)$ *or* $-h_1(r_3) < b_1 < -h_1(r_4)$ *or* $b_1 \geq 0$*, system* (2.1) *has seven saddle points and one center point* $(0,0)$*, and the phase portraits are plotted in Fig.2.*

② *When* $-h_1(r_4) < b_1 < 0$*, system* (2.1) *has fourteen saddle points and eight centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.3.*

③ *When* $b_1 = -h_1(r_3)$*, system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point* $(0,0)$*, and the the phase portraits are plotted in Fig.4.*

④ *When* $b_1 = -h_1(r_4)$*, system* (2.1) *has seven degenerate saddle points and one center point* $(0,0)$*, and the phase portraits are plotted in Fig.5.*

(5) When $\frac{5}{4} < b_1 < -4 + \frac{3}{2}\sqrt{21}$*, then we have* $r_4 < r_1 < r_2 < r_3$*, where* $r_1 = 0$*,* $r_2 = 1$*,* $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$*,* $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When* $b_1 < -h_1(r_2)$ *or* $-h_1(r_3) < b_1 < -h_1(r_4)$ *or* $b_1 \geq 0$, *system* (2.1) *has seven saddle points and one center point* $(0, 0)$, *and the phase portraits are plotted in Fig.2.*

② *When* $-h_1(r_2) < b_1 < -h_1(r_3)$ *or* $-h_1(r_4) < b_1 < 0$, *system* (2.1) *has fourteen saddle points and eight centers (including point* $(0, 0)$*), and the phase portraits are plotted in Fig.3.*

③ *When* $b_1 = -h_1(r_3)$ *or* $b_1 = -h_1(r_2)$ *or* $b_1 = -h_1(r_4)$, *system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point* $(0, 0)$, *and the phase portraits are plotted in Fig.4.*

(6) When $b_2 = -4 + \frac{3}{2}\sqrt{21}$, *then we have* $r_4 < r_1 < r_2 < r_3$, *where* $r_1 = 0$, $r_2 = 1$, $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$, $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When* $b_1 < -h_1(r_2)$ *or* $b_1 > -h_1(r_3)$, *system* (2.1) *has seven saddle points and one center point* $(0, 0)$, *and the phase portraits are plotted in Fig.2.*

② *When* $-h_1(r_2) < b_1 < -h_1(r_4)$, *system* (2.1) *has fourteen saddle points and eight centers (including point* $(0, 0)$*), and the the phase portraits are plotted in Fig.3.*

③ *When* $b_1 = -h_1(r_3)$ *or* $b_1 = -h_1(r_2)$, *system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point* $(0, 0)$, *and the phase portraits are plotted in Fig.4.*

④ *When* $b_1 = -h_1(r_4)$, *system* (2.1) *has fourteen saddle points, seven degenerate saddle points and eight centers (including point* $(0, 0)$*), and the phase portraits are plotted in Fig.6.*

⑤ *When* $-h_1(r_4) < b_1 < 0$, *system* (2.1) *has twenty-one saddle points and fifteen centers (including point* $(0, 0)$*), and the phase portraits are plotted in Fig.7.*

(7) When $b_2 > -4 + \frac{3}{2}\sqrt{21}$, *then we have* $r_4 < r_1 < r_2 < r_3$, *where* $r_1 = 0$, $r_2 = 1$, $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$, $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When* $b_1 < -h_1(r_2)$ *or* $b_1 > -h_1(r_3)$, *system* (2.1) *has seven saddle points and one center point* $(0, 0)$, *and the phase portraits are plotted in Fig.2.*

② *When* $-h_1(r_2) < b_1 < -h_1(r_4)$ *or* $0 \leq b_1 < -h_1(r_3)$, *system* (2.1) *has fourteen saddle points and eight centers (including point* $(0, 0)$*), and the phase portraits are plotted in Fig.3.*

③ *When* $b_1 = -h_1(r_3)$ *or* $b_1 = -h_1(r_2)$, *system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point* $(0, 0)$, *and the phase portraits are plotted in Fig.4.*

④ *When* $b_1 = -h_1(r_4)$, *system* (2.1) *has fourteen saddle points, seven degenerate saddle points and eight centers (including point* $(0, 0)$*), and the phase portraits are plotted in Fig.6.*

⑤ *When* $-h_1(r_4) < b_1 < 0$, *system* (2.1) *has twenty-one saddle points and fifteen centers (including point* $(0, 0)$*), and the phase portraits are plotted in Fig.7.*

(8) When $-4 - \frac{3}{2}\sqrt{21} < b_1 < -10$, *then we have* $r_4 < r_3 < r_1 < r_2$, *where* $r_1 = 0$, $r_2 = 1$, $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$, $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When* $b_1 > -h_1(r_3)$ *or* $-h_1(r_2) < b_1 < -h_1(r_4)$ *or* $b_1 \leq 0$, *system* (2.1) *has seven saddle points and one center point* $(0,0)$, *and the phase portraits are plotted in Fig.2.*

② *When* $-h_1(r_4) < b_1 < -h_1(r_3)$ *or* $0 < b_1 < -h_1(r_2)$, *system* (2.1) *has fourteen saddle points and eight centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.3.*

③ *When* $b_1 = -h_1(r_3)$ *or* $b_1 = -h_1(r_2)$ *or* $b_1 = -h_1(r_4)$, *system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point* $(0,0)$, *and the phase portraits are plotted in Fig.4.*

(9) When $b_1 = -4 - \frac{3}{2}\sqrt{21}$, *then we have* $r_4 < r_3 < r_1 < r_2$, *where* $r_1 = 0$, $r_2 = 1$, $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$, $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When* $b_1 < -h_1(r_4)$ *or* $b_1 > -h_1(r_3)$, *system* (2.1) *has seven saddle points and one center point* $(0,0)$, *and the phase portraits are plotted in Fig.2.*

② *When* $-h_1(r_2) < b_1 < -h_1(r_3)$, *system* (2.1) *has fourteen saddle points and eight centers (including point* $(0,0)$*), and the the phase portraits are plotted in Fig.3.*

③ *When* $b_1 = -h_1(r_3)$ *or* $b_1 = -h_1(r_4)$, *system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point* $(0,0)$, *and the phase portraits are plotted in Fig.4.*

④ *When* $b_1 = -h_1(r_2)$, *system* (2.1) *has fourteen saddle points, seven degenerate saddle points and eight centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.6.*

⑤ *When* $0 < b_1 < -h_1(r_2)$, *system* (2.1) *has twenty-one saddle points and fifteen centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.7.*

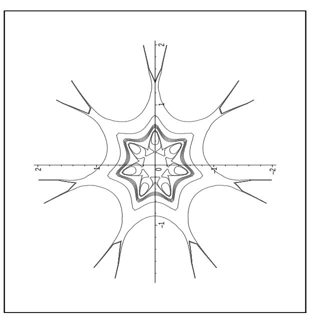

Figure 6.

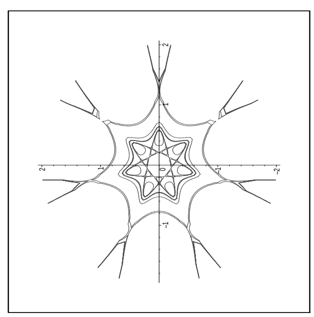

Figure 7.

(10) When $b_1 < -4 - \frac{3}{2}\sqrt{21}$, *then we have* $r_4 < r_3 < r_1 < r_2$, *where* $r_1 = 0$, $r_2 = 1$, $r_3 = \frac{b_2+\sqrt{b_2^2+10b_2}}{5}$, $r_4 = \frac{b_2-\sqrt{b_2^2+10b_2}}{5}$.

① *When* $b_1 < -h_1(r_4)$ *or* $b_1 > -h_1(r_3)$, *system* (2.1) *has seven saddle points and one center point* $(0,0)$, *and the phase portraits are plotted in Fig.2.*

② *When* $-h_1(r_2) < b_1 < -h_1(r_3)$ *or* $-h_1(r_4) < b_1 \leq 0$, *system* (2.1) *has fourteen saddle points and eight centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.3.*

③ *When* $b_1 = -h_1(r_3)$ *or* $b_1 = -h_1(r_4)$*, system* (2.1) *has seven saddle points, seven degenerate saddle points and one center point* $(0,0)$*, and the phase portraits are plotted in Fig.4.*

④ *When* $b_1 = -h_1(r_2)$*, system* (2.1) *has fourteen saddle points, seven degenerate saddle points and eight centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.6.*

⑤ *When* $0 < b_1 < -h_1(r_2)$*, system* (2.1) *has twenty-one saddle points and fifteen centers (including point* $(0,0)$*), and the phase portraits are plotted in Fig.7.*

Proof of case (1) in the Theorem 2.3.

Let $h_1(r) = kr^5 + b_3r^4 + b_2r^2$ and $h_2(r) = -kr^5 + b_3r^4 + b_2r^2$. For $b_3 = -\frac{1}{4}(5k + 2b_2)$, we have

$$h_1(r) = kr^5 - \frac{1}{4}(5k+2b_2)r^4 + b_2r^2,\ h_2(r) = -kr^5 - \frac{1}{4}(5k+2b_2)r^4 + b_2r^2.$$

If $b_2 > 0$ or $b_2 < -10k$, then $g_1'(r) = h_1'(r) = 0$ has four solutions:

$$r_1 = 0,\quad r_2 = 1,\quad r_3 = \frac{b_2 + \sqrt{b_2^2 + 10kb_2}}{5k},\quad r_4 = \frac{b_2 - \sqrt{b_2^2 + 10kb_2}}{5k}.$$

For $k > 0$, particularly we choose $k = 1$ in the following. Then we have $r_3 > 0$ and $r_4 < 0$. Consider the relation between r_2 and r_3, and we find that as $0 < b_2 < \frac{5}{4}$, it holds $r_4 < 0 < r_3 < 1$. Furthermore, when $0 < b_2 < \frac{5}{4}$, we have $h_1(r_3) > h_1(r_4)$, but we do not know the relation between $h_1(r_1)$ and $h_1(r_2)$. By calculation, we find that when $0 < b_2 < \frac{1}{2}$, it holds $h_1(r_1) > h_1(r_2)$.

Particularly, we choose $b_2 = \frac{1}{4}$, then we have

$$h_1(0) = 0,\quad h_1(1) = -\frac{1}{8},\quad h_1(r_3) = \frac{9949 + 369\sqrt{41}}{800000},\quad h_1(r_4) = \frac{9949 - 369\sqrt{41}}{800000}.$$

From [10], we have the following conclusions:

① When $b_1 < -h_1(r_3)$, $h_1(r) = -b_1$, system (2.1) has one singular point which is a saddle point.

② When $b_1 = -h_1(r_3)$, $h_1(r) = -b_1$, system (2.1) has two singular points which are respectively one saddle point and one degenerate saddle point.

③ When $-h_1(r_3) < b_1 < 0$, $h_1(r) = -b_1$, system (2.1) has three singular points which are respectively two saddle points and one center.

④ When $0 \le b_1 < \frac{1}{8}$, $h_1(r) = -b_1$, system (2.1) has two singular points which are respectively one saddle point and one center.

⑤ When $b_1 = \frac{1}{8}$, $h_1(r) = -b_1$, system (2.1) has one singular point which is a degenerate saddle point.

If $b_2 > 0$ or $b_2 < -10k$, then $g_2'(r) = h_2'(r) = 0$ has four solutions:

$$r_1^* = 0,\quad r_2^* = -1,\quad r_3^* = \frac{b_2 + \sqrt{b_2^2 + 10kb_2}}{-5k},\quad r_4^* = \frac{b_2 - \sqrt{b_2^2 + 10kb_2}}{-5k}.$$

For $k > 0$, particularly we choose $k = 1$ in the following. Then we have $r_3^* < 0$ and $r_4^* > 0$. We consider the relation between r_2^* and r_3^*, and we find that as

$0 < b_2 < \frac{5}{4}$, it holds $-1 < r_3^* < 0 < r_4^*$. Furthermore when $0 < b_2 < \frac{5}{4}$, we have $h_2(r_3^*) > h_2(r_4^*)$, but we do not know the relation between $h_2(r_1^*)$ and $h_2(r_2^*)$. By calculation, we find that when $0 < b_2 < \frac{1}{2}$, it holds $h_2(r_1^*) > h_2(r_2^*)$.

Particularly, we choose $b_2 = \frac{1}{4}$, then we have

$$h_2(0) = 0, \quad h_2(-1) = -\frac{1}{8}, \quad h_2(r_3^*) = \frac{9949 + 369\sqrt{41}}{800000}, \quad h_2(r_4^*) = \frac{9949 - 369\sqrt{41}}{800000}.$$

And we have the following conclusions:

① When $b_1 = -h_2(r_4^*)$, system (2.1) has one singular point which is a degenerate saddle point.

② When $-h_2(r_4^*) < b_1 < 0$, system (2.1) has two singular points which are respectively one saddle point and one center.

③ When $b_1 \geq 0$, system (2.1) has one singular point which is a saddle point.

Combining the above analysis, we get the conclusions of case (1) in the Theorem 2.3. The proof is completed. □

3. The property of infinite singular points of system (2.1)

Consider system (2.1). Let $r = \frac{1}{u}$, $d\tau = \frac{dt}{u^5}$, then we have

$$\begin{cases} du/d\tau = -uk\sin 7(\theta + \varphi), \\ d\theta/d\tau = k\cos 7(\theta + \varphi) + b_3 u + b_2 u^3 + b_1 u^5. \end{cases} \tag{3.1}$$

As $r \to \infty$, then $u \to 0$. Let $k\cos 7(\theta+\varphi) = 0$, we get $\theta + \varphi = \frac{\pi}{14}$ or $\theta + \varphi = \frac{3\pi}{14}$. Consider roots of the following characteristic equation of Jacobian matrix

$$|J(u,\theta) - \lambda| = (\lambda + \sin 7(\theta + \varphi))(\lambda + 7k\sin 7(\theta + \varphi)) = 0.$$

When $\theta + \varphi = \frac{\pi}{14}$, two eigenvalues of $J(u,\theta)$ satisfy $\lambda_1 < 0$, $\lambda_2 < 0$, that means the system (3.1) has seven stable nodes. In other words, system (2.1) has seven stable infinite singular points which are nodes. When $\theta + \varphi = \frac{3\pi}{14}$, $\lambda_1 > 0$, $\lambda_2 > 0$, the system has seven unstable nodes. In other words, system (2.1) has fourteen infinite singular points which are both nodes.

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