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SMOOTH SOLUTION OF PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS USING RADIAL BASIS FUNCTIONS

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Abstract In this work, we present the method based on radial basis functions to solve partial integro-differential equations. We focus on the parabolic type of integro-differential equations as the most common forms including the "*memory*" of the systems. We propose to apply the collocation scheme using radial basis functions to approximate the solutions of partial integro-differential equations. Due to the presented technique, system of linear or nonlinear equations is made instead of primary problem. The method is efficient because the rate of convergence of collocation method based on radial basis functions is exponential. Some numerical examples and investigation of the experimental results show the applicability and accuracy of the method.

Keywords Partial integro-differential equations, radial basis functions, collocation scheme, integration method.

MSC(2000) 35R09, 45K05, 97N50.

1. Introduction

Many different fields of engineering and science include the system mixed of a partial differential equation and the integral terms involving the unknown function [1, 4, 18, 44, 53]. Such terms appear when we must consider the effects of the "*memory*" of the system. Corresponding to the partial differential equations, there are many different types of these equations such as parabolic and hyperbolic forms [2, 3, 52].

These class of equations called partial integro-differential equations (PIDE) can describe some phenomena in compression of poro-viscoelastic media [22], reaction-diffusion problems [17] and nuclear reactor dynamics [41–43], geophysics [19], plasma physics [21], electromagnetic theory [10].

Although these equations are solved by using some different methods [20, 32, 33, 35, 51, 52], there are some new research to find much more fast and efficient method through the new methods [2, 3, 25, 45, 54]. In this paper, we apply radial basis functions to approximate solution which lead to the continuous solutions.

The radial basis function (RBF) methodology was introduced by Hardy [23] and became popular in multivariate interpolation [5–9, 26, 38–40]. In 1990, Kansa introduced a way to use it for solving parabolic, hyperbolic and elliptic partial differential

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equations [30]. After that, radial basis functions have been widely discussed [5, 15] and applied in numerous fields. In this study, we propose using RBFs for solving the parabolic partial integro-differential equations.

As Kansa proposed the radial basis functions technique based on collocation scheme [30], RBFs have been coupled with some other techniques [12–14, 27, 48–50].

Here, we use the collocation method based on radial basis functions to solve the PIDE problems. The most privileges of the method are making a continuous solution and exponentially convergence.

This work is organized as follows. In Section 2, we introduce the parabolic partial integro-differential equations as the most common operators. In Section 3, we apply the radial basis functions as the effective tool to approximate the solution. This section is devoted to describing the collocation method applied for solving partial integro-differential equations. In Section 4, some illustrative examples and numerical results are shown. Also, we investigate the validity of the method by analysis of the experimental results. An error analysis for the proposed method is introduced in Section 5. Finally, we review some important points of the numerical findings in Section 6.

2. Parabolic partial integro-differential equation

As mentioned before, partial integro-differential equations appear in many different types. Here, we focus only on parabolic partial integro-differential equation which is one of the most important types.

Consider the following partial integro-differential equation

$$u_t(x,t) = g(x,t) + u_{xx}(x,t) + \int_0^t k(x,t,s,u(x,s))ds, \;\; a \le x \le b, \;\; 0 \le t \le T, \tag{2.1}$$

with the following initial and boundary conditions

$$u(x,0) = f(x), \qquad x \in [a,b], \tag{2.2}$$

$$u(a,t) = h(t), \;\; u(b,t) = l(t), \qquad t \in [0,T], \tag{2.3}$$

where $u_t = \frac{\partial u}{\partial t}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$. The functions g and k are continuous on $\Omega = \{(x,t) \in \mathbb{R}^2 : x \in I, t \in J\}$ and $I \times S \times \mathbb{R}$, respectively, with $I = [a,b]$, $J = [0,T]$ and $S = \{(t,s) \in J \times J : s \le t\}$.

In the general form, we can write these operators as

$$\mathcal{L}(u(\mathbf{x})) = \lambda(\mathbf{x}), \qquad \text{in}\, \Omega, \tag{2.4}$$

where $\mathbf{x} = (x,t)$, with the following initial and boundary conditions

$$\mathcal{B}(u(\mathbf{x})) = \mu(\mathbf{x}), \qquad \text{in}\, \partial\Omega, \tag{2.5}$$

However, such equations can handle describing of some physical phenomena in the real world, they are heavy and complex in computation.

3. Description of the RBF collocation method

Consider the general partial integro-differential equation (2.1) with the initial and boundary conditions (2.2), (2.3), respectively. As it is illustrated in Fig.1, Ω is a rectangle surrounded in $\partial\Omega$. The initial and boundary conditions are defined on three sides of this rectangle which are numbered.

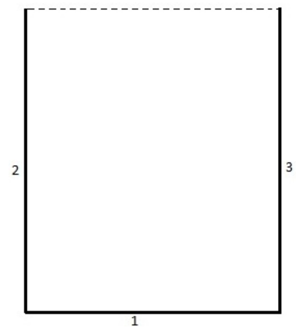


Figure 1. Ω is a rectangle surrounded in $\partial\Omega$. The initial and boundary conditions are defined on three sides of this rectangle which is numbered 1, 2, 3. The first side is correspond to $a \leq x \leq b$ and the second and third sides are correspond to $0 \leq t \leq T$.

Now we use the RBFs for discretization of both time and space variables. Let

$$\Omega = \{(x_i, t_j),\, a \leq x_i \leq b,\, 0 \leq t_j \leq T,\, i, j = 1, 2, \cdots, n\}, \tag{3.1}$$

be a set of scattered nodes. Then the approximated solution of the problem (2.1)-(2.3) is considered as follows:

$$\tilde{u}(x,t) = \sum_{i=1}^{N} \alpha_i \phi_i(x,t), \tag{3.2}$$

where $\phi_i(x,t) = \phi(||(x,t) - (x_k, t_l)||)$, $k, l = 1, 2, \cdots, n, i = 1, 2, \cdots, N = n^2$, for a radial function ϕ and α_i, $i = 1, 2, \cdots, N$ are unknown coefficients that must be found. There are many different RBFs can be used as the multiquadratics (MQ) $\varphi(r) = (r^2 + c^2)^{\frac{\beta}{2}}, \beta \neq 0, \beta \neq 2\mathbb{N}$, inverse multiquadratics (IMQ) $\varphi(r) = \dfrac{1}{\sqrt{r^2 + c^2}}$, Gaussians (GA) $\varphi(r) = e^{-c^2 r^2}$ and etc.

The collocation technique is used for finding unknowns α_i, $i = 1, 2, \cdots, N$. Let

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4, \tag{3.3}$$

where

$$\begin{aligned} 1\,th \text{ side}: \quad & \Omega_1 = \{(x_i, t_j), a \leq x_i \leq b, t_j = 0, i, j = 1, 2, \cdots, n\}, \\ 2\,th \text{ side}: \quad & \Omega_2 = \{(x_i, t_j), x_i = a, 0 < t_j \leq T, i, j = 1, 2, \cdots, n\}, \\ 3\,th \text{ side}: \quad & \Omega_3 = \{(x_i, t_j), x_i = b, 0 < t_j \leq T, i, j = 1, 2, \cdots, n\}, \\ \text{Inner area}: \quad & \Omega_4 = \{(x_i, t_j), a < x_i < b, 0 < t_j \leq T, i, j = 1, 2, \cdots, n\}. \end{aligned}$$

Also we assume $\Omega_i \neq \varnothing$ for $i = 1, 2, 3, 4$. Now (2.1)-(2.3) are approximated by using

(3.2). Thus we have

$$\sum_{i=1}^{N} \alpha_i \phi_i(x_k, t_l) = f(x_k), \quad (x_k, t_l) \in \Omega_1, \tag{3.4}$$

$$\sum_{i=1}^{N} \alpha_i \phi_i(x_k, t_l) = h(t_l), \quad (x_k, t_l) \in \Omega_2, \tag{3.5}$$

$$\sum_{i=1}^{N} \alpha_i \phi_i(x_k, t_l) = l(t_l), \quad (x_k, t_l) \in \Omega_3, \tag{3.6}$$

$$\sum_{i=1}^{N} \alpha_i \left[\frac{\partial}{\partial t}\phi_i(x_k, t_l) - \frac{\partial^2}{\partial x^2}\phi_i(x_k, t_l)\right] - A(x_k, t_l) = g(x_k, t_l), \ (x_k, t_l) \in \Omega_4, \tag{3.7}$$

where

$$A(x_k, t_l) = \int_0^{t_l} k(x_k, t_l, s, \sum_{i=1}^{N} \alpha_i \phi_i(x_k, s))ds, \quad (x_k, t_l) \in \Omega_4, \tag{3.8}$$

which results in a system of equations that can be solved via Newton's iteration method or other efficient method to obtain the coefficients $\alpha_i, i = 1, 2, ..., N$ [28].

Remark 3.1. (Legendre-Gauss nodes and weights) Let $L_{M+1}(\xi)$ be the Legendre polynomial of order $M+1$ on $[-1, 1]$. Then the Legendre-Gauss nodes are

$$-1 < \xi_0 < \xi_1 < \cdots < \xi_M < 1, \tag{3.9}$$

where $\{\xi_i\}_{i=0}^{M}$ are the zeros of $L_{M+1}(\xi)$. No explicit formulas are known for the points ξ_i, and so they are computed numerically using subroutines [11,31]. Also we approximate the integral of f on $[-1, 1]$ as

$$\int_{-1}^{1} f(\xi)d\xi \simeq \sum_{i=0}^{M} w_i f(\xi_i), \tag{3.10}$$

where ξ_i are Legendre-Guass nodes in (3.9) and the weights w_i given in [11,31]

$$w_i = \frac{2}{(1-\xi_i^2)[L'_{M+1}(\xi_i)]^2}, \quad i = 0, 1, \cdots, M. \tag{3.11}$$

As it is mentioned in [11,31], the integration in (3.10) is exact whenever $f(\xi)$ is a polynomial of degree $\leq 2M+1$.

For obtaining $A(x_k, t_l), \ (x_k, t_l) \in \Omega_4$, by change of variable $s = \frac{t_l}{2}(\xi+1)$, (3.8) can be written as:

$$A(x_k, t_l) = \frac{t_l}{2}\int_{-1}^{1} k\Big(x_k, t_l, \frac{t_l}{2}(\xi+1), \sum_{i=1}^{N} \alpha_i \phi_i(x_k, \frac{t_l}{2}(\xi+1))\Big)d\xi, \ (x_k, t_l) \in \Omega_4. \tag{3.12}$$

By applying numerical integration method given in (3.10), we can approximate the integral in (3.12) and we can get:

$$A(x_k, t_l) \simeq \frac{t_l}{2}\sum_{j=0}^{M} w_j k\Big(x_k, t_l, \frac{t_l}{2}(\xi_j+1), \sum_{i=1}^{N} \alpha_i \phi_i(x_k, \frac{t_l}{2}(\xi_j+1))\Big)d\xi, \ (x_k, t_l) \in \Omega_4. \tag{3.13}$$

In case $\mathcal{L}$ and $\mathcal{B}$ are the linear operators, we have a linear system that can be solved to obtain the coefficient $\alpha_i, i = 1, 2, ..., N$.

4. Numerical examples

To show the efficiency of the present method on the partial integro-differential equation, we give some computational results to illustrate our theoretical discussion. These experiments are chosen such that the relevant analytical solutions are known. The computations associated with the examples were performed using Matlab 7 software on a Personal Computer.

Example 4.1 ([25]). We consider the partial integro-differential equation with the initial condition

$$u(x,0) = 0, \qquad -1 \leq x \leq 1,$$

and boundary conditions

$$\begin{array}{lll} u(-1,t) & = 0, & \quad 0 \leq t \leq 1, \\ u(1,t) & = 0, & \quad 0 \leq t \leq 1, \end{array}$$

where $k(x,t,s,u) = -\exp(x(t-s))u$,

$$g(x,t) = \left(1 - x^2\right)\cos(t) + \frac{\left(x^2 - 1\right)\left(x\sin(t) - \mathrm{e}^{xt} + \cos(t)\right)}{x^2 + 1} + 2\sin(t),$$

and $u(x,t) = (1 - x^2)\sin(t)$ is the exact solution.

In Figure 2, the exact solution, numerical solution and the corresponding absolute error function $|u - \widetilde{u}|$ are plotted by using GA-RBF with $c = 0.1$ and the uniform set of collocation points $x_i = -1 + (i-1)/5$ and $t_i = (i-1)/10,\ i = 1, 2, \cdots, 11$.

Example 4.2 ([25]). In this example, we consider the partial integro-differential equation with the initial condition

$$u(x,0) = 1 - x^2, \qquad -1 \leq x \leq 1,$$

and boundary conditions

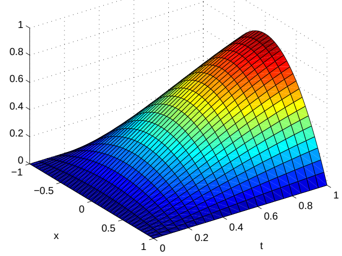
$$\begin{array}{lll} u(-1,t) & = 0, & 0 \leq t \leq 1, \\ u(1,t) & = 0, & 0 \leq t \leq 1, \end{array}$$

where $k(x,t,s,u) = -\frac{1+t-s}{1+x}u$,

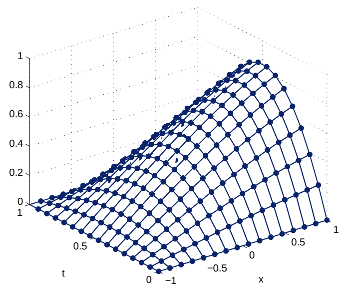
$$g(x,t) = \frac{\left(2x^2 - 2\right)t}{\left(t^2 + 1\right)^2} + \left(\frac{x-1}{2}\right)\ln\left(t^2 + 1\right) + \left(-xt + t - x + 1\right)\arctan(t) + \frac{2}{1 + t^2},$$

and $u(x,t) = \frac{1-x^2}{1+t^2}$ is the exact solution.

In Figure 3, the exact solution, numerical solution and the corresponding absolute error function $|u - \widetilde{u}|$ are plotted by using GA-RBF with $c = 0.1$ and the uniform set of collocation points $x_i = -1 + 2(i-1)/15$ and $t_i = (i-1)/15,\ i = 1, 2, \cdots, 16$.



(a)



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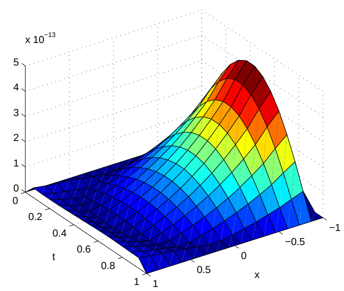


(c)

Figure 2. Exact solution (a), approximated solution (b) and absolute error function $|u - \widetilde{u}|$ for $N = 121$ using GA-RBF with $c = 0.1$. See Example 4.1.

Example 4.3 ([25])**.** We consider the partial integro-differential equation with the initial condition

$$u(x,0) = 1 - x^2, \qquad -1 \leq x \leq 1,$$

and boundary conditions

$$\begin{aligned} u(-1,t) &= 0, \quad 0 \leq t \leq 1, \\ u(1,t) &= 0, \quad 0 \leq t \leq 1, \end{aligned}$$

where $k(x,t,s,u) = -\sin(x(t-s))u$,

$$g(x,t) = \left(1 - x^2\right) \mathrm{e}^t + \frac{\left(x^2 - 1\right)\left(x \cos\left(xt\right) + \sin\left(xt\right) - x\mathrm{e}^t\right)}{x^2 + 1} + 2\,\mathrm{e}^t,$$

and $u(x,t) = (1 - x^2)\exp(t)$ is the exact solution.

In Figure 4, the exact solution, numerical solution and the corresponding absolute error function $|u - \widetilde{u}|$ are plotted by using GA-RBF with $c = 0.1$ and the uniform set of collocation points $x_i = -1 + (i-1)/5$ and $t_i = (i-1)/10,\ i = 1, 2, \cdots, 11$.

Example 4.4. We consider the nonlinear partial integro-differential equation with the initial condition

$$u(x,0) = x^2, \qquad 0 \leq x \leq 1,$$

and boundary conditions

$$\begin{aligned} u(0,t) &= t^2, \qquad & 0 \leq t \leq 1, \\ u(1,t) &= 1 + t^2, \qquad & 0 \leq t \leq 1, \end{aligned}$$

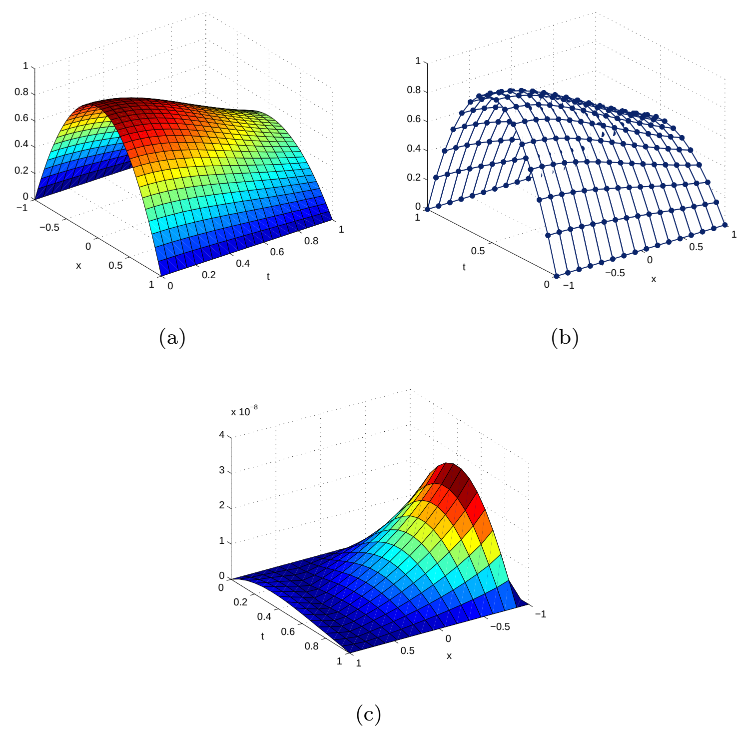


Figure 3. Exact solution (a), approximated solution (b) and absolute error function $|u - \widetilde{u}|$ for $N = 256$ using GA-RBF with $c = 0.1$. See Example 4.2.

where $k(x, t, s, u) = \sin(xt)u^2$,

$$g(x,t) = 2\,t - 2 - \frac{1}{5}\,\sin{(xt)}\,t^5 - \frac{2}{3}\,\sin{(xt)}\,x^2t^3 - \sin{(xt)}\,x^4t,$$

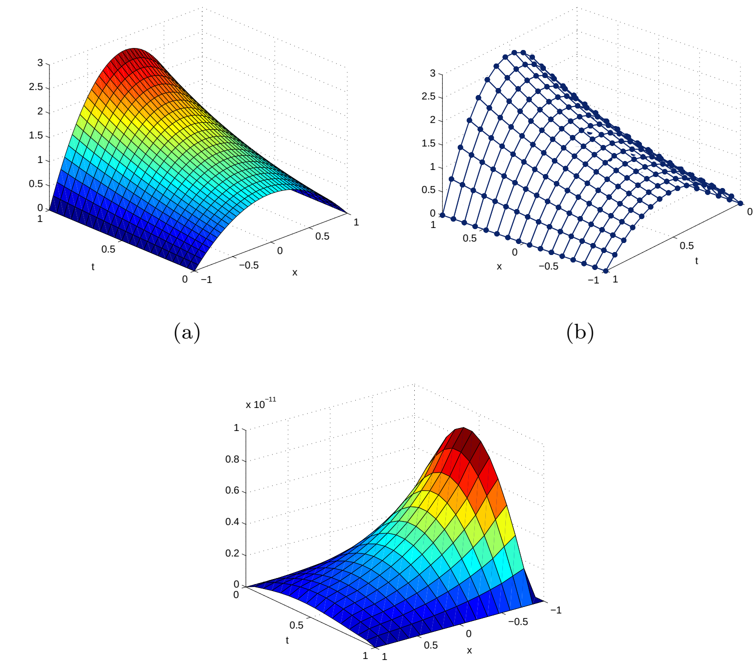
and $u(x,t) = x^2 + t^2$ is the exact solution.

In Figure 5, the exact solution, numerical solution and the corresponding absolute error function $|u - \widetilde{u}|$ are plotted by using GA-RBF with $c = 0.01$ and the uniform set of collocation points $x_i = (i-1)/4$ and $t_i = (i-1)/4,\ i = 1, 2, \cdots, 5$.

5. Error Analysis

Madych have proven exponential convergence property of multiquadratic approximation [36]. He has shown that under certain conditions, the interpolation error is $\varepsilon = O(\lambda^{\frac{c}{h}})$ where c is the shape parameter, h is the mesh size and $0 < \lambda < 1$ is a constant. It implies we can improve the approximated solution either by reducing the size of h or by increasing the magnitude of c. It means that if $c \to \infty$ then $\varepsilon \to 0$. Since increasing of c can improve the accuracy exponentially without extra computation [16, 24, 36, 37], it is preferred to decrease error rather than reducing h.

However, according to 'uncertainty principle' of Schaback [47], as the error becomes smaller, the matrix becomes more ill-conditioned; hence the solution will break down as c becomes too large. The experimental results confirm such behavior of the error values as c becomes larger. The numerical results for Examples



(a) (b)

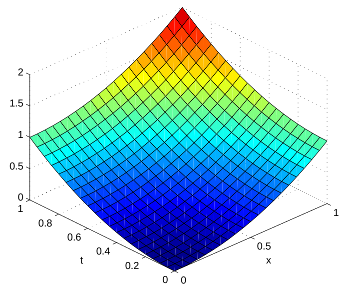
(c)

Figure 4. Exact solution (a), approximated solution (b) and absolute error function $|u - \widetilde{u}|$ for $N = 121$ using GA-RBF with $c = 0.1$. See Example 4.3.

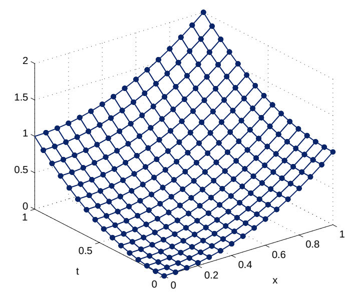
4.1, 4.2 and 4.3 are demonstrated in Figures 6, 7 and Tables 1 and 2 which show according to the findings of Madych, the error functions decrease exponentially as c becomes larger in bounded interval. After that according to the research of Schaback the error values decline as c becomes too large. The best c is different for various problems and not the same RBFs.

Table 1. The maximum absolute error by using GA-RBF with different c for Examples 4.1, 4.2 and 4.3.

c	Example 1 $N = 121$	Example 2 $N = 256$	Example 3 $N = 121$
0.001	7.45×10^{-8}	1.12×10^{-1}	3.49×10^{-4}
0.01	8.55×10^{-12}	1.72×10^{-5}	8.51×10^{-12}
0.1	4.52×10^{-13}	3.11×10^{-8}	9.25×10^{-12}
0.5	2.51×10^{-7}	1.14×10^{-7}	8.38×10^{-7}
1	1.19×10^{-4}	6.58×10^{-7}	3.63×10^{-4}
1.2	4.52×10^{-4}	5.52×10^{-6}	1.59×10^{-3}
2	8.51×10^{-3}	7.16×10^{-4}	2.82×10^{-2}
3	2.62×10^{-2}	7.13×10^{-3}	8.55×10^{-2}



(a)



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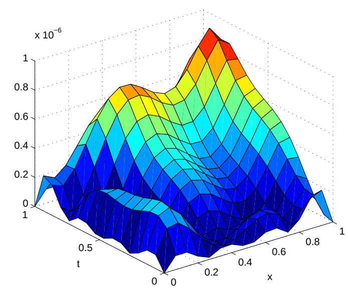

(c)

Figure 5. Exact solution (a), approximated solution (b) and absolute error function $|u - \widetilde{u}|$ for $N = 25$ using GA-RBF with $c = 0.01$. See Example 4.4.

6. Conclusion

In this work, we apply the radial basis functions method to solve the parabolic partial integro-differential equations. The presented method is flexible to solve the nonlinear problems. The numerical experiments show the applicability and accuracy of method. All experiments are performed within some seconds, running on DELL with 4Gb RAM 2.4GHz CPU. This method can be utilized for solving the other types of partial integro-differential equations such as hyperbolic partial integro-differential equations.

Table 2. The maximum absolute error by using MQ-RBF with different c for Examples 4.1, 4.2 and 4.3.

c	Example 1 $N = 121$	Example 2 $N = 256$	Example 3 $N = 121$
0.1	5.22×10^{-2}	1.81×10^{-2}	1.31×10^{-1}
1	2.01×10^{-3}	1.35×10^{-4}	3.16×10^{-3}
5	6.38×10^{-6}	2.51×10^{-6}	1.82×10^{-5}
19	5.43×10^{-13}	2.42×10^{-7}	3.75×10^{-11}
25	1.61×10^{-12}	3.24×10^{-7}	3.31×10^{-11}
30	1.91×10^{-11}	8.51×10^{-7}	2.54×10^{-11}
35	2.19×10^{-11}	2.11×10^{-6}	8.01×10^{-11}

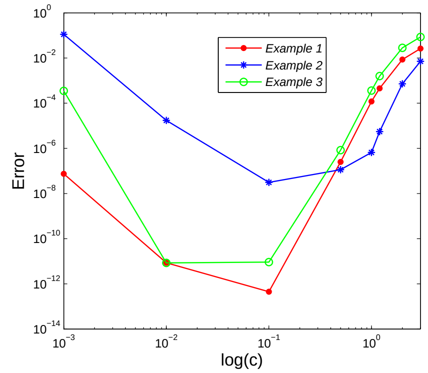

Figure 6. Horizontal axis is related to shape parameter (c) with *log* mode and vertical axis shows absolute error values with *log* mode when the solutions are approximated by using GA-RBF.

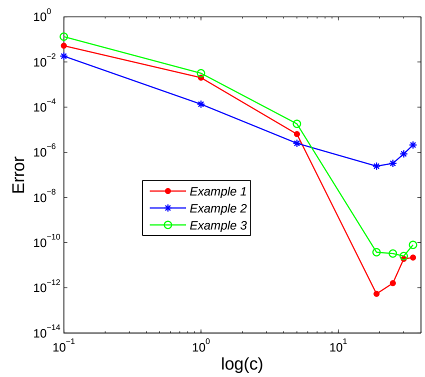

Figure 7. Horizontal axis is related to shape parameter (c) with *log* mode and vertical axis shows absolute error values with *log* mode when the solutions are approximated by using MQ-RBF.

References

[1] J.M. Appell, A.S. Kalitvin and P.P. Zabreiko, *Partial Integral Operators and Integro-differential Equations*, Marcel Dekker, New york, 2000.

[2] Z. Avazzadeh, Z. Beygi Rizi, F.M. Maalek Ghaini and G.B. Loghmani, *A semi-discrete scheme for solving nonlinear hyperbolic-type partial integro-differential equations using radial basis functions*, J. Math. Phys., 52 (2011), 063520.

[3] Z. Avazzadeh, Z. Beygi Rizi, F.M. Maalek Ghaini and G.B. Loghmani, *A numerical solution of nonlinear parabolic-type Volterra partial integro-differential equations using radial basis functions*, Eng. Anal. Boundary Elem., 36 (2012), 881-893.

[4] S. Behzadi, S. Abbasbandy, T. Allahviranloo and A. Yildirim, *Application of homotopy analysis method for solving a class of nonlinear Volterra-Fredholm integro-differential equations*, J. Appl. Anal. Comput., 2 (2012), 127-136.

[5] M.D. Buhmann, *Radial Basis Functions: Theory and Implementations*, Cambridge University Press, 2003.

[6] M.D. Buhmann and C.A. Micchelli, *Multiquadric interpolation improved*, Comput. Math. Appl., 24 (1992), 21-26.

[7] M.D. Buhmann and C.A. Micchelli, *Multivariate interpolation in odd-dimensional Euclidean spaces using multiquadrics*, Const. Approx., 6 (1990), 21-34.

[8] M.D. Buhmann, *Multivariate cardinal interpolation with radial basis functions*, Constr. Approx., 6 (1990), 225-255.

[9] M.D. Buhmann, *Multivariable Interpolation Using Radial Basis Functions, Ph.D. Dissertation*, University of Cambridge, 1989.

[10] F. Bloom, *Ill-posed Problems for Integrodifferential Equations in Mechanics and Electromagnetic Theory*, SIAM, 1981.

[11] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York, 1988.

[12] W. Chen, *New RBF collocation schemes and kernel RBFs with applications*, Lecture Notes in Computational Science and Engineering, Springer-Verlag., 26 (2002), 75-86.

[13] W. Chen and M. Tanaka, *A meshless, exponential convergence, integration-free, and boundary-only RBF technique*, Comput. Math. Appl., 43 (2002), 379-391.

[14] W. Chen, Z. Fu and B.T. Jin, *A truly boundary-only meshfree method for inhomogeneous problems based on recursive composite multiple reciprocity technique*, Eng. Anal. Bound. Elem., 34 (2010), 196-205.

[15] W. Cheney and W. Light, *A Course in Approximation Theory*, William Allan, Newyork, 1999.

[16] A.H.-D. Cheng, M.A. Golberg, E.J. Kansa and G. Zammito,*Exponential convergence and h-c multiquadric collocation method for partial differential equations*, Numer. Methods Partial Differential Eq., 19 (2003), 571-594.

[17] H. Engler, *On some parabolic integro-differential equations: Existence and asymptotics of solutions*, Proc. Int. Conf. Equadiff, Würzburg 1982, Lect. Notes Math., 1017 (1983), 161-167.

[18] D.X. Guo, *A semi-lagrangian Runge-Kutta method for time-dependent partial differential equations*, J. Appl. Anal. Comput., 3 (2013), 251-263.

[19] M. Grasselli, S.I. Kabanikhin and A. Lorenzi, *An inverse hyperbolic integrodifferential problem arising in geophysics II*, Nonlinear Anal., 15 (1990), 283-298.

[20] C.E. Greenwell-Yanik and G. Fairweather, *Analyses of spline collocation methods for parabolic and hyperbolic problems in the two space variables*, SIAM J. numer. Analysis., 23 (1986), 282-296.

[21] Y.N. Grigoriev, N.H. Ibragimov, V.F. Kovalev and S.V. Meleshko, *Symmetries of Integro-Differential Equations: With Applications in Mechanics and Plasma Physics*, Springer, 2010.

[22] G.T. Habetler and R.L. Schiffman, *A finite difference method for analyzing the compression of poro-viscoelastic media*, Comput., 6 (1970), 342-348.

[23] R.L. Hardy, *Multiquadratic equation of topology and other irregular surface*, J. Geophys. Res., 76 (1971), 1905-1915.

[24] C.-S. Huang, C.F. Lee and A.H.-D. Cheng, *Error estimate, optimal shape factor, and high precision computation of multiquadric collocation method*, Eng. Anal. Boundary Elem., 31 (2007), 614-623.

[25] F. F. Izadi and M. Dehghan, *The spectral methods for parabolic Volterra integro-differential equations*, J. Comput. Appli. Math., 235 (2011), 4032-4046.

[26] I.R.H. Jackson, *"Radial Basis Function Methods for Multivariable Approximation"*, Ph.D. Dissertation, University of Cambridge, 1988.

[27] L. Jianyu, L. Siwei, Q. Yingjian and H. Yaping, *Numerical solution of elliptic partial differential equation using radial basis function neural networks*, Neural Networks., 16 (2003), 729-734.

[28] B. Kaltenbacher, A. Neubauer and O. Scherzer, *Iterative Regularization Methods for Nonlinear Ill-Posed Problems*, Walter de Gruyter, 2008.

[29] E.J. Kansa and Y.C. Hon, *Circumventing the ill-conditioning problem with multiquadric radial basis functions: applications to elliptic partial differential equations*, Comput. Math. Appl., 39 (2000), 123-137.

[30] E.J. Kansa, *Multiquadrics: A scattered data approximation scheme with applications to computational fluid dynamics: II. Parabolic, hyperbolic, and elliptic partial differential equations*, Comput. Math. Appl., 19 (1990), 146-161.

[31] P. K. Kythe and M. R. Schäferkotter, *Handbook of Computational Method for Integration*, Chapman and Hall/CRC Press, 2005.

[32] J.-P. Kauthen, *The method of lines for parabolic partial integro-differential equations*, Journal of Integral equations and Application., 4 (1992), 69-81.

[33] S. Larsson, V. Thomée and L.B. Wahlbin, *Numerical solution of parabolic integro-differential equations by the discontinuous Galerkin method*, Math. Comput., 67 (1998), 45-71.

[34] S. Lowitzsch, *Error estimates for matrix-valued radial basis function interpolation*, J. Approx. Theory., 137 (2005), 238-249.

[35] J. Ma, *Finite element methods for partial Volterra integro-differential equations on two-dimensional unbounded spatial domains*, Appl. Math. Comput., 186 (2007), 598-609.

[36] W.R. Madych, *Miscellaeous error bounds for multiquadratic and related interpolators*, Comput. Math. Appl., 24 (1992), 121-138.

[37] W.R. Madych, *Bounds on multivariate polynomials and exponential error estimates for multiquadric interpolation*, J. Approx. Theory., 70 (1992), 94-114.

[38] W.R. Madych and S.A. Nelson, *Multivariate interpolation and conditionally positive definite functions, II*, Math. Comput., 54 (1990), 211-230.

[39] W.R. Madych and S.A. Nelson, *Multivariate interpolation and conditionally positive definite functions*, Approx. Theory Appl., 4 (1988), 77-89.

[40] C.A. Micchelli, *Interpolation of scattered data: distance matrices and conditionally positive definite functions*, Constr. Approx., 2 (1986), 11-22.

[41] B.G. Pachpatte, *On a nonlinear diffusion system arising in reactor dynamics*, J. Math. Analysis Applic., 94 (1983), 501-508.

[42] C.V. Pao, *Bifurcation analysis of a nonlinear diffusion system in reactor dynamics*, Appl. Analysis., 9 (1979), 107-119.

[43] C.V. Pao, *Solution of a nonlinear integrodifferential system arising in nuclear reactor dynamics*, J. Math. Analysis Applic., 48 (1974), 470-492.

[44] S. Serovajsky, *Optimal control for systems described by hyperbolic equation with strong nonlinearity*, J. Appl. Anal. Comput., 3 (2013), 183-195.

[45] E.W. Sachs and A.K. Strauss, *Efficient solution of a partial integro-differential equation in finance*, Appl. Numer. Math., 58 (2008), 1687-1703.

[46] R. Schaback, *Improved error bounds for scattered data interpolation by radial basis functions*, Math. Comp., 68 (1999), 201-216.

[47] R. Schaback, *Error estimate and condition numbers for radial basis function interpolation*, Adv. Comput. Math., 3 (1995), 251-264.

[48] C. Shu, H. Ding and K.S. Yeo, *Solution of partial differential equations by a global radial basis function-based differential quadrature method*, Eng. Anal. Bound. Elem., 28 (2004), 1217-1226.

[49] C. Shu, H. Ding and N. Zhao, *Numerical comparison of least square-based finite-difference (LSFD) and radial basis function-based finite-difference (RBFFD) methods*, Comput. Math. Appl., 51 (2006), 1297-1310.

[50] W.X. Wu, C. Shu and C.M. Wang, *Vibration analysis of arbitrarily shaped membranes using local radial basis function-based differential quadrature method*, J. Sound Vib., 306 (2007), 252-270.

[51] Y. Yan and G. Fairweather, *Orthogonal spline collocation methods for some partial integrodifferential equations*, SIAM J. Numer. Anal., 29 (1992), 755-768.

[52] E.G. Yanik and G. Fairweather, *Finite element methods for parabolic and hyperbolic partial integro-differential equations*, Nonlinear Anal., 12 (1988), 785-809.

[53] M. Zarebnia, *A numerical solution of nonlinear Volterra-Fredholm integral equations*, J. Appl. Anal. Comput., 3 (2013), 95-104.

[54] R. Zacher, *Boundedness of weak solutions to evolutionary partial integro-differential equations with discontinuous coefficients*, J. Math. Anal. Appl., 348 (2008), 137-149.