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OSCILLATION CRITERIA FOR A CLASS OF FIRST ORDER NEUTRAL IMPULSIVE DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract Sufficient conditions are established for oscillation of all solutions of a class of nonlinear neutral impulsive differential-difference equations of first order with deviating argument and fixed moments of impulse effect.

Keywords Oscillation, nonoscillation, neutral, impulsive differential-difference equations, non-linear, delay.

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1. Introduction

In recent years the impulsive differential equations are an object of intensive study [11, 14, 22, 25, 26, see for e.g.]. These equations describe processes which are characterized as continuous, as jump-wise change of the phase variables describing process. There are adequate mathematical models of processes and phenomena studied in theoretical physics, chemical technology, population dynamics, rhythmical beating, merging of solutions and noncontinuity of solutions. Moreover, the theory of impulsive differential equations is emerging as an important area of investigation, since it is much richer than the corresponding theory of differential equations without impulse effect.

Oscillatory and asymptotic behaviour of the solutions of various classes of functional differential equations has taken the shape of a well-developed theory presented in the monographs [13, 15] and for recent information we refer the reader to some of the works [9, 19, 20, 24]. We may note that, in the present years much effort has been devoted to the study of functional differential equations of neutral type. However, the impulsive differential equations of neutral type is not well studied. Hence in this work, the author has made an attempt to study the oscillatory behaviour of solutions of a class of nonlinear neutral first order impulsive differential equations of the form

$$\begin{aligned} &(y(t)+p(t)y(t-\tau))'+q(t)G(y(t-\sigma))=0, && t\neq\tau_k,\ k\in\mathbb{N}, \\ &\triangle(y(\tau_k)+p_ky(\tau_k-\tau))+q_kG(y(\tau_k-\sigma))=0, && k\in\mathbb{N}, \end{aligned} \tag{1.1}$$

where $\tau,\sigma\in\mathbb{R}_+=(0,+\infty)$; $\tau_1,\tau_2,...,\tau_k,...$ are the moments of impulse effect; p_k and q_k are constants ($k\in\mathbb{N}$); $G\in C(\mathbb{R},\mathbb{R})$ such that $xG(x)>0$ for, $x\neq 0$;

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$q \in C(\mathbb{R}_+, \mathbb{R}_+)$; $p \in PC(\mathbb{R}_+, \mathbb{R})$, and

$$\begin{aligned}&\triangle(y(\tau_k) + p_k y(\tau_k - \tau))\\ =&y(\tau_k + 0) + p_k y(\tau_k - \tau + 0) - y(\tau_k - 0) - p_k y(\tau_k - \tau - 0);\end{aligned}$$

$y(\tau_k - 0) = y(\tau_k)$; and $y(\tau_k - \tau - 0) = y(\tau_k - \tau)$, $k \in \mathbb{N}$.

The study of stability and asymptotic behaviour of certain ordinary differential equations with impulses have been appeared in [7, 12, 21, 23]. It has been noticed that almost there is no such work of studying the impulsive differential-difference equations of the type (1.1) and the objective of this work is to establish sufficient conditions for oscillation of solutions of (1.1) with deviating argument and fixed moments of impulse effect. In this direction, we refer the readers to some of the works [1–6, 8, 16–18].

Definition 1.1. A function $y : [-\rho, +\infty) \to \mathbb{R}$ is said to be a solution of (1.1) with initial function $\o \in C([-\rho, 0], \mathbb{R})$, $y(t) = \o(t)$ for $t \in [-\rho, 0]$, $y \in PC(\mathbb{R}_+, \mathbb{R}), z(t) = y(t) + p(t)y(t - \tau)$ is continuously differentiable for $t \in \mathbb{R}_+$, and $y(t)$ satisfies (1.1) for all sufficiently large $t \geq 0$, where $\rho = max\{\tau, \sigma\}$ and $PC(\mathbb{R}_+, \mathbb{R})$ is the set of all functions $U : \mathbb{R}_+ \to \mathbb{R}$ which are continuous for $t \in \mathbb{R}_+, t \neq \tau_k, k \in \mathbb{N}$, continuous from the left- side for $t \in \mathbb{R}_+$, and have discontinuity of the first kind at the points $\tau_k \in \mathbb{R}_+, k \in \mathbb{N}$.

Definition 1.2. A nontrivial solution $y(t)$ of (1.1) is said to be nonoscillatory, if there exists a point $t_0 \geq 0$ such that $y(t)$ has a constant sign for $t \geq t_0$. Otherwise, the solution $y(t)$ is said to be oscillatory.

Definition 1.3. A solution $y(t)$ of (1.1) is said to be regular, if it is defined on some interval $[T_y, +\infty) \subset [t_0, +\infty)$ and

$$sup\{|y(t)| : t \geq T\} > 0$$

for every $T \geq T_y$. A regular solution $y(t)$ of (1.1) is said to be eventually positive (eventually negative), if there exists $t_1 > 0$ such that $y(t) > 0 (y(t) < 0)$, for $t \geq t_1$.

2. Oscillation Results

In this section, we study the impulsive differential-difference equations of the form (1.1) and its corresponding inequalities

$$\begin{aligned}&(y(t) + p(t)y(t - \tau))' + q(t)G(y(t - \sigma)) \leq 0,\ t \neq \tau_k, && k \in \mathbb{N},\\ &\triangle(y(\tau_k) + p_k y(\tau_k - \tau)) + q_k G(y(\tau_k - \sigma)) \leq 0, && k \in \mathbb{N},\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&(y(t) + p(t)y(t - \tau))' + q(t)G(y(t - \sigma)) \geq 0,\ t \neq \tau_k, && k \in \mathbb{N},\\ &\triangle(y(\tau_k) + p_k y(\tau_k - \tau)) + q_k G(y(\tau_k - \sigma)) \geq 0, && k \in \mathbb{N}.\end{aligned} \tag{2.2}$$

We introduce the following assumptions:

(A_1) $0 < \tau_1 < \tau_2 < ...$ and $\lim_{k \to \infty} = +\infty$,

(A_2) $p \in PC(\mathbb{R}_+, \mathbb{R})$, $p_k = p(\tau_k - 0) = p(\tau_k)$ and $q_k = q(\tau_k - 0) = q(\tau_k)$, $k \in \mathbb{N}$.

Theorem 2.1. *Let $-\infty < -M \leq p(t) \leq -1,\ M > 0,\ t \in \mathbb{R}_+$ and $\tau > \sigma$. Assume that (A_1) and (A_2) hold. If*

(A_3) $G(uv) = G(u)G(v),\ u, v \in \mathbb{R}$,

(A_4) G is superlinear and

$$\int\limits_0^{\pm\infty} \frac{dx}{G(x)} < \infty, \quad \sum_{k=1}^{\infty} \int\limits_{\gamma(\tau_k)}^{\gamma(\tau_k+0)} \frac{dx}{G(x)} < +\infty,$$

(A_5) $\int\limits_0^{\infty} q(t)dt + \sum_{k=1}^{\infty} q_k = \infty$

hold, then

1. *the inequality* (2.1) *has no eventually positive solutions,*
2. *the inequality* (2.2) *has no eventually negative solutions,*
3. *all solutions of* (1.1) *are oscillatory.*

Proof. Let $y(t)$ be a regular solution of (2.1). Hence, there exists a $t_0 > 0$ such that $y(t) > 0,\ y(t - \tau) > 0$ and $y(t - \sigma) > 0$, for $t \geq t_0 > max\{\sigma, \tau\}$. Set

$$\begin{aligned} z(t) &= y(t) + p(t)y(t - \tau), && t \geq t_0, \\ z(\tau_k) &= y(\tau_k) + p(\tau_k)y(\tau_k - \tau), && k \in \mathbb{N}. \end{aligned} \tag{2.3}$$

From (2.1) and the assumption (A_2), it follows that

$$\begin{aligned} &z'(t) < -q(t)G(y(t - \sigma)) < 0, \\ &\triangle z(\tau_k) < -q_k G(y(\tau_k - \sigma)) < 0, \quad k \in \mathbb{N} \end{aligned}$$

hold and hence z is a decreasing function for $t \geq t_0$. We claim that $z(t) < 0$, for $t \geq t_0$. If not, let $z(t) \geq 0$, for $t \geq t_1 > t_0$. Consequently,

$$\begin{aligned} &y(t) \geq -p(t)y(t - \tau) \geq y(t - \tau), \\ &y(\tau_k) \geq -p_k y(\tau_k - \tau) \geq y(\tau_k - \tau), \quad k \in \mathbb{N} \end{aligned}$$

implies that y is bounded from below by $m > 0$. Integrating (2.1) from t_1 to $t(t \geq t_1)$, we obtain

$$z(t) - z(t_1) - \sum_{t_1 \leq \tau_k < t} \triangle z(\tau_k) + \int\limits_{t_1}^{t} q(s)G(y(s - \sigma))ds < 0,$$

that is,

$$z(t) - z(t_1) + \sum_{t_1 \leq \tau_k < t} q_k G(y(\tau_k - \sigma)) + \int\limits_{t_1}^{t} q(s)G(y(s - \sigma))ds < 0.$$

Therefore,

$$z(t) < z(t_1) - G(m)[\sum_{t_1 \le \tau_k < t} q_k + \int_{t_1}^{t} q(s)ds] < 0, k \in \mathbb{N}$$
$$\to -\infty \;\; as \;\; t \to \infty \;\; (by A_5),$$

a contradiction to the fact that $z(t) > 0$ on $[t_1, \infty)$. So our claim holds. Upon using (2.3), it follows that $z(t+\tau-\sigma) > p(t+\tau-\sigma)y(t-\sigma)$ and $z(\tau_k+\tau-\sigma) > p(\tau_k+\tau-\sigma)y(\tau_k-\sigma)$, for $k \in \mathbb{N}$. Hence, the inequality (2.1) becomes

$$\begin{aligned} &z^{'}(t) + \frac{q(t)}{G(-M)}G(z(t+\tau-\sigma)) \le 0, && t \ne \tau_k, \;\; k \in \mathbb{N}, \\ &\triangle z(\tau_k) + \frac{q_k}{G(-M)}G(z(\tau_k+\tau-\sigma)) \le 0, && k \in \mathbb{N} \end{aligned} \tag{2.4}$$

due to (A_3). Because z is decreasing on $[t_1, \infty)$, then

$$\begin{aligned} &z'(t) + \frac{q(t)}{G(-M)}G(z(t)) \le 0, && t \ne \tau_k, \;\; k \in \mathbb{N}, \\ &\triangle z(\tau_k) + \frac{q_k}{G(-M)}G(z(\tau_k)) \le 0, && k \in \mathbb{N}. \end{aligned}$$

Integrating the first inequality from t_1 to $t (t \ge t_1)$, we get

$$\int_{t_1}^{t} \frac{z'(s)}{G(z(s))}ds + \frac{1}{G(-M)}\int_{t_1}^{t} q(s)ds > 0,$$

that is,

$$\lim_{t\to\infty}\int_{t_1}^{t} q(s)ds \le -G(-M)\lim_{t\to\infty}\int_{z(t_1)}^{z(t)} \frac{z'(s)}{G(z(s))}ds < \infty. \tag{2.5}$$

On the other hand, if $z(\tau_k+0) < x < z(\tau_k-0)$, then $G(x) > G(z(\tau_k+0))$ and the second inequality implies that

$$q_k < -G(-M)\frac{\triangle z(\tau_k)}{G(x)} = -G(-M)\int_{z(\tau_k)}^{z(\tau_k+0)} \frac{dx}{G(x)}, \;\; k \in \mathbb{N}.$$

Hence,

$$\sum_{k=1}^{\infty} q_k \le -G(-M)\sum_{k=1}^{\infty}\int_{z(\tau_k)}^{z(\tau_k+0)} \frac{dx}{G(x)} < \infty. \tag{2.6}$$

(2.5) and (2.6) together gives a contradiction to (A_5). Thus (2.1) has no eventually positive solution.

In order to prove that (2.2) has no eventually negative solution, it is enough to note that $-y(t)$ is a solution of (2.1) when $y(t)$ is a solution of (2.2). This is because of (A_3), where it is easy to verify the fact that $G(-1) = -G(1)$.

It follows from the assertions 1 and 2 that the (1.1) has neither eventually positive nor eventually negative solution. Hence, each regular solution of (1.1) oscillates. This completes the proof of the theorem. □

Theorem 2.2. *Let* $-\infty < -M \le p(t) \le -1,\ M > 0,\ t \in \mathbb{R}_+$. *Assume that* (A_1), (A_2), (A_3) *and* (A_5) *hold. Then*

1. *the inequality* (2.1) *has no eventually positive bounded solutions,*
2. *the inequality* (2.2) *has no eventually negative bounded solutions,*
3. *all bounded solutions of* (1.1) *are oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.1, we have that $z(t) < 0$, for $t \ge t_1$. Hence the inequality (2.4) holds. Because z is decreasing, there exist $t_2 > t_1$ and $C > 0$ such that $z(t) \le -C$, for $t \ge t_2$. Moreover, the inequality (2.4) reduces to

$$z'(t) + \frac{G(-C)}{G(-M)} q(t) < 0, \tag{2.7}$$

$$\triangle z(\tau_k) + \frac{G(-C)}{G(-M)} q_k < 0, \quad k \in \mathbb{N}$$

for $t \ge t_2$. Integrating (2.7) from t_2 to $t(t \ge t_2)$, we get

$$z(t) - z(t_2) - \sum_{t_2 \le \tau_k < t} \triangle z(\tau_k) + \frac{G(-C)}{G(-M)} \int_{t_2}^{t} q(s)ds < 0,$$

that is,

$$z(t) - z(t_2) + \frac{G(-C)}{G(-M)} [\sum_{t_2 \le \tau_k < t} q_k + \int_{t_2}^{t} q(s)ds] < 0. \tag{2.8}$$

Since $y(t)$ is bounded, then $z(t)$ is bounded and hence the inequality (2.8) becomes

$$\lim_{t\to\infty} \frac{G(-C)}{G(-M)} [\sum_{t_2 \le \tau_k < t} q_k + \int_{t_2}^{t} q(s)ds] \le -\lim_{t\to\infty} z(t) + z(t_2) < \infty,$$

a contradiction to (A_5). The rest of the proof follows from Theorem 2.1. Hence the theorem is proved. □

Theorem 2.3. *Let* $-1 < -a \le p(t) \le 0, a > 0, t \in \mathbb{R}_+$ *and* $\tau > \sigma$. *If* (A_1), (A_2), (A_3), (A_5) *and*

(A_4) *G is sublinear and*

$$\int_0^{\pm C} \frac{dx}{G(x)} < \infty, \quad \sum_{k=1}^{\infty} \int_{\gamma(\tau_k)}^{\gamma(\tau_k+0)} \frac{dx}{G(x)} < +\infty, \quad C > 0, \quad \lim_{k\to\infty} \gamma(\tau_k) < \infty$$

hold, then the conclusion of Theorem 2.1 is true.

Proof. Proceeding as in Theorem 2.1, we may note that z is monotonic decreasing on $[t_0, \infty)$, $t_0 > \tau$. Hence there exists $t_1 > t_0$ such that z is of one sign on $[t_1, \infty)$. Suppose that $z(t) > 0$ for the inequality (2.1) on $[t_1, \infty)$. Using (2.3), it follows that $z(t) \leq y(t)$ and $z(\tau_k) \leq y(\tau_k)$, $k \in \mathbb{N}$ on $[t_1, \infty)$. Consequently, the inequality (2.1) becomes

$$\begin{aligned} &z'(t) + q(t)G(z(t-\sigma)) < 0, \\ &\triangle z(\tau_k) + q_k G(z(\tau_k - \sigma)) < 0, \quad k \in \mathbb{N}, \end{aligned}$$

that is,

$$\begin{aligned} &\frac{z'(t)}{G(z(t))} + q(t) < 0, \\ &\frac{\triangle z(\tau_k)}{G(z(\tau_k))} + q_k < 0, \quad k \in \mathbb{N}. \end{aligned}$$

Because z is decreasing and $\lim_{t\to\infty} z(t) < \infty$, then the above inequality yields that

$$\begin{aligned} &\lim_{t\to\infty} \int_{t_1}^{t} q(s)ds < -\lim_{t\to\infty} \int_{z(t_1)}^{z(t)} \frac{dx}{G(x)} < \infty, \\ &\sum_{k=1}^{\infty} q_k < -\sum_{k=1}^{\infty} \frac{\triangle z(\tau_k)}{G(x)} = -\sum_{k=1}^{\infty} \int_{z(\tau_k - 0)}^{z(\tau_k + 0)} \frac{dw}{G(w)} < \infty, \quad k \in \mathbb{N} \end{aligned}$$

due to $\lim_{k\to\infty} z(\tau_k) < +\infty$, if $z(\tau_k + 0) < w < z(\tau_k - 0)$ and $G(w) < G(z(\tau_k))$, for $k \in \mathbb{N}$. Thus we have a contradiction to (A_5) and hence $z(t) < 0$, for $t \geq t_1$. From (2.3), it is easy to verify that $y(t) < y(t-\tau)$ and $y(\tau_k) < y(\tau_k - \tau)$, $k \in \mathbb{N}$ on $[t_2, \infty)$, $t_2 > t_1$. Further, we can write

$$\begin{aligned} &y(t) < y(t-\tau) < y(t-2\tau) < \ldots < y(t_2), \\ &y(\tau_k) < y(\tau_k - \tau) < y(\tau_k - 2\tau) < \ldots < y(t_2), \end{aligned}$$

that is, y is bounded on $[t_2, \infty)$. Hence z is bounded on $[t_2, \infty)$. The rest of the proof follows from Theorem 2.2. This completes the proof of the theorem. □

Theorem 2.4. *Let $0 \leq p(t) \leq b < \infty$, for $t \in \mathbb{R}_+$ and $\tau \leq \sigma$. Assume that $(A_1), (A_2)$ and (A_3) hold. Furthermore, assume that*

(A_7) there exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u+v)$, $u, v \in \mathbb{R}_+$,

(A_8) G is sublinear and

$$\begin{aligned} &\int_{0}^{\pm c} \frac{dx}{G(x)} < \infty, && c > 0, \\ &\sum_{k=1}^{\infty} \Big[\int_{\gamma(\tau_k)}^{\gamma(\tau_k + 0)} \frac{dx}{G(x)} + \int_{\gamma(\tau_k - \tau)}^{\gamma(\tau_k - \tau + 0)} \frac{dw}{G(w)} \Big] < +\infty, && \lim_{k\to\infty} \gamma(\tau_k) < \infty, \end{aligned}$$

(A₉) $\int_0^\infty Q(t)dt + \sum_{k=1}^\infty Q_k = \infty$

hold, where $Q(t) = min\{q(t), q(t-\tau)\},\ t \geq \tau;\ Q(\tau_k) = min\{q(\tau_k), q(\tau_k - \tau)\},\ \tau_k > \tau,\ k \in \mathbb{N}$. *Then the conclusion of Theorem 2.1 is true.*

Proof. Proceeding as in Theorem 2.1, we have that z is monotonic decreasing on $[t_0, \infty), t_0 > \sigma$. Hence there exists $t_1 > t_0$ such that $z(t) > 0$ on $[t_1, \infty)$. Using (2.1) and (2.3), it is easy to verify that

$$\begin{aligned} 0 &> z'(t) + q(t)G(y(t-\sigma)) + G(b)z'(t-\tau) + G(b)q(t-\tau)G(y(t-\tau-\sigma)) \\ &\geq z'(t) + G(b)z'(t-\tau) + Q(t)[G(y(t-\sigma)) + G(by(t-\tau-\sigma))] \\ &\geq z'(t) + G(b)z'(t-\tau) + \lambda Q(t)G(y(t-\sigma) + by(t-\tau-\sigma)) \\ &\geq z'(t) + G(b)z'(t-\tau) + \lambda Q(t)G(z(t-\sigma)) \end{aligned}$$

due to $(A_3), (A_7)$ and (A_9), where $z(t) \leq y(t) + by(t-\tau)$. Similarly, we obtain

$$\triangle z(\tau_k) + G(b)\triangle z(\tau_k - \tau) + \lambda Q_k G(z(\tau_k - \sigma)) < 0, \quad k \in \mathbb{N}.$$

Consequently, there exists $t_2 > t_1$ such that

$$\begin{aligned} &\frac{z'(t)}{G(z(t-\sigma))} + G(b)\frac{z'(t-\tau)}{G(z(t-\sigma))} + \lambda Q(t) < 0, \\ &\frac{\triangle z(\tau_k)}{G(z(\tau_k-\sigma))} + G(b)\frac{\triangle z(\tau_k-\tau)}{G(z(\tau_k-\sigma))} + \lambda Q_k < 0, \quad k \in \mathbb{N}. \end{aligned} \tag{2.9}$$

Because z is decreasing on $[t_2, \infty)$ and $\tau \leq \sigma$, the inequalities in (2.9) become

$$\begin{aligned} &\frac{z'(t)}{G(z(t))} + G(b)\frac{z'(t-\tau)}{G(z(t-\tau))} + \lambda Q(t) < 0, \\ &\frac{\triangle z(\tau_k)}{G(z(\tau_k))} + G(b)\frac{\triangle z(\tau_k-\tau)}{G(z(\tau_k-\tau))} + \lambda Q_k < 0, \quad k \in \mathbb{N}, \end{aligned}$$

that is,

$$\begin{aligned} &\int_{t_2}^{t} \frac{z'(s)}{G(z(s))}ds + G(b)\int_{t_2}^{t} \frac{z'(s-\tau)}{G(z(s-\tau))}ds + \lambda\int_{t_2}^{t} Q(s)ds < 0, \\ &\frac{\triangle z(\tau_k)}{G(x)} + G(b)\frac{\triangle z(\tau_k-\tau)}{G(u)} + \lambda Q_k < 0, \quad k \in \mathbb{N}, \end{aligned} \tag{2.10}$$

where $\tau_k + 0 < x < \tau_k - 0$ and $\tau_k - \tau + 0 < u < \tau_k - \tau - 0$. As a result, the inequalities in (2.10) yield

$$\begin{aligned} &\lambda \lim_{t\to\infty}\int_{t_2}^{t} Q(s)ds < -\lim_{t\to\infty}[\int_{z(t_2)}^{z(t)} \frac{z'(s)}{G(z(s))}ds + G(b)\int_{z(t_2-\tau)}^{z(t-\tau)} \frac{z'(s-\tau)}{G(z(s-\tau))}ds], \\ &\lambda Q_k < -\int_{z(\tau_k-0)}^{z(\tau_k+0)} \frac{dx}{G(x)} - G(b)\int_{z(\tau_k-\tau-0)}^{z(\tau_k-\tau+0)} \frac{du}{G(u)}, \quad k \in \mathbb{N}. \end{aligned}$$

Hence,

$$\int_{t_2}^{\infty} Q(s)ds < \infty,$$

$$\lambda \sum_{k=1}^{\infty} Q_k < -\sum_{k=1}^{\infty}\Big[\int_{z(\tau_k-0)}^{z(\tau_k+0)} \frac{dx}{G(x)} + G(b)\int_{z(\tau_k-\tau-0)}^{z(\tau_k-\tau+0)} \frac{du}{G(u)}\Big], \quad k \in \mathbb{N},$$

a contradiction to (A_9) due to (A_8), $\lim_{t\to\infty} z(t) < \infty$ and $\lim_{k\to\infty} z(\tau_k) < \infty$. The rest of the proof follows from Theorem 2.1. Hence the theorem is proved. □

Theorem 2.5. *Let $p(t) \leq -1$, for $t \in \mathbb{R}_+$ and $\tau > \sigma$. Assume that $(A_1), (A_2), (A_3)$ and (A_5) hold. Furthermore, assume that*

(A_{10}) there exists $\gamma > 0$ such that $G(u) \geq \gamma|u|$, $u \in \mathbb{R}$,

(A_{11}) $\limsup_{k\to\infty} \left[\int_{\tau_k-\tau+\sigma}^{\tau_k} \frac{-q(u)du}{p(u+\tau-\sigma)} + \sum_{\tau_k-\tau+\sigma\leq\tau_i<\tau_k} \frac{-q(\tau_i)}{p(\tau_i+\tau-\sigma)}\right] > \frac{1}{\gamma}$, $k \in \mathbb{N}$

hold. Then the conclusion of Theorem 2.1 is true.

Proof. Let $y(t)$ be an eventually positive solution of (2.1), for $t \geq t_0 > \tau$. Proceeding as in Theorem 2.1, we get a contradiction when $z(t) > 0$ on $[t_1, \infty)$. Hence $z(t) < 0$, for $t \geq t_1$. From (2.3), it follows that $z(t+\tau-\sigma) > p(t+\tau-\sigma)y(t-\sigma)$, that is,

$$G(y(t-\sigma)) > G\left(\frac{z(t+\tau-\sigma)}{p(t+\tau-\sigma)}\right) \geq \frac{\gamma z(t+\tau-\sigma)}{p(t+\tau-\sigma)}.$$

Therefore, the inequality (2.1) reduces to

$$\begin{aligned} &z'(t) + \frac{\gamma q(t)}{p(t+\tau-\sigma)} z(t+\tau-\sigma) < 0, && t \geq t_1,\ t \neq \tau_k, \\ &\triangle z(\tau_k) + \frac{\gamma q(\tau_k)}{p(\tau_k+\tau-\sigma)} z(\tau_k+\tau-\sigma) < 0, && k \in \mathbb{N}, \end{aligned}$$

that is,

$$\begin{aligned} &z'(t) + \eta(t) z(t+l) < 0, && t \geq t_1,\ t \neq \tau_k, \\ &\triangle z(\tau_k) + \eta_k z(\tau_k + l) < 0, && k \in \mathbb{N}, \end{aligned} \tag{2.11}$$

where $l = \tau - \sigma$ and $\eta(t) = \frac{\gamma q(t)}{p(t+\tau-\sigma)}$, for $t \geq t_1$. Integrating (2.11) from $\tau_k - l$ to $\tau_k (\tau_k \geq t_1 + l, k \in \mathbb{N})$, we obtain

$$z(\tau_k) - z(\tau_k - l) - \sum_{\tau_k-l\leq\tau_i<\tau_k} \triangle z(\tau_i) + \int_{\tau_k-l}^{\tau_k} \eta(u) z(u+l) du < 0,$$

that is,

$$z(\tau_k) + \sum_{\tau_k-l\leq\tau_i<\tau_k} \eta_i z(\tau_i + l) + \int_{\tau_k-l}^{\tau_k} \eta(u) z(u+l) du < 0.$$

As a result, the last inequality reduces to

$$z(\tau_k) + z(\tau_k)\left[\sum_{\tau_k - l \le \tau_i < \tau_k} \eta_i + \int_{\tau_k - l}^{\tau_k} \eta(u)du\right] < 0,$$

that is,

$$\left[1 + \sum_{\tau_k - l \le \tau_i < \tau_k} \frac{\gamma q(\tau_i)}{p(\tau_i + \tau - \sigma)} + \int_{\tau_k - l}^{\tau_k} \frac{\gamma q(u)du}{p(u + \tau - \sigma)}\right] > 0,$$

which contradicts our assumption (A_{11}). The rest of the proof follows from Theorem 2.1. Hence the theorem is proved. □

Theorem 2.6. *Let* $-1 < -a \le p(t) \le 0,\ a > 0$, *for* $t \in \mathbb{R}_+$. *If* $(A_1), (A_2), (A_3), (A_5)$, (A_{10}) *and*

(A_{12}) $\quad \limsup_{k\to\infty}\left[\int\limits_{\tau_k-\sigma}^{\tau_k} q(t)dt + \sum_{\tau_k-\sigma\le\tau_i<\tau_k} q_i\right] > \frac{1}{\gamma}, \quad k \in \mathbb{N}$

hold, then the conclusion of Theorem 2.1 is true.

Proof. Proceeding as in Theorem 2.3, we have two cases viz., $z(t) > 0$ and $z(t) < 0$, for $t \ge t_1$. The case $z(t) < 0$, for $t \ge t_1$ follows from the proof of Theorem 2.3. Consider the former case for $t \ge t_1$. Integrating (2.1) from $\tau_k - \sigma$ to $\tau_k(\tau_k \ge t_1 + \sigma, k \in \mathbb{N})$, we get

$$z(\tau_k) - z(\tau_k - \sigma) - \sum_{\tau_k - \sigma \le \tau_i < \tau_k} \triangle z(\tau_i) + \int_{\tau_k - \sigma}^{\tau_k} q(s)G(z(s-\sigma))ds < 0,$$

that is,

$$z(\tau_k) - z(\tau_k - \sigma) + \sum_{\tau_k - \sigma \le \tau_i < \tau_k} q_i G(z(\tau_i - \sigma)) + \int_{\tau_k - \sigma}^{\tau_k} q(s)G(z(s-\sigma))ds < 0.$$

Using the fact that z is decreasing, the last inequality yields

$$-z(\tau_k - \sigma) + \gamma z(\tau_k - \sigma) \sum_{\tau_k - \sigma \le \tau_i < \tau_k} q_i + \gamma z(\tau_k - \sigma)\int_{\tau_k - \sigma}^{\tau_k} q(s)ds < -z(\tau_k)$$

due to (A_{10}). Consequently,

$$\limsup_{k\to\infty}\left[\int\limits_{\tau_k-\sigma}^{\tau_k} q(t)dt + \sum_{\tau_k-\sigma\le\tau_i<\tau_k} q_i\right] \le \frac{1}{\gamma}, \quad k \in \mathbb{N}$$

a contradiction to (A_{12}). The rest of the proof follows from Theorem 2.1 and hence the proof is complete. □

Theorem 2.7. *Let* $0 \le p(t) \le b < \infty$, *for* $t \in \mathbb{R}_+$ *and* $\sigma \ge 2\tau$. *Assume that* $(A_1), (A_2)$ *and* (A_{10}) *hold. If* $G(-u) = -G(u)$ *and*

(A_{13}) $\quad \limsup_{k\to\infty}\left[\int\limits_{\tau_k-\tau}^{\tau_k} Q(t)dt + \sum_{\tau_k-\tau\le\tau_i<\tau_k} Q(\tau_i)\right] > \frac{1+b}{\gamma}, \quad k \in \mathbb{N}$

hold, where $Q(t)$ *is defined in Theorem 2.4, then the conclusion of Theorem 2.1 is true.*

Proof. Proceeding as in Theorem 2.4, we obtain

$$\begin{aligned} &z'(t) + bz'(t-\tau) + \gamma Q(t)z(t-\sigma) < 0, && t \geq t_1,\ t \neq \tau_k, \\ &\triangle z(\tau_k) + b\triangle z(\tau_k - \tau) + \gamma Q_k z(\tau_k - \sigma) < 0, && k \in \mathbb{N}. \end{aligned} \tag{2.12}$$

Integrating (2.12) from $\tau_k - \tau$ to $\tau_k(\tau_k \geq t_1 + \tau, k \in \mathbb{N})$, we get

$$\begin{aligned} &z(\tau_k) + bz(\tau_k - \tau) - z(\tau_k - \tau) - bz(\tau_k - 2\tau) \\ &- \sum_{\tau_k - \tau \leq \tau_i < \tau_k} \triangle z(\tau_i) + \gamma \int_{\tau_k - \tau}^{\tau_k} Q(t)z(t-\sigma)dt < 0, \end{aligned}$$

that is,

$$\begin{aligned} &z(\tau_k) + bz(\tau_k - \tau) - z(\tau_k - \tau) - bz(\tau_k - 2\tau) \\ &+ \gamma \sum_{\tau_k - \tau \leq \tau_i < \tau_k} Q(\tau_i)z(\tau_i - \sigma) + \gamma \int_{\tau_k - \tau}^{\tau_k} Q(t)z(t-\sigma)dt < 0. \end{aligned}$$

Using the fact that z is decreasing on $[t_1, \infty)$, it follows from the last inequality that

$$-z(\tau_k - \tau) - bz(\tau_k - 2\tau) + \gamma z(\tau_k - \sigma)\left[\sum_{\tau_k - \tau \leq \tau_i < \tau_k} Q(\tau_i) + \int_{\tau_k - \tau}^{\tau_k} Q(t)dt\right] < 0,$$

that is,

$$-(1+b)z(\tau_k - 2\tau) + \gamma z(\tau_k - \sigma)\left[\sum_{\tau_k - \tau \leq \tau_i < \tau_k} Q(\tau_i) + \int_{\tau_k - \tau}^{\tau_k} Q(t)dt\right] < 0.$$

As $\sigma \geq 2\tau$, the last inequality becomes

$$\left[-(1+b) + \gamma\left\{\sum_{\tau_k - \tau \leq \tau_i < \tau_k} Q(\tau_i) + \int_{\tau_k - \tau}^{\tau_k} Q(t)dt\right\}\right] z(\tau_k - 2\tau) < 0$$

and hence

$$\limsup_{k \to \infty}\left[\int_{\tau_k - \tau}^{\tau_k} Q(t)dt + \sum_{\tau_k - \tau \leq \tau_i < \tau_k} Q(\tau_i)\right] \leq \frac{1+b}{\gamma},$$

a contradiction to (A_{13}). The rest of the proof follows from Theorem 2.1. This completes the proof of the theorem. □

3. Discussion and Example

In [10], Graef et al. have studied the oscillatory and asymptotic behavior of solutions of the equations of the form

$$(y(t) + p(t)y(t-\tau))' + q(t)G(y(t-\sigma)) = 0. \tag{3.1}$$

They have established the sufficient conditions for oscillation of all solutions of (3.1), when

$$p(t) \leq 0 \;\; and \;\; \int_0^\infty q(t)dt = \infty.$$

Taking the present work into account, it reveals that (3.1) could be oscillatory with impulses irrespective of $p(t) \neq 0$ following to the differential-difference equations.

The following example illustrates the main result of our work:

Example 3.1. Consider the differential-difference equations of the form

$$\left(y(t) + (1+e^{-t})y(t-\frac{1}{2})\right)' + q(t)(8+|y(t-1)|)|y(t-1)|sgn\, y(t-1) = 0,$$

$$\triangle\left(y(\tau_k) + (1+e^{-\tau_k})y(\tau_k-\frac{1}{2})\right) + q_k(8+|y(\tau_k-1)|)|y(\tau_k-1)|sgn\, y(\tau_k-1) = 0,$$

where $0 \leq p(t) = 1+e^{-t} \leq 2$, $q(t) = t$, $\tau_k = 2^k$, for $k > 1$ and $\gamma = 8$. Clearly, $Q(t) = t-1$ and

$$\int_{2^k-\frac{1}{2}}^{2^k} Q(t)dt + \sum_{2^k-\frac{1}{2}\leq\tau_i<2^k} Q(\tau_i) > \int_{2^k-\frac{1}{2}}^{2^k} Q(t)dt$$

$$= \int_{2^k-\frac{1}{2}}^{2^k}(t-1)dt = \frac{2^k}{2} - \frac{5}{8} > \frac{3}{8}, \qquad if\ k > 1$$

implies that (A_{13}) holds. Hence by Theorem 2.7, all solutions of the above differential-difference equations are oscillatory.

We conclude this section with the following existence result without proof:

Theorem 3.1. *Let* $-1 < -a \leq p(t) \leq 0, a > 0, t \in \mathbb{R}_+$. *Assume that* $(A_1), (A_2)$ *and* (A_3) *hold. If*

$$\int_0^\infty q(t)dt + \sum_{k=1}^\infty q_k < \infty,$$

then the differential-difference equations (1.1) *admits positive bounded solutions.*

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