Journal of Applied Analysis and Computation
Volume 4, Number 1, February 2014
Website:http://jaac-online.com/
pp. 103–114

ON THE EQUIVALENCE OF DIFFERENTIAL EQUATIONS*

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Abstract In this article we use the reflecting function of Mironenko to study some complicated differential equations which are equivalent to the Riccati equation and some polynomial differential equations. The results are applied to discussion of the qualitative behavior of periodic solutions of these complicated differential equations.

Keywords Polynomial equation, reflective function, equivalence, periodic solution.

MSC(2000) 34A12.

1. Introduction

Now, let us consider differential system

$$x' = X(t, x), \quad t \in R, \quad x \in D \subset R^n. \tag{1.1}$$

We assume that system (1.1) has continuously differentiable right-hand sides and which has a general solution $\phi(t; t_0, x_0)$. In [7] there was an elaborated method of the reflecting function which give us an opportunity to research the qualitative behavior of solutions of (1.1). The reflecting function for system (1.1) is defined in some region near the hyperplane $t = 0$ by the formula $F(t, x) := \phi(-t; t, x)$. If system (1.1) is 2ω-periodic with respect to t, then $F(-\omega, x)$ is its Poincaré mapping [1,7,8]. Therefore, the solution $\phi(t; -\omega, x_0)$ which can be extended to $[-\omega, \omega]$ is 2ω-periodic if and only if $F(-\omega, x_0) = x_0$.

The reflective function $F(t, x)$ of system (1.1) can be found sometimes even for the case when the system (1.1) cannot be integrated by quadrature. For example, every system (1.1) for which $X(-t, x) = -X(t, x)$ has a reflective function given by the formula $F(t, x) \equiv x$. We know this due to the following property. A differentiable function $F(t, x)$ is the reflective function of system (1.1) if and only if the following basic relation

$$F'_t + F'_x X(t, x) + X(-t, F) = 0, F(0, x) = x \tag{1.2}$$

holds.

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* The authors were supported by National Natural Science Foundation of China (11271026,61374010) and National Science Foundation of Jiangsu (BK2012682).

So if we can find the solution of the basic relation (1.2), then we can find the initial data for periodic solutions of (1.1) and investigate the character of the stability for those solutions.

If system

$$x' = Y(t,x) \tag{1.3}$$

has the same reflective function $F(t,x)$ as the system (1.1), then $Y(0,x) = X(0,x)$ and systems

$$\begin{aligned} &F'_t + F'_x X(t,x) + X(-t,F) = 0,\\ &F'_t + F'_x Y(t,x) + Y(-t,F) = 0,\\ &F(0,x) = x \end{aligned}$$

are compatible. At this moment, we call system (1.3) is equivalent to system (1.1).

To check whether the above systems are compatible we can use the Frobenius theorem ([5]). Doing this in practice, however, is a very hard task.

If we can neither solve the system (1.1) nor the problem (1.2), then it is good enough to construct any system (1.3) which is equivalent to (1.1). To do this, sometimes we can use:

Lemma 1.1. *[9] Let the vector functions $\triangle_k$ $(k = 1, 2, ..., m)$ be solutions of the equation*

$$\triangle'_t + \triangle'_x X(t,x) = X'_x \triangle \tag{1.4}$$

and $\alpha_k(t)(k = 1, 2, ..., m)$ be any scalar continuous odd functions. Then every system of the form

$$x' = X(t,x) + \sum_{k=1}^{m} \alpha_k(t)\triangle_k(t,x) \tag{1.5}$$

is equivalent to system (1.1) *(here m is any natural number or even $m = \infty$). So if we find some solutions of equation* (1.4)*, we can construct system* (1.5)*, which has the same reflective function as system* (1.1)*.*

In this paper, we will discuss the form of $\triangle(t,x)$ on the condition equation

$$x' = \sum_{k=0}^{m} a_k(t)x^k + \alpha(t)\triangle(t,x) \tag{1.6}$$

is equivalent to

$$x' = \sum_{k=0}^{m} a_k(t)x^k =: X(t,x). \tag{1.7}$$

By the equivalence, we can reduce the analysis of properties of solutions of the perturbed equation (1.6) to the investigation of behavior of the solutions of polynomial equation (1.7).

There are some planar polynomial differential systems can be transformed to an equation of the form (1.7), where the $a_k(t)(k = 0, 1, 2..., m)$ are polynomials in $\cos t$ and $\sin t$. The fact that systems with a homogeneous nonlinearity can be transformed to (1.7) with $N = 3$ has been exploited in a number of previous papers [2,4,15]. To study equation (1.7) is closely related to research the qualitative behavior of the solutions of planar polynomial differential systems.

Other results concerning the reflective function and its applications can be found in works of Mironenko [7–10], Musafirov and others [3, 6, 11–14, 16–20].

2. Main results

Now, consider the polynomial differential equation (1.7), where $a_k := a_k(t)$ $(k = 0, 1, 2, .., m)$ are continuously differentiable functions.

Theorem 2.1. *Suppose that $\alpha(t)$ and $b_k := b_k(t)$ $(k = 0, 1, 2, ..., m)$ are continuously differentiable functions and satisfy the following conditions:*

$$b_0' = a_1 b_0 - a_0 b_1, \tag{2.1}$$

$$b_k' = (k+1)(a_{k+1} b_0 - a_0 b_{k+1}) + (k-1)(a_k b_1 - a_1 b_k), \tag{2.2}$$

$$k = 1, 2, ..., m, \qquad a_{m+1} = b_{m+1} = 0,$$

$$a_{m-k} b_m = a_m b_{m-k}, \qquad k = 1, 2, ..., m-2. \tag{2.3}$$

Then the differential equation (1.6) with $\triangle(t, x) = \sum_{k=0}^{m} b_k(t) x^k$ is equivalent to equation (1.7), where $\alpha(t) + \alpha(-t) = 0$.

In addition, if the equations (1.7) and (1.6) are $2\omega-$periodic with respect to t, then the qualitative behavior of periodic solutions of their defined on $[-\omega, \omega]$ coincide.

Proof. In the relation (1.4), taking

$$\triangle(t, x) = \sum_{k=0}^{m} b_k(t) x^k \text{ and } X(t, x) = \sum_{k=0}^{m} a_k(t) x^i,$$

it implies

$$\sum_{k+0}^{m} b_k' x^k = \sum_{k=0}^{2m-1} (\sum_{i=0}^{k+1} (k+1-2i) a_{k+1-i} b_i) x^k.$$

Equating the coefficients of like powers of x, we get

$$b_k' = \sum_{i=0}^{k+1} (k+1-2i) a_{k+1-i} b_i,$$

$$k = 0, 1, 2, ..., 2m-2, \quad a_i = b_i = 0, \text{ when } i > m.$$

By calculating the above relations, we obtain

$$a_{m-k} b_m = a_m b_{m-k} \quad (k = 1, 2, ..., m-2).$$

and identifies (2.1) (2.2).

Therefore, when the conditions (8-10) are satisfied, the identity (1.4) is valid. By the literature [7,9], we know equation (1.6) is equivalent to equation (1.7). □

Corollary 2.1. *If $(b_{0i}, b_{1i}, ..., b_{mi})$ $(i = 1, 2, .., p)$ satisfy the relation (2.1)-(2.3), $\beta_i(t)$ $(i = 1, 2, ..., p)$ are arbitrary continuously differentiable odd functions, then equation*

$$x' = X(t, x) + \sum_{i=1}^{p} \beta_i(t) \Delta_i(t, x) \tag{2.4}$$

is equivalent to equation (1.7), in which $\Delta_i(t, x) = \sum_{k=0}^{m} b_{ki}(t) x^k$.

Proof. Let $F(t,x)$ be a reflective function of (1.7). In view of the assumptions, we know $\Delta_i(t,x)$ $(i=1,2,...,p)$ satisfy the relation (1.4). By the literature [9], we have

$$F'_x(t,x)\Delta_i(t,x)=\Delta_i(-t,F(t,x)).$$

In the following, we will verify $F(t,x)$ is also a reflective function of equation (2.4).

In fact, as

$$\begin{aligned}&F'_t(t,x)+F'_x(t,x)(X(t,x)+\sum_{i=1}^{p}\beta_i(t)\Delta_i(t,x))+X(-t,F(t,x))\\&+\sum_{i-1}^{p}\beta_i(-t)\Delta_i(-t,F(t,x))=F'_t(t,x)+F'_x(t,x)X(t,x)+X(-t,F(t,x))\\&+\sum_{i=1}^{p}\beta_i(t)(F'_x(t,x)\Delta_i(t,x)-\Delta_i(-t,F(t,x)))=0.\end{aligned}$$

Thus, equation (2.4) is equivalent to equation (1.7). □

Example 2.1. It is easy to check that differential equation

$$x'=x^m\cos t$$

($m>1, m$ is a positive integral) is equivalent to equation

$$x'=x^m\cos t+\sum_{i=1}^{m}\beta_i(t)(k_{i1}x+((m-1)k_{1i}\cos t+k_{2i})x^m),$$

where $\beta_i(t)$ $(i=1,2,...,m)$ are arbitrary odd functions, k_{1i},k_{2i} $(i=1,2,...,m)$ are constants. When $\beta_i(t)$ $(i=1,2,...,m)$ are 2π-periodic continuously differentiable odd functions, all the solutions of above equations are $2\pi-$periodic.

Now, let us consider

$$\Delta(t,x)=\frac{\sum_{k=0}^{m+2}b_kx^k}{c_0+c_2x^2},\tag{2.5}$$

where $c_0c_2\neq 0$ and $c_i:=c_i(t)$, $b_j:=b_j(t)$ $(i=0,2,\ j=0,1,2,...,m+2)$ are continuously differentiable functions.

Denote

$$\begin{aligned}&d_k=c_0a_k+c_2a_{k-2},\quad k=0,1,2,...,m+2,\\&A_k=\frac{1}{c_0^2}\sum_{i=0}^{k+1}(k+1-2i)b_id_{k+1-i}+\frac{b_{k-2}}{c_0}(\frac{c_2}{c_0})',\quad k=0,1,2,3,4,\\&A_k=\frac{1}{c_0^2}\sum_{i=0}^{3}(k+1-2i)(b_id_{k+1-i}-d_ib_{k+1-i})+\frac{b_{k-2}}{c_0}(\frac{c_2}{c_0})',\\&k=5,6,...,m+4,\quad b_i=d_i=0,\ \text{when}\ i>m+2\ \text{or}\ i<0.\end{aligned}$$

Theorem 2.2. *If c_0, c_2 and b_k $(k=0,1,2,...,m+2)$ satisfy the following relations*

$$(\frac{b_k}{c_0})' = \sum_{i=0}^{[\frac{k}{2}]}(-\frac{c_2}{c_0})^i A_{k-2i},$$
$$k=0,1,2,...,m+4. \quad b_i = 0, \text{ when } i > m+2, \tag{2.6}$$
$$b_{m+2-k}d_{m+2} = b_{m+2}d_{m+2-k}, \quad k=1,2,...,m-2. \tag{2.7}$$

Then differential equation

$$x' = X(t,x) + \alpha(t)\Delta(t,x) \tag{2.8}$$

is equivalent to (1.7)*, where $\alpha(t)$ is an arbitrary continuously differentiable odd function, $\Delta(t,x)$ is in the form of* (2.5)*.*

In addition, if equations (1.7) *and* (2.8) *are 2ω-periodic with respect to t, then the qualitative behavior of 2ω-periodic solutions of* (1.7) *and* (2.8) *coincide.*

Where and in the following $[k]$ stands for the integer part of k.

Proof. In the relation (1.4), taking

$$\Delta(t,x) = \frac{\sum_{k=0}^{m+2} b_k(t)x^k}{c_0 + c_2x^2} \text{ and } X(t,x) = \sum_{k=0}^{m} a_k(t)x^k,$$

we obtain

$$\begin{aligned}
&\sum_{k=0}^{m+2} b_k'x^k(c_0+c_2x^2) - (c_0'+c_2'x^2)\sum_{k=0}^{m+2} b_kx^k \\
&+ (\sum_{k=1}^{m+2} kb_kx^{k-1}(c_0+c_2x^2) - 2c_2x\sum_{k=0}^{m+2} b_kx^k)\sum_{k=0}^{m} a_kx^k \\
=&(c_0+c_2x^2)\sum_{k=1}^{m} ka_kx^{k-1}\sum_{k=0}^{m+2} b_kx^k,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\sum_{k=0}^{m+2} b_k'x^k(c_0+c_2x^2) - (c_0'+c_2'x^2)\sum_{k=0}^{m+2} b_kx^k \\
=&\sum_{k=1}^{m+2} kd_kx^{k-1}\sum_{k=0}^{m+2} b_kx^k - \sum_{k=1}^{m+2} kb_kx^{k-1}\sum_{k=0}^{m+2} d_kx^k.
\end{aligned}$$

Equating the coefficients of like powers of x, we have

$$b_k'c_0 + b_{k-2}'c_2 - c_0'b_k - c_2'b_{k-2} = \sum_{i=0}^{k+1}(k+1-2i)b_id_{k+1-i},$$
$$k=0,1,2,...,m+4, \ b_i = d_i = 0, \text{ when } i > m+2, \text{ or } i < 0, \tag{2.9}$$
$$\sum_{i=k}^{m+2}(m+k+2-2i)b_id_{m+k+2-i} = 0, \quad k=4,5,...,m+1. \tag{2.10}$$

From (2.10) follows

$$b_{m+2-k}d_{m+2} = b_{m+2}d_{m+2-k}, \quad k = 1, 2, ..., m-2.$$

Substituting it into (2.9) and performing simple computations, we obtain

$$(\frac{b_k}{c_0})' + \frac{c_2}{c_0}(\frac{b_{k-2}}{c_0})' = A_k,$$
$$k = 0, 1, 2, ..., m+4, \ b_i = d_i = 0, \text{ when } i < 0, \text{ or } i > m+2. \tag{2.11}$$

Solving it implies

$$(\frac{b_k}{c_0})' = \sum_{i=0}^{[\frac{k}{2}]}(-\frac{c_2}{c_0})^i A_{k-2i}, \quad k = 0, 1, ..., m+4, \quad b_i = 0, \ i > m+2.$$

Therefore, when the relations (2.6)-(2.7) are satisfied, the identity (1.4) is valid. By the literature [7,9], we know that equation (2.8) is equivalent to equation (1.7). The proof is finished. □

Similar discuss as corollary 2.1, we get

Corollary 2.2. *If*

$$(c_{0i}, c_{1i}, b_{0i}, b_{1i}, ..., b_{mi}), \ \ i = 1, 2, ..., p$$

satisfy the relations (2.6)-(2.7), β_i $(i = 1, 2, ..., p)$ *are arbitrary continuously differentiable odd functions, then differential equation*

$$x' = X(t, x) + \sum_{i=1}^{p}\beta_i(t)\Delta_i(t, x)$$

is equivalent to equation (1.7)*, where*

$$\Delta_i(t, x) = \frac{\sum_{k=0}^{m+2} b_{ki}(t)x^k}{c_{0i} + c_{1i}x^2}, \ \ c_{0i}c_{1i} \neq 0.$$

Denote

$$\delta_1 = \sum_{i=0}^{[\frac{m+1}{2}]}(-\frac{c_2}{c_0})^i\frac{b_{m+1-2i}}{c_0}, \quad \delta_2 = \sum_{i=0}^{[\frac{m+2}{2}]}(-\frac{c_2}{c_0})^i\frac{b_{m+2-2i}}{c_0},$$

$$W = [(\frac{c_0}{c_2})' - 2\frac{c_0}{c_2}\sum_{i=0}^{[\frac{m-1}{2}]}(-\frac{c_0}{c_2})^i a_{1+2i}]^2 + \frac{c_0}{c_2}[2\sum_{i=0}^{[\frac{m}{2}]}(-\frac{c_0}{c_2})^i a_{2i}]^2.$$

Theorem 2.3. *If all the conditions of Theorem 2.2 are satisfied, and* $W \neq 0$*, then equation* (2.8) *is equivalent to equation* (1.7)*. In this case, equation* (2.8) *is a polynomial differential equation, too.*

Proof. According to theorem 2.2, we know equation (2.8) is equivalent to equation (1.7). By calculating (2.6), we get

$$\delta_1[(\frac{c_0}{c_2})^i - 2\frac{c_0}{c_2}\sum_{i=0}^{[\frac{m-1}{2}]}(-\frac{c_0}{c_2})^i a_{1+2i}] - \delta_2[2\sum_{i=0}^{[\frac{m}{2}]}(-\frac{c_0}{c_2})^i a_{2i}]\frac{c_0}{c_2} = 0, \tag{2.12}$$

$$\delta_1[2\sum_{i=0}^{[\frac{m}{2}]}(-\frac{c_0}{c_2})^i a_{2i}] + \delta_2[(\frac{c_0}{c_2})' - 2\frac{c_0}{c_2}\sum_{i=0}^{[\frac{m-1}{2}]}(-\frac{c_0}{c_2})^i a_{1+2i}] = 0. \tag{2.13}$$

As $W \neq 0$, the algebraic equations (2.12)-(2.13) has a unique solution: $\delta_1 = \delta_2 = 0$. In this case,

$$\Delta(t,x) = \frac{\sum_{k=0}^{m+2} b_k x^k}{c_0 + c_2 x^2} = \sum_{k=0}^{m} e_k x^k, \tag{2.14}$$

where $e_k = \sum_{i=0}^{[\frac{k}{2}]} (-\frac{c_2}{c_0})^i \frac{b_{k-2i}}{c_0}$.

Therefore, equation (2.8) is a polynomial differential equation, too. □

Theorem 2.4. *If all the conditions of Theorem 2.2 are satisfied, and $\delta_1^2(0)+\delta_2^2(0) = 0$, then the differential equation (2.8) is equivalent to equation (1.7). In this case, equation (2.8) is a polynomial differential equation, too.*

Proof. According to theorem 2.2, we know that equation (2.8) is equivalent to equation (1.7).

Differentiating δ_1 and δ_2 and using relations (2.6)-(2.7) and by performing simple computations, we obtain

$$\delta_1' = -\delta_1 \sum_{i=0}^{[\frac{m-1}{2}]} (m-4i) a_{1+2i} (-\frac{c_0}{c_i})^i + \delta_2 \sum_{i=0}^{[\frac{m}{2}]} (m+2-4i) a_{2i} (-\frac{c_0}{c_i})^i,$$

$$\delta_2' = \frac{c_2}{c_0} \sum_{i=0}^{[\frac{m}{2}]} (m+3-4i) a_{2i} (-\frac{c_0}{c_i})^i \delta_1 - \delta_2 \sum_{i=0}^{[\frac{m-1}{2}]} (m+1-4i) a_{2i+1} (-\frac{c_0}{c_i})^i.$$

Applying the hypothesis of the present theorem, the linear system of above has a unique solution satisfying initial value $\delta_1(0) = \delta_2(0) = 0$, i.e. $\delta_1(t) = \delta_2(t) = 0$. Thus, $\Delta(t,x)$ is in the form of (2.14) and equation (2.8) is a polynomial differential equation, too. □

Now, we consider the Riccati equation

$$x' = a_0(t) + a_1(t)x + a_2(t)x^2 =: Y(t,x), \tag{2.15}$$

where $a_i(t)$ $(i = 0,1,2, t \in R)$ are continuously differentiable functions.

Corollary 2.3. *Suppose that $(b_{0i}(t), b_{1i}(t), b_{2i}(t))$, $i = 1,2,...m$ are solutions of equations:*

$$b_0' = a_1 b_0 - a_0 b_1, \quad b_1' = 2(a_2 b_0 - a_0 b_2),$$
$$b_2' = a_2 b_1 - a_1 b_2.$$

$\beta_i(t)$ $(i = 1,2,...,m)$ are continuously differentiable odd functions. Then equation

$$x' = Y(t,x) + \sum_{i=1}^{m} \beta_i(t)(b_{0i}(t) + b_{1i}(t)x + b_{2i}(t)x^2)$$

is equivalent to the Ricatti equation (2.15), and they have the same in period $[-\omega,\omega]$ transformation (have the same shift operator $x_0 \to \varphi(\omega; -\omega, x_0)$, where $\varphi(t;t_0,x_0)$ is general solution of (2.15)).

Example 2.2. The Riccati equation

$$x' = -\cos t e^{-\sin t} + \cos t e^{\sin t} x^2 \tag{2.16}$$

is equivalent to

$$x' = -\cos t e^{-\sin t} + \cos t e^{\sin t} x^2 + \alpha(t)(-e^{-\sin t} + x + e^{\sin t} x^2). \tag{2.17}$$

According to the Theorem 2.25 which is in literature [7, p102], we know that the Riccati equation (2.16) has at least one $2\pi-$periodic solution. Therefore, when $\alpha(t)$ is a $2\pi-$periodic continuously differentiable odd function, the equation (2.17) has at least one $2\pi-$periodic solution, too.

Example 2.3. The Riccati equation

$$x' = \sin t e^{-\sin t} - x\cos t + \sin t e^{\sin t} x^2 \tag{2.18}$$

is equivalent to

$$x' = \sin t e^{-\sin t} - x\cos t + \sin t e^{\sin t} x^2 + \alpha(t)(e^{-\sin t} + e^{\sin t} x^2). \tag{2.19}$$

It is easy to verify that $F(t,x) = e^{2\sin t} x$ is the reflective function of equation (2.18). So, its Poincaré mapping can be expressed by $T(x) = F(-\pi, x) \equiv x$. Thus, all the solutions of (2.18) are $2\pi-$periodic. Therefore, when $\alpha(t)$ is a 2π-periodic continuously differentiable odd function, all the solutions of equation (2.19) are also $2\pi-$periodic.

Now let us denote

$$\Delta(t,x) := \frac{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}{c_0 + c_1 x + c_2 x^2}, \tag{2.20}$$

here $c_0 \cdot c_2 \neq 0$, and $c_i := c_i(t),\ b_j := b_j(t)\ (i = 0,1,2,\ j = 0,1,...,4)$ are continuously differentiable functions,

$$\delta_1 := \frac{b_1}{c_0} - \frac{b_3}{c_2} - \frac{c_1 b_0}{c_0^2} + \frac{c_1 b_4}{c_2^2}, \qquad \delta_2 := \frac{b_2}{c_0} - \frac{b_4}{c_2} - \frac{c_1 b_3}{c_0 c_2} - \frac{c_2 b_0}{c_0^2} + \frac{c_1^2 b_4}{c_2^2 c_0},$$

$$W := [2a_1 - \frac{c_1}{c_2}a_2 - \frac{c_1}{c_0}a_0 - \frac{c_2}{c_0}(\frac{c_0}{c_2})'][(\frac{c_0}{c_2})' - \frac{c_1}{c_2}(\frac{c_1}{c_2})' - \frac{c_1}{c_2}a_0 + (\frac{c_1^2}{c_2^2} - 2\frac{c_0}{c_2})a_1$$
$$+ (3\frac{c_1 c_0}{c_2^2} - \frac{c_1^3}{c_2^3})a_2] - [2a_0 + (\frac{c_1}{c_2})' - \frac{c_1}{c_2}a_1 + \frac{c_1^2}{c_2^2}a_2 - 2\frac{c_0}{c_2}a_2]^2,$$

$$\delta_1[2a_0 + (\frac{c_1}{c_2})' - \frac{c_1}{c_2}a_1 + \frac{c_1^2}{c_2^2}a_2 - 2\frac{c_0}{c_2}a_2] + \delta_2[(\frac{c_0}{c_2})' - \frac{c_1}{c_2}(\frac{c_1}{c_2})' - \frac{c_1}{c_2}a_0$$
$$+ (\frac{c_1^2}{c_2^2} - 2\frac{c_0}{c_2})a_1 + (3\frac{c_1 c_0}{c_2^2} - \frac{c_1^3}{c_2^3})a_2] = 0, \tag{2.21}$$

$$\delta_1[2a_1 - \frac{c_1}{c_2}a_2 - \frac{c_1}{c_0}a_0 - \frac{c_2}{c_0}(\frac{c_0}{c_2})']\delta_2[2a_0 + (\frac{c_1}{c_2})' - \frac{c_1}{c_2}a_1 + \frac{c_1^2}{c_2^2}a_2 - 2\frac{c_0}{c_2}a_2] = 0, \tag{2.22}$$

$$(\frac{b_0}{c_0})' = a_0(\frac{c_1 b_0}{c_0^2} - \frac{b_1}{c_0}) + a_1\frac{b_0}{c_0}, \tag{2.23}$$

$$(\frac{b_1}{c_0})' = (\frac{c_1}{c_0})'\frac{b_0}{c_0} + a_0(2\frac{c_2 b_0}{c_0^2} + \frac{c_1 b_1}{c_0^2} - 2\frac{b_2}{c_0} - \frac{c_1^2 b_0}{c_0^3}) + a_1\frac{c_1 b_0}{c_0^2} + 2a_2\frac{b_0}{c_0}, \tag{2.24}$$

$$(\frac{b_2}{c_2})' = (\frac{c_1}{c_2})'\frac{b_3}{c_2} + (\frac{c_0}{c_2})'\frac{b_4}{c_2} - (\frac{c_1}{c_2})'\frac{c_1 b_4}{c_2^2} - a_0(\frac{b_3}{c_2} + \frac{c_1 b_4}{c_2^2}) + a_1(\frac{b_2}{c_2} + \frac{c_1^2 b_4}{c_2^3} - \frac{c_1 b_3}{c_2^2}$$
$$- 2\frac{c_0 b_4}{c_2^2}) + a_2(3\frac{b_1}{c_2} + 3\frac{c_0 c_1 b_4}{c_2^3} + \frac{c_1^2 b_3}{c_2^3} - \frac{c_1 b_2}{c_2^2} - 2\frac{c_0 b_3}{c_2^2} - \frac{c_1^3 b_4}{c_2^4}), \tag{2.25}$$

$$(\frac{b_3}{c_2})' = (\frac{c_1}{c_2})'\frac{b_4}{c_2} - 2a_0\frac{b_4}{c_2} - a_1\frac{c_1b_4}{c_2^2} + a_2(2\frac{b_2}{c_2} - 2\frac{c_0b_4}{c_2^2} - \frac{c_1b_3}{c_2^2} + \frac{c_1^2b_4}{c_2^3}), \tag{2.26}$$

$$(\frac{b_4}{c_2})' = -a_1\frac{b_4}{c_2} + a_2(\frac{b_3}{c_2} - \frac{c_1b_4}{c_2^2}). \tag{2.27}$$

Theorem 2.5. *(1) If the relations* (2.21)-(2.27) *are satisfied, then the Riccati equation* (2.15) *is equivalent to equation*

$$x' = Y(t,x) + \alpha(t)\Delta(t,x), \tag{2.28}$$

in which the function $\Delta(t,x)$ *is in the form* (2.20).

(2) If the relations (2.21)-(2.27) *are satisfied, and* $W \neq 0$*, then the Riccati equation* (2.15) *is equivalent to the equation* (2.28). *Moreover,* (2.28) *is a Riccati equation, too.*

(3) If the relations (2.21)-(2.27) *are satisfied,* $(\frac{c_0}{c_2})'$ *and* $(\frac{c_1}{c_2})'$ *are continuous, and* $\delta_1^2(0)+\delta_2^2(0) = 0$*, then the Riccati equation* (2.15) *is equivalent to the equation* (2.28). *Furthermore,* (2.28) *is a Riccati equation, too.*

Proof. Substituting the formula (2.20) into the relation (1.4) and equating the coefficients of like powers of x, we obtain

$$b_0'c_0 - c_0'b_0 = a_0(c_1b_0 - c_0b_1) + a_1c_0b_0, \tag{2.29}$$

$$b_0'c_1 + b_1'c_0 - b_0c_1' - b_1c_0' = 2a_0(c_2b_0 - c_0b_2) + 2a_1c_1b_0 + 2a_2c_0b_0, \tag{2.30}$$

$$\begin{aligned} &b_0'c_2 + b_1'c_1 + b_2'c_0 - b_0c_2' - b_1c_1' - b_2c_0' = a_0(c_2b_1 - b_2c_1 - 3c_0b_3) \\ &+ a_1(c_1b_1 - c_0b_2 + 3c_2b_0) + a_2(c_0b_1 + 3c_1b_0), \end{aligned} \tag{2.31}$$

$$\begin{aligned} &b_1'c_2 + b_2'c_1 + b_3'c_0 - b_1c_2' - b_2c_1' - b_3c_0' = -a_0(2c_1b_3 + 4c_0b_4) \\ &+ a_1(2c_2b_1 - 2c_0b_3) + a_2(2c_1b_1 + 4c_2b_0), \end{aligned} \tag{2.32}$$

$$\begin{aligned} &b_2'c_2 + b_3'c_1 + b_4'c_0 - b_2c_2' - b_3c_1' - b_4c_0' = -a_0(c_2b_3 + 3c_1b_4) \\ &+ a_1(c_2b_2 - c_1b_3 - 3c_0b_4) + a_2(3c_2b_1 - c_0b_3 + c_1b_2), \end{aligned} \tag{2.33}$$

$$b_3'c_2 + b_4'c_1 - b_3c_2' - b_4c_1' = -2a_0c_2b_4 - 2a_1c_1b_4 + 2a_2(c_2b_2 - c_0b_4), \tag{2.34}$$

$$b_4'c_2 - c_2'b_4 = -a_1c_2b_4 + a_2(c_2b_3 - c_1b_4). \tag{2.35}$$

Taking account of relations (2.29)-(2.30) and (2.34)-(2.35) and by performing simple computations, we obtain the identities (2.23)-(2.24) and (2.26)-(2.27), substituting them into (2.33), we get the relation (2.25). Putting (2.23)-(2.27) into (2.31)-(2.32) we obtain

$$\begin{aligned} (\frac{b_0}{c_0})' =& -\frac{c_2b_0}{c_0^2}(\frac{c_0}{c_2})' + \frac{b_1}{c_2}(\frac{c_1}{c_0})' - \frac{c_0b_2}{c_2^2}(\frac{c_2}{c_0})' + (\frac{c_0}{c_2})'\frac{c_1b_1}{c_0^2} - (\frac{c_1}{c_2})'\frac{c_1b_2}{c_2c_0} + (\frac{c_2}{c_0})'\frac{c_0c_1b_3}{c_2^3} \\ &+ (\frac{c_1}{c_2})'\frac{c_1^2b_3}{c_2^2c_0} + (\frac{c_0}{c_2})'\frac{c_1^2b_4}{c_2^2c_0} - (\frac{c_1}{c_2})'\frac{c_1^3b_4}{c_2^3c_0} + (\frac{c_1}{c_2})'\frac{c_1b_4}{c_2^2} - (\frac{c_1}{c_2})'\frac{b_3}{c_2} - (\frac{c_0}{c_2})'\frac{b_4}{c_2} \\ &+ (\frac{c_1}{c_2})'\frac{c_1b_4}{c_2^2} + a_0(\frac{b_1}{c_0} - 2\frac{b_3}{c_2} + 3\frac{c_1b_4}{c_2^2} - \frac{c_1b_2}{c_0c_2} + \frac{c_1^2b_3}{c_2^2c_0} - \frac{c_1^3b_4}{c_2^3c_0}) \\ &+ a_1(3\frac{b_0}{c_0} - 2\frac{b_2}{c_2} + 2\frac{c_0b_4}{c_2^2} + 3\frac{c_1b_3}{c_2^2} - \frac{c_1b_1}{c_0c_2} - 4\frac{c_1^2b_4}{c_2^3} + \frac{c_1^2b_2}{c_2^2c_0} - \frac{c_1^3b_3}{c_2^3b_2} + \frac{c_1^4b_4}{c_2^4c_0}) \\ &+ a_2(5\frac{c_1^3b_4}{c_2^4} - \frac{c_1^3b_2}{c_2^3c_0} + \frac{c_1^4b_3}{c_2^4c_0} - \frac{c_1^5b_4}{c_2^5c_0} - 4\frac{c_1^2b_3}{c_2^3} + \frac{c_1^2b_1}{c_2^2c_0} + 3\frac{c_1b_2}{c_2^2} - 5\frac{c_0c_1b_4}{c_2^3} \end{aligned}$$

$$+ 2\frac{c_0 b_3}{c_2^2} - 2\frac{b_1}{c_2} - \frac{c_1 b_0}{c_0 c_2}), \tag{2.36}$$

$$\begin{aligned}(\frac{b_1}{c_0})' =& -(\frac{c_0}{c_2})'\frac{c_2 b_1}{c_0^2} + (\frac{c_1}{c_2})'\frac{b_2}{c_0} - (\frac{c_2}{c_0})'\frac{c_0 b_3}{c_2^2} - (\frac{c_1}{c_2})'\frac{c_1 b_3}{c_0 c_2} - (\frac{c_0}{c_2})'\frac{c_1 b_4}{c_0 c_2} + (\frac{c_1}{c_2})'\frac{c_1^2 b_4}{c_2^2 c_0} \\ & - (\frac{c_1}{c_2})'\frac{b_4}{c_2} + a_0(\frac{c_1^2 b_4}{c_2^2 c_0} - \frac{c_1 b_3}{c_0 c_2} - 2\frac{b_4}{c_2}) + a_1(-\frac{c_1^3 b_4}{c_2^3 c_0} - \frac{c_1 b_2}{c_0 c_2} + \frac{c_1^2 b_3}{c_2^2 c_0} + 3\frac{c_1 b_4}{c_2^2} \\ & + 2\frac{b_1}{c_0} - 2\frac{b_3}{c_2}) + a_2(\frac{c_1^2 b_2}{c_2^2 c_0} - 4\frac{c_1^2 b_4}{c_2^3} - \frac{c_1^3 b_3}{c_2^3 c_0} + \frac{c_1^4 b_4}{c_2^4 c_0} + 3\frac{c_1 b_3}{c_2^2} - \frac{c_1 b_2}{c_0 c_2} - 2\frac{b_2}{c_2} \\ & + 2\frac{c_0 b_4}{c_2^2} + 4\frac{b_0}{c_0}). \end{aligned} \tag{2.37}$$

Computing (2.23) and (2.36), (2.24) and (2.34), it implies that the identities (2.21)-(2.22) are valid. Therefore, when the relations (2.21)-(2.27) are satisfied, the identity (1.4) is valid for $\Delta(t, x)$ which is in the form (2.20). Thus, the equation (2.15) and (2.28) are equivalent.

When $W \neq 0$, the algebraic equations (2.21)-(2.22) has a unique solution, i.e. $\delta_1(t) = \delta_2(t) = 0$. In this case,

$$\Delta(t, x) = \frac{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}{c_0 + c_1 x + c_2 x^2} = e_0 + e_1 x + e_2 x^2, \tag{2.38}$$

here $e_0 = \frac{b_0}{c_0}, e_1 = \frac{b_3}{c_2} - \frac{c_1 b_4}{c_2^2}, e_2 = \frac{b_4}{c_2}$. Moreover, the equation (2.28) is a Riccati equation, too.

Applying the relations (30-34) and computing, we obtain

$$\begin{aligned}\delta_1' &= \delta_1(-\frac{c_2}{c_0}(\frac{c_0}{c_2})' + a_0\frac{c_1}{c_0} + 2a_1 - a_2\frac{c_1}{c_2}) + \delta_2((\frac{c_1}{c_2})' - a_1\frac{c_1}{c_2} + a_2(\frac{c_1^2}{c_2^2} - 4\frac{c_0}{c_2})), \\ \delta_2' &= \delta_1(a_0\frac{c_2}{c_0} + 3a_2) + \delta_2(a_1 - \frac{c_2}{c_0}(\frac{c_0}{c_2})' - 3a_2\frac{c_1}{c_2}).\end{aligned}$$

By the assumptions of the present theorem, above the linear system has a unique solution satisfying initial value $\delta_1(0) = 0,\ \delta_2(0) = 0$, i.e. $\delta_1(t) = 0,\ \delta_2(t) = 0$. Thus, $\Delta(t, x)$ is in the form (2.20) and the equation (2.28) is a Riccati equation, too. □

Example 2.4. The Riccati equation

$$x' = e^{\frac{1}{2}\sin t} + \frac{x}{2}\cos t + e^{-\frac{1}{2}\sin t}x^2$$

is equivalent to equation

$$x' = e^{\frac{1}{2}\sin t} + \frac{x}{2}\cos t + e^{-\frac{1}{2}\sin t}x^2 + \alpha(t)\Delta(t, x),$$

here

$$\begin{aligned}\Delta(t, x) &= \frac{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}{e^{\sin t} + x^2}, \\ b_0 &= \mu(t)e^{\frac{3}{2}\sin t}, \quad b_1 = -\mu'(t)e^{\sin t}, \\ b_2 &= 2\mu(t)e^{\frac{1}{2}\sin t} + \frac{1}{2}\mu''(t)e^{\frac{1}{2}\sin t}, \\ b_3 &= -\frac{5}{3}\mu'(t) - \frac{1}{6}\mu'''(t), \\ b_4 &= \mu(t)e^{-\frac{1}{2}\sin t} + \frac{2}{3}\mu''(t)e^{-\frac{1}{2}\sin t} + \frac{1}{24}\mu^{(4)}(t)e^{-\frac{1}{2}\sin t},\end{aligned}$$

$$\mu(t) = k_1 \cos\lambda_1 t + k_2 \sin\lambda_1 t + k_3 \cos\lambda_2 t + k_4 \sin\lambda_2 t,$$
$$\lambda_1 = \sqrt{20 - 4\sqrt{21}}, \quad \lambda_2 = \sqrt{20 + 4\sqrt{21}},$$
$$k_2(\lambda_1^3 - 4\lambda_1) + k_4(\lambda_2^3 - 4\lambda_2) \neq 0,$$

k_i $(i = 1, 2, 3, 4)$ are constants.

In fact, in the Theorem 2.5, taking $a_1 = 0$, $a_2 = 1$, the relations (2.21)-(2.27) are equivalent to

$$\delta_1(2a_1 - \frac{c_0'}{c_0}) + 2\delta_2(a_0 - c_0 a_2) = 0,$$
$$2\delta_1(a_0 - c_0 a_2) + \delta_2(c_0' - 2c_0 a_1) = 0,$$
$$b_0' = (\frac{c_0'}{c_0} + a_1)b_0 - a_0 b_1,$$
$$b_1' = (2a_2 + 2\frac{a_0}{c_0})b_0 + \frac{c_0'}{c_0}b_1 - 2a_0 b_2,$$
$$b_2' = 3a_2 b_1 + a_1 b_2 - (a_0 + 2c_0 a_2)b_3 + (c_0' - 2c_0 a_1)b_4,$$
$$b_3' = 2a_2 b_2 - 2(a_0 + c_0 a_2)b_4,$$
$$b_4' = a_2 b_3 - a_1 b_4.$$

It is not difficult to verify that $b_i(i = 0, 1, ..., 4)$, in the present example, satisfy these relations with $a_0 = e^{\frac{1}{2}\sin t}, a_1 = \frac{\cos t}{2}, a_2 = e^{-\frac{1}{2}\sin t}, c_0 = e^{\sin t}$, and $\delta_1(0) \neq 0$. It means that the present conclusion is true.

Example 2.5. The Riccati equation

$$x' = 1 + x^2$$

is equivalent to equation

$$x' = 1 + x^2 + \alpha(t)\frac{\sin 4t - 4x\cos 4t - 6x^2 \sin 4t + 4x^3 \cos 4t + x^4 \sin 4t}{1 + x^2},$$

here $\alpha(t)$ is an arbitrary odd continuous function. This equation does not have one $\frac{\pi}{2}$-periodic solution.

Corollary 2.4. *If*

$$\Delta_k(t, x) = \frac{b_{k0} + b_{k1}x + b_{k2}x^2 + b_{k3}x^3 + b_{k4}x^4}{c_{k0} + c_{k1}x + c_{k2}x^2}$$

satisfies the conditions of Theorem 2.5, then the Riccati equation (2.15) *is equivalent to equation*

$$x' = a_0(t) + a_1(t)x + a_2(t)x^2 + \Sigma\beta_k(t)\Delta_k(t, x),$$

where $\beta_k(t)$ are arbitrary continuously differentiable odd functions.

From this corollary, we see that the properties of solution of above the complected equation also can be determined by the Riccati equation (2.15).

References

[1] V. Arnold, *Ordinary differential equation*, Science Press, Moscow, 1971, 198-240.

[2] M. Alwash and N. Lloyd, *Non-autonomous equations related to polynomial two-dimensional systems*, Proc. Roy. Soc. Edinburgh Sect.A, 105(1987), 129-152.

[3] Zhibo Cheng and Jingli Ren, *Periodic solution for high-order differential system*, J. Appl. Anal. Comput., 3(3)(2013), 239-249.

[4] J. Decline, N. G. Lloyd, and J. M. Pearson, *Cubic systems and abel equation*, J. Diff. Equ., 147(1998), 435-454.

[5] Ph. Hartman, *Ordinary differential equations*, Johns Hopkins University, New York, London, Sydney, 1964.

[6] Noel G. Lloyd and Jane M. Pearson, *A cubic differential system with nine limit cycles*, J. Appl. Anal. Comput., 2(3), 293-304.

[7] V. I. Mironenko, *Reflecting function and discussion of many-dimensional differential system*, Gomel University, Belarus, 2004.

[8] V. I. Mironenko, *The reflecting function of a family of functions*, Differ. Equ., 36(12)(2000), 1636-1641.

[9] V. V. Mironenko, *Time symmetry preserving perturbations of differential systems*, Differ. Equ., 40(20)(2004), 1395-1403.

[10] V. I. Mironenko and V. V. Mironenko, *Time symmetries and in-period transformations*, Appl. Math. Lett., 24(2011), 1721-1723.

[11] E. V. Musafirov, *Differential systems, the mapping over period for which is represented by a product of three exponential matrixes*, J. Math. Anal. Appl., 329(2007), 647-654.

[12] E. V. Musafirov, *The reflecting function and the small parameter method*, Appl. Math. Lett., 21(2008), 1064-1068.

[13] V. Maiorovskaya,*Quadratic system with linear reflecting function*, Differ. Equ., 45(2)(2009), 271-273.

[14] P. P. Veresovich,*Nonautonomous second order quadric system equivalent to linear system*, Differ. Equ., 34(12),(1998), 2257-2259.

[15] L. Yang and T. Yuan, *Some New Results on Abel Equations*, J. Math. Anal. Appl., 261(2001), 100-112.

[16] Zhiyan Yang, Tao Jiang and Zhujun Jing, *Bifurcations of periodic solutions and chaos in Duffing-van der Pol equation with one external forcing*,J. Appl. Anal. Comput., 3(4)(2013), 405-431.

[17] Zhengxin Zhou, *On the Poincare mapping and periodic solutions of nonautonomous differential systems*, Commun. Pure Appl. Anal., 2(2007), 541-547.

[18] Zhengxin Zhou, *The structure of reflective function of polynomial differential systems*, Nonlinear Analysis , 71(2009), 391-398.

[19] Zhengxin Zhou, *On the relationship between quadratic polynomial differential system and the Bernoulli equation*, Appl. Math. Comput., 21(2011), 8716-8721.

[20] Zhengxin Zhou,*On the equivalence of differential systems*, J. Appl. Anal. Comput., 2(2)(2012), 241-249.