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TRAVELING WAVE SOLUTIONS OF NONLINEAR SCALAR INTEGRAL DIFFERENTIAL EQUATIONS ARISING FROM SYNAPTICALLY COUPLED NEURONAL NETWORKS*

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Abstract Consider the following nonlinear scalar integral differential equations arising from synaptically coupled neuronal networks

$$\frac{\partial u}{\partial t} + u = \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c + \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \Theta) \mathrm{d}y \right] \mathrm{d}\tau,$$

and

$$\frac{\partial u}{\partial t} + u = \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c + \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \Theta) \mathrm{d}y \right] \mathrm{d}\tau - w_0.$$

These model equations generalize many important nonlinear scalar integral differential equations arising from synaptically coupled neuronal networks. The kernel functions K and W represent synaptic couplings between neurons in synaptically coupled neuronal networks. The synaptic couplings can be very general, including not only pure excitations (modeled with nonnegative kernel functions), lateral inhibitions (modeled with Mexican hat kernel functions), lateral excitations (modeled with upside down Mexican hat kernel functions), but also synaptic couplings which may change sign for finitely many times or even infinitely many times. The function $H = H(u-\theta)$ represents the Heaviside step function, which is defined by $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = \frac{1}{2}$ and $H(u-\theta) = 1$ for all $u > \theta$.

The functions ξ and η represent probability density functions defined on $(0, \infty)$. The parameter $c > 0$ represents the speed of an action potential and the parameter $\tau > 0$ represents a constant delay. In these equations, $u = u(x,t)$ stands for the membrane potential of a neuron at position x and time t. The positive constants $\alpha > 0$ and $\beta > 0$ represent synaptic rates. The positive constants $\theta > 0$ and $\Theta > 0$ represent thresholds for excitation of neurons. The positive constant $w_0 > 0$ is to be given.

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The authors will establish the existence and stability of traveling wave solutions of these nonlinear scalar integral differential equations by coupling together speed index functions, stability index functions (often called Evans functions, that is, complex analytic functions), implicit function theorem, intermediate value theorem, mean value theorem, global strong maximum principle for Evans functions, linearized stability criterion and many other important techniques in dynamical systems. They will find sufficient conditions satisfied by the synaptic couplings, by the probability density functions, by the synaptic rate constants and by the thresholds so that the traveling wave solutions and their wave speeds exist, and the stability of the traveling wave solutions is true. The main results obtained in this paper greatly improve many previous results.

Keywords synaptically coupled neuronal networks, nonlinear scalar integral differential equations, traveling wave solutions, existence, stability, speed index functions, stability index functions.

MSC(2000) 92C20.

1. Introduction

The spatial temporal dynamics of nonlinear scalar integral differential equations arising from synaptically coupled neuronal networks have attracted much attention in recent years, see [2, 5, 6, 8–10, 13, 15–17, 21, 23, 34, 36–40, 42, 45, 46, 59, 60, 68, 77–80]. They are powerful models to explain many phenomena and data observed in neurobiology medical science: such as pathological visual hallucination patterns [9,22], cortical epilepsy [14], general anesthesia [35,64], stimuli in turtle visual cortex [52] and cat visual cortex [66], migraine [43,53], and encephalographic data [54]. In addition, spatial temporal propagation of electrical activity has been observed in neural tissues [55] and motivates the theoretical study of traveling wave solutions (representing nerve impulses).

The synaptically coupled neuronal networks involve axonal couplings between single neurons. Since the transmission speed of action potentials along axons is finite, the axonal propagation of action potentials involve a certain transmission delay. This delay depends strongly on the axonal branching architecture [62] and the degree of myelination of axonal branches [56]. For instance, unmyelinated axons exhibit a small transmission speed in the range of one tenth to one meter per second and take place mainly in short-range intra-cortical couplings [48]. In long-range axonal fibers, such as cortical to cortical couplings, the axons are myelinated leading to a faster transmission speed in the range of one meter per second to one hundred meters per second [19]. Consequently, the resulting transmission delay between two spatial positions depends on the axon and changes between one half millisecond and one hundred millisecond. Since these delay time are in the same range as time constants of synaptic responses, effects of finite transmission delays on the spatial temporal evolution of activity take place. Although one may estimate such mean transmission delay time along axonal fibers, physiological results suggest that the transmission speed depends on the specific path the action potential takes and hence changes in a single axonal branching structure from one neuron to another. In addition, the branching structure shows plasticity effects [65] and hence changes the transmission speed. Consequently, the neuronal network does not exhibit a single transmission speed but rather a distribution of speeds. The proposed models are equations involving such a distribution of transmission speeds and extends previous studies assuming a single speed only.

The model equations under consideration involves two kinds of delays. In intracortical couplings, the lengths of axon may change due to the absence of nerve fibers of fixed length producing a transmission delay proportional to the distance between two spatial positions. In contrast, cortical to cortical feedback couplings may exhibit nerve fibers with fixed length producing a constant feedback delay [61]. By virtue of the distribution of transmission speeds, these two pathways are modeled with a distribution of transmission speeds and distributed feedback delays. Most recent research papers [17, 36, 49, 77, 79] in synaptically coupled neuronal networks considered either transmission speeds in intra-area couplings or delay in feedback couplings, although experimental observations suggest the presence of both couplings, a distribution of transmission speeds and feedback delays. Only very few previous research groups [6, 47, 78, 80] considered both a single transmission delay in intra-area couplings and the feedback delay. The proposed equations extends these previous model equations by a rigorous mathematical analysis of traveling wave solutions in the presence of both distributed and delayed feedback interactions to gain insights into the existence and stability of traveling wave solutions and their wave speeds.

The model equations under consideration describes the propagation of traveling wave solutions in synaptically coupled neuronal networks on a mesoscopic spatial and temporal scale with typical spatial range of five hundred millimeters and temporal time constants of five to ten milliseconds. The corresponding spatial domain is coarse-grained and exhibits spatial patches which are motivated by the macrocolumnar structure in primary sensory areas [31, 33]. Such neural field model equations allow the successful reproduction of electroencephalographic activity on the head [35] and the successful description of spiral waves in neural tissue [32, 44].

1.1. The Mathematical Model Equations

Consider the following nonlinear scalar integral differential equations arising from synaptically coupled neuronal networks ([16, 21, 24, 69, 70, 78, 80])

$$\begin{aligned}\frac{\partial u}{\partial t} + u = \alpha \int_0^{\infty} \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ + \beta \int_0^{\infty} \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \Theta) \mathrm{d}y \right] \mathrm{d}\tau, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned}\frac{\partial u}{\partial t} + u = \alpha \int_0^{\infty} \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ + \beta \int_0^{\infty} \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \Theta) \mathrm{d}y \right] \mathrm{d}\tau - w_0. \end{aligned} \tag{1.2}$$

These equations may be obtained by setting $\varepsilon = 0$, $w = 0$ and $\varepsilon = 0$, $w = w_0$ respectively, in the following nonlinear singularly perturbed system of integral differential equations

$$\frac{\partial u}{\partial t} + u + w = \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c$$

$$+ \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \Theta) \mathrm{d}y \right] \mathrm{d}\tau, \tag{1.3}$$

$$\frac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \tag{1.4}$$

These model equations generalize many important nonlinear scalar integral differential equations arising from synaptically coupled neuronal networks. In these equations, $u = u(x,t)$ stands for the membrane potential of a neuron at position x and time t and $w = w(x,t)$ represents a leaking current, a slow process that controls the excitation of the neuronal network. The kernel functions K and W stand for synaptic couplings between neurons in synaptically coupled neuronal networks. The synaptic couplings can be very general, including not only pure excitations (modeled with nonnegative kernel functions), lateral inhibitions (modeled with Mexican hat kernel functions), lateral excitations (modeled with upside down Mexican hat kernel functions), but also synaptic couplings which may change sign for finitely many times or even infinitely many times. The gain function is given by the Heaviside step function $H = H(u-\theta)$, which is defined by $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = \frac{1}{2}$ and $H(u-\theta) = 1$ for all $u > \theta$. The probability density functions ξ and η are defined on $(0, \infty)$. The function ξ represents a biological distribution of action potential speeds and ξ may have a compact support $[c_1, c_2]$, where $c_1 > 0$ and $c_2 > 0$ are positive constants. The function η represents a biological distribution of constant delays and may also have a compact support $[\tau_1, \tau_2]$, where $\tau_1 \geq 0$ and $\tau_2 > 0$ are constants. The parameter $c > 0$ represents the finite propagation speed of an action potential along the axon and $\frac{1}{c}|x-y|$ denotes the spatial temporal delay. The parameter $\tau > 0$ represents a constant delay. Moreover, the positive constants $\alpha > 0$ and $\beta > 0$ represent synaptic rates, and the positive constants $\theta > 0$ and $\Theta > 0$ represent thresholds for excitation of neurons in synaptically coupled neuronal networks. The integrals represent nonlocal spatial temporal interactions between neurons. The derivative $\frac{\partial u}{\partial t}$ represents longitudinal current along an axon. Sodium channels are voltage gated channels. In other words, sodium conductance is a function of the membrane potential. The sodium current is derived by using Ohm's law and should be a nonlinear smooth function of u, just like the nonlinear function $f(u) = u(u-1)(u-a)$ in the Fitzhugh-Nagumo equations [41] or in the Hodgkin-Huxley equations [30]. The linear function $f(u) = u$ stands for a good approximation of the sodium current, where $m > 0$ is a positive constant and n is a real constant. We may interpret the constant m as the sodium conductance and the constant n as the sodium reversal potential. The model equation may be derived by using Kirchhoff's second law for closed circuit. For more neurobiological backgrounds of the nonlinear scalar integral differential equations (1.1) and (1.2), please see [7, 20, 24, 29, 78, 80]. The positive constant $w_0 > 0$ is to be given.

1.2. The Main Goals

In this paper, we will accomplish the existence and stability of traveling wave solutions of the nonlinear scalar integral differential equations (1.1) and (1.2). We will couple together speed index functions, stability index functions (often called

Evans functions, that is, complex analytic functions), implicit function theorem, intermediate value theorem, mean value theorem, global strong maximum principle for stability index functions, linearized stability criterion and many other important techniques in dynamical systems to accomplish the existence and stability of the traveling wave solutions of the nonlinear scalar integral differential equations (1.1) and (1.2).

We will generalize previous results on the existence and stability of traveling wave solutions of nonlinear scalar integral differential equations involving one convolution product with simple synaptic couplings (the synaptic coupling may change sign at most once, from positive to negative or from negative to positive) to nonlinear scalar integral differential equations involving two convolution products with very general synaptic couplings (the synaptic couplings may change signs from positive to negative for finitely many times or even infinitely many times, such as $K(x) = \frac{\rho^2+\omega^2}{2\rho}\exp(-\rho|x|)\cos(\omega x)$ and $W(x) = \frac{\rho^2+\omega^2}{2\omega}\exp(-\rho|x|)\sin(\omega|x|)$, where $\rho > 0$ and $\omega > 0$ are positive constants). The existence and stability of traveling wave solutions of the complicated cases are very important in mathematical neuroscience but they have been open for a long time. The main results (Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4) obtained in this paper are new and they greatly improve previous results. The paper shows that adding complexity to the nonlinear scalar integral differential equations to account for more accurate neurobiological mechanisms in synaptically coupled neuronal networks does not affect the existence and stability of the traveling wave solutions.

1.3. The Mathematical Assumptions

Suppose that the kernel functions K and W are at least piecewise smooth on $\mathbb{R}$. Suppose that the probability density functions $\xi \geq 0$ and $\eta \geq 0$ are either nonnegative finite at least piecewise smooth functions or linear combinations of Dirac delta impulse functions defined on $(0,\infty)$. Suppose that there exists a positive constant $c_0 > 0$, such that $\xi = 0$ on $(0, c_0)$ and $\xi \geq 0$ on (c_0,∞). Without loss of generality, let

$$c_0 = \sup\{c > 0 : \xi = 0 \text{ on } (0, c) \text{ and } \xi \geq 0 \text{ on } (c,\infty)\}.$$

Suppose that there exist two positive integers $M \geq 1$ and $N \geq 1$ and there exist two positive constants $C > 0$ and $\rho > 0$, such that the kernel functions (K,W), the probability density functions (ξ, η), the synaptic rate constants (α, β) and the thresholds (θ, Θ) satisfy the following conditions

$$|K(x)| + |W(x)| \leq C \exp(-\rho|x|), \text{ on } \mathbb{R}, \tag{1.5}$$

$$\int_{-\infty}^{0} |x|K(x)\mathrm{d}x \geq 0, \qquad \int_{-\infty}^{0} |x|W(x)\mathrm{d}x \geq 0, \tag{1.6}$$

$$\int_{z}^{0}\left\{\int_{y_M}^{0}\left[\cdots\int_{y_3}^{0}\int_{y_2}^{0} K(y_1)\mathrm{d}y_1\mathrm{d}y_2\cdots\right]\mathrm{d}y_{M-1}\right\}\mathrm{d}y_M \geq 0, \text{ for } z < 0, \tag{1.7}$$

$$\int_{z}^{0}\left\{\int_{y_N}^{0}\left[\cdots\int_{y_3}^{0}\int_{y_2}^{0} W(y_1)\mathrm{d}y_1\mathrm{d}y_2\cdots\right]\mathrm{d}y_{N-1}\right\}\mathrm{d}y_N \geq 0, \text{ for } z < 0, \tag{1.8}$$

$$\int_{-\infty}^{0} \exp\left(\frac{x}{c_0}\right)\left[\alpha K(x) + \beta W(x)\right]\mathrm{d}x \geq \frac{\alpha+\beta}{2} - \theta, \tag{1.9}$$

$$\alpha K(0) + \beta W(0) \int_0^\infty (1+\tau)\eta(\tau)\mathrm{d}\tau > 0, \tag{1.10}$$

$$\int_{\mathbb{R}} K(x)\mathrm{d}x = 1, \qquad \int_{\mathbb{R}} W(x)\mathrm{d}x = 1, \tag{1.11}$$

$$\int_{-\infty}^0 K(x)\mathrm{d}x = \frac{1}{2}, \qquad \int_{-\infty}^0 W(x)\mathrm{d}x = \frac{1}{2}, \tag{1.12}$$

$$\int_0^\infty \xi(c)\mathrm{d}c = 1, \qquad \int_0^\infty \eta(\tau)\mathrm{d}\tau = 1, \tag{1.13}$$

$$\int_0^\infty \frac{1}{c}\xi(c)\mathrm{d}c < \infty, \qquad \int_0^\infty \eta(\tau)e^\tau\mathrm{d}\tau < \infty, \tag{1.14}$$

$$\text{either } 0 < 2\theta < \alpha < \Theta < \frac{1}{2}(\alpha+\beta), 0 < \alpha(\Theta-\alpha) < \beta\theta, \tag{1.15}$$

$$\text{or} \qquad \theta = \Theta, \qquad 0 < 2\theta < \frac{\alpha+\beta}{2}. \tag{1.16}$$

The kernel functions in the following three classes satisfy these conditions.

(A) The first class consists of all nonnegative kernel functions (representing pure excitations in neuronal networks). For examples, $K(x) = \frac{\rho}{2}\exp(-\rho|x|)$ and $W(x) = \sqrt{\frac{\rho}{\pi}}\exp(-\rho|x|^2)$ may represent pure excitations, where $\rho > 0$ is a constant. Here, ρ has a biological meaning. It indicates how the excitation of a synaptic coupling is distributed. If ρ is large, then a neuron is strongly coupled with neurons in a relatively small region; if ρ is small, then a neuron is strongly coupled with all neurons in a relatively large region.

(B) The second class consists of all Mexican-hat kernel functions (representing lateral inhibitions in neuronal networks). Each kernel function satisfies the conditions $K \geq 0$ on $(-M, M)$ and $K \leq 0$ on $(-\infty, -M) \cup (M, \infty)$, for a positive constant $M > 0$. For example, $K(x) = A\exp(-a|x|^2) - B\exp(-b|x|^2)$ may represent a lateral inhibition, where $A > B > 0$ and $a > b > 0$ are positive constants, such that

$$\frac{A}{a} \geq \frac{B}{b}, \qquad A\sqrt{\frac{\pi}{a}} - B\sqrt{\frac{\pi}{b}} = 1.$$

(C) The third class consists of all upside down Mexican-hat kernel functions (representing lateral excitations in neuronal networks). Each kernel function satisfies the conditions $K \leq 0$ on $(-M, M)$ and $K \geq 0$ on $(-\infty, -M) \cup (M, \infty)$, for a positive constant $M > 0$. For example, $K(x) = A\exp(-a|x|) - B\exp(-b|x|)$ may represent a lateral excitation, where $0 < A < B$ and $0 < a < b$ are positive constants, such that

$$\frac{A}{a^2} \geq \frac{B}{b^2}, \qquad \frac{A}{a} - \frac{B}{b} = \frac{1}{2}.$$

1.4. The Main Results

To make the statements of the main results and the mathematical analysis as simple as possible, we define the sign function $s = s(x)$ by: $s(x) = -1$ for all $x < 0$, $s(0) = 0$ and $s(x) = 1$ for all $x > 0$.

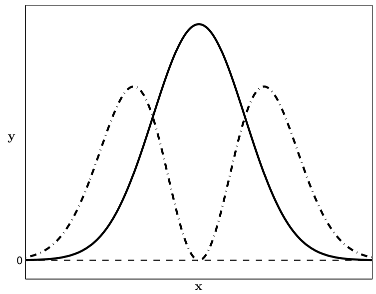

Figure 1. The graph of two nonnegative synaptic couplings. The solid curve is an on-center kernel function and the dashed one is an off-center kernel function.

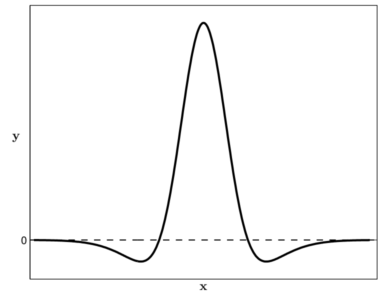

Figure 2. The graph of a lateral inhibition synaptic coupling (that is, a Mexican hat kernel function).

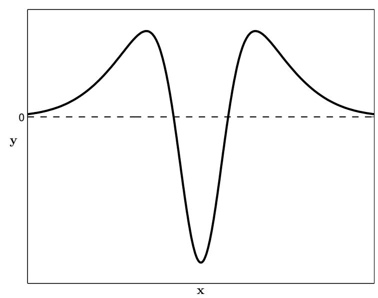

Figure 3. The graph of a lateral excitation synaptic coupling (that is, an upside down Mexican hat kernel function).

Theorem 1.1. *(Existence of the traveling wave fronts) There exist exactly three traveling wave fronts to the nonlinear scalar integral differential equation (1.1). The first two traveling wave fronts are called small fronts because they cross one threshold only. The third traveling wave front is called a large front because it crosses both thresholds.*

(I) *The first small traveling wave front* $u(x,t) = U_{\text{front}-1}(z)$ *is given by*

$$\begin{aligned} U(z) &= \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_1)} K(x)\mathrm{d}x \right] \mathrm{d}c \\ &\quad - \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu_1}\right) \frac{c}{c+s(x)\mu_1} K\left(\frac{cx}{c+s(x)\mu_1}\right) \mathrm{d}x \right] \mathrm{d}c, \\ U'(z) &= \frac{\alpha}{\mu_1} \int_0^\infty \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu_1}\right) \frac{c}{c+s(x)\mu_1} K\left(\frac{cx}{c+s(x)\mu_1}\right) \mathrm{d}x \right] \mathrm{d}c, \\ &\lim_{z\to-\infty} U(z) = 0, \qquad \lim_{z\to\infty} U(z) = \alpha, \qquad \lim_{z\to\pm\infty} U'(z) = 0, \end{aligned}$$

where the first wave speed $\mu_1 = \mu_1(\alpha, \xi, K, \theta)$ *is the unique solution of the first* *speed equation*

$$\alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c = \frac{\alpha}{2} - \theta.$$

(II) *The second small traveling wave front* $u(x,t) = U_{\text{front}-2}(z)$ *is given by*

$$\begin{aligned} U(z) &= \alpha + \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu_2\tau} W(x) \mathrm{d}x \right] \mathrm{d}\tau \\ &\quad - \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{z-\mu_2\tau} \exp\left(\frac{x-z}{\mu_2} \right) W(x) \mathrm{d}x \right] \mathrm{d}\tau, \\ U'(z) &= \frac{\beta}{\mu_2} \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{z-\mu_2\tau} \exp\left(\frac{x-z}{\mu_2} \right) W(x) \mathrm{d}x \right] \mathrm{d}\tau, \\ \lim_{z\to-\infty} U(z) &= \alpha, \qquad \lim_{z\to\infty} U(z) = \alpha+\beta, \qquad \lim_{z\to\pm\infty} U'(z) = 0, \end{aligned}$$

where the second wave speed $\mu_2 = \mu_2(\beta, \eta, W, \Theta)$ *is the unique solution of the second speed equation*

$$\begin{aligned} &\beta \int_0^\infty \eta(\tau) \left[\int_{-\mu\tau}^0 W(x) \mathrm{d}x \right] \mathrm{d}\tau + \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu\tau} \exp\left(\frac{x}{\mu} \right) W(x) \mathrm{d}x \right] \mathrm{d}\tau \\ =&\alpha + \frac{\beta}{2} - \Theta. \end{aligned}$$

(III) *The large traveling wave front* $u(x,t) = U_{\text{front}-3}(z)$ *is given by*

$$\begin{aligned} U(z) &= \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_3)} K(x) \mathrm{d}x \right] \mathrm{d}c \\ &\quad - \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu_3} \right) \frac{c}{c+s(x)\mu_3} K\left(\frac{cx}{c+s(x)\mu_3} \right) \mathrm{d}x \right] \mathrm{d}c \\ &\quad + \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu_3\tau - Z_3} W(x) \mathrm{d}x \right] \mathrm{d}\tau \\ &\quad - \beta \exp\left(\frac{Z_3}{\mu_3} \right) \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{z-\mu_3\tau-Z_3} \exp\left(\frac{x-z}{\mu_3} \right) W(x) \mathrm{d}x \right] \mathrm{d}\tau, \\ U'(z) &= \frac{\alpha}{\mu_3} \int_0^\infty \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu_3} \right) \frac{c}{c+s(x)\mu_3} K\left(\frac{cx}{c+s(x)\mu_3} \right) \mathrm{d}x \right] \mathrm{d}c \\ &\quad + \frac{\beta}{\mu_3} \exp\left(\frac{Z_3}{\mu_3} \right) \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{z-\mu_3\tau-Z_3} \exp\left(\frac{x-z}{\mu_3} \right) W(x) \mathrm{d}x \right] \mathrm{d}\tau, \\ \lim_{z\to-\infty} U(z) &= 0, \qquad \lim_{z\to\infty} U(z) = \alpha+\beta, \qquad \lim_{z\to\pm\infty} U'(z) = 0, \end{aligned}$$

where, if $\theta = \Theta$, *then* $Z_3 = 0$ *and the third wave speed* $\mu_3 = \mu_3(\alpha, \beta, \xi, \eta, K, W, \theta)$ *is the unique solution of the third speed equation*

$$\begin{aligned} &\alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c + \beta \int_0^\infty \eta(\tau) \left[\int_{-\mu\tau}^0 W(x) \mathrm{d}x \right] \mathrm{d}\tau \\ &+ \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu\tau} \exp\left(\frac{x}{\mu} \right) W(x) \mathrm{d}x \right] \mathrm{d}\tau = \frac{\alpha+\beta}{2} - \theta. \end{aligned}$$

If $0<\theta<\Theta$*, then* $(\mu_3, Z_3) = \big(\mu_3(\alpha,\beta,\xi,\eta,K,W,\theta,\Theta), Z_3(\alpha,\beta,\xi,\eta,K,W,\theta,\Theta)\big)$ *is the unique solution of the system of speed equations*

$$\begin{aligned}
&\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu\tau-Z}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&+\beta\exp\left(\frac{Z}{\mu}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu\tau-Z}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau = \frac{\alpha+\beta}{2}-\theta,
\end{aligned}$$

and

$$\begin{aligned}
&\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{x}{\mu}\right)\frac{c}{c+s(x+Z)\mu}K\left(\frac{c(x+Z)}{c+s(x+Z)\mu}\right)\mathrm{d}x\right]\mathrm{d}c\\
&-\alpha\int_0^\infty \xi(c)\left[\int_0^{cZ/(c+s(Z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c+\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&+\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu\tau}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau = \frac{\alpha+\beta}{2}-\Theta.
\end{aligned}$$

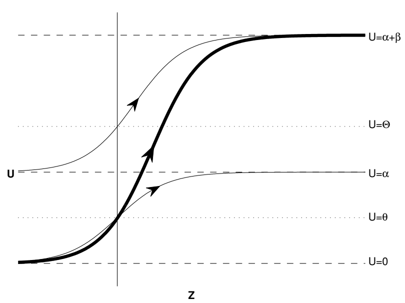

Figure 4. The graphs of the three traveling wave fronts of equation (1.1).

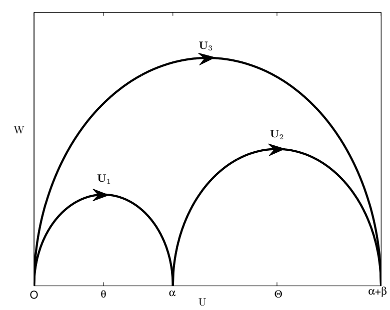

Figure 5. The phase plane portrait of the three traveling wave fronts of equation (1.1).

Theorem 1.2. *(Existence of the traveling wave backs) There exist exactly three traveling wave backs* $u(x,t) = U_{\text{back}-1}(z)$*,* $u(x,t) = U_{\text{back}-2}(z)$ *and* $u(x,t) = U_{\text{back}-3}(z)$ *to the nonlinear scalar integral differential equation (1.2) with the same speeds as the corresponding traveling wave fronts* $u(x,t) = U_{\text{front}-1}(z)$*,* $u(x,t) = U_{\text{front}-2}(z)$ *and* $u(x,t) = U_{\text{front}-3}(z)$*, respectively. Moreover*

$$\begin{aligned}
&U_{\text{back}-1}(z) = 2\theta - U_{\text{front}-1}(z),\\
&U_{\text{back}-2}(z) = 2\Theta - U_{\text{front}-2}(z),\\
&U_{\text{back}-3}(z) = 2\theta - U_{\text{front}-3}(z), \qquad \text{if } \theta=\Theta,\\
&U_{\text{back}-3}(z) = -W_3+\alpha\int_0^\infty \xi(c)\left[\int_{cz/(c+s(z)\mu_3)}^\infty K(x)\mathrm{d}x\right]\mathrm{d}c
\end{aligned}$$

$$+\beta\int_0^\infty \eta(\tau)\left[\int_{z+Z_3-\mu_3\tau}^\infty \exp\left(\frac{x-z}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau$$
$$+\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^z \exp\left(\frac{x-z}{\mu_3}\right)\frac{c}{c+s(x)\mu_3}K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c$$
$$+\beta\exp\left(\frac{Z_3}{\mu_3}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{z+Z_3-\mu_3\tau}\exp\left(\frac{x-z}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau,\ \textit{if } \theta<\Theta,$$

where $W_3 > 0$ and $Z_3 > 0$ are positive constants. These traveling wave backs cross their thresholds in the following way.

(I) *The first small traveling wave back $U = U_{\text{back}-1}(z)$ crosses the small threshold θ exactly once and it does not cross the large threshold Θ.*

(II) *The second small traveling wave back $U = U_{\text{back}-2}(z)$ crosses the large threshold Θ exactly once and it does not cross the small threshold θ.*

(III) *The large traveling wave back $U = U_{\text{back}-3}(z)$ crosses the small threshold θ exactly once and it crosses the large threshold Θ exactly once.*

Theorem 1.3. *(The exponential stability of the traveling wave fronts) Let $k = 1, 2, 3$. Consider the following Cauchy problem for the nonlinear scalar integral differential equation*

$$\frac{\partial P}{\partial t}+\mu_k\frac{\partial P}{\partial z}+P$$
$$=\alpha\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(z-y)H\left(P\left(y-\frac{\mu_k}{c}|z-y|, t-\frac{1}{c}|z-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c$$
$$+\beta\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-y)H(P(y-\mu_k\tau, t-\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau, \tag{1.17}$$
$$P(z,0)=P_0(z). \tag{1.18}$$

Let $U_k = U_{\text{front}-\text{k}}(z)$ represent the traveling wave front of the nonlinear scalar integral differential equation (1.1). There exist three positive constants $C_k > 0$, $M_k > 0$ and $\rho_k > 0$, such that if the initial function P_0 satisfies the condition $\|P_0 - U_k\|_{L^\infty(\mathbb{R})} < C_k$, then the global solution of (1.17)-(1.18) enjoys the decay estimate

$$\|P(\cdot,t)-U_k(\cdot+h_k)\|_{L^\infty(\mathbb{R})}\le M_k\exp(-\rho_k t)\|P_0-U_k\|_{L^\infty(\mathbb{R})},$$

for all $t > 0$, where $h_k \neq 0$ is a real time-independent constant, satisfying the estimate $|h_k| \le M_k\|P_0 - U_k\|_{L^\infty(\mathbb{R})}$. In another word, t The traveling wave fronts of the nonlinear scalar integral differential equation (1.1) are exponentially stable.

Theorem 1.4. *(The exponential stability of the traveling wave backs) Let $k = 1, 2, 3$. Consider the following Cauchy problem for the nonlinear scalar integral differential equation*

$$\frac{\partial P}{\partial t}+\mu_k\frac{\partial P}{\partial z}+P$$
$$=\alpha\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(z-y)H\left(P\left(y-\frac{\mu_k}{c}|z-y|, t-\frac{1}{c}|z-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c$$
$$+\beta\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-y)H(P(y-\mu_k\tau, t-\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau-w_k, \tag{1.19}$$

$$P(z,0) = P_0(z). \tag{1.20}$$

Let $U_k = U_{\text{back}-\text{k}}(z)$ represent the traveling wave back of the nonlinear scalar integral differential equation (1.2). There exist three positive constants $C_k > 0$, $M_k > 0$ and $\rho_k > 0$, such that if the initial function P_0 satisfies the condition $\|P_0 - U_k\|_{L^\infty(\mathbb{R})} < C_k$, then the global solution of (1.19)-(1.20) enjoys the decay estimate

$$\|P(\cdot,t) - U_k(\cdot + h_k)\|_{L^\infty(\mathbb{R})} \le M_k \exp(-\rho_k t)\|P_0 - U_k\|_{L^\infty(\mathbb{R})},$$

for all $t > 0$, where $h_k \neq 0$ is a real time-independent constant, satisfying the estimate $|h_k| \le M_k \|P_0 - U_k\|_{L^\infty(\mathbb{R})}$. In another word, the traveling wave backs of the nonlinear scalar integral differential equation (1.2) are exponentially stable.

These results will be rigorously proved in the next two sections.

Remark 1.1. The conditions $0 < 2\theta < \alpha < \Theta < \frac{1}{2}(\alpha + \beta)$ guarantee that the existence and stability of each of the first two small traveling wave fronts of (1.1) as well as the existence and stability of each of the first two small traveling wave backs of (1.2) are true; and the conditions $0 < \alpha(\Theta - \alpha) < \beta\theta$ guarantee that the existence and stability of the large traveling wave front of (1.1) and the existence and stability of the large traveling wave back of (1.2) are true.

The study of the existence and stability of the traveling wave fronts of (1.1) and the traveling wave backs of (1.2) are the preparation for the study of the existence and stability of traveling pulse solutions of the nonlinear singularly perturbed system of integral differential equations (1.3)-(1.4).

1.5. The Existence of Wave Speeds and the Speed Index Functions

Wave speeds play a very important role in the study of traveling wave solutions of nonlinear scalar integral differential equations. Once a wave speed is found, the traveling wave solution is easy to obtain by using usual techniques in ordinary differential equations. Moreover, wave speeds may be closely related to the stability of traveling wave solutions. Intuitively, stable traveling wave solutions are the most important solutions. Let us motivate the definition of a speed index function by using a very simple model in synaptically coupled neuronal networks.

Consider the following nonlinear scalar integral differential equation

$$\frac{\partial u}{\partial t} + u = \alpha \int_{\mathbb{R}} K(x-y) H(u(y,t) - \theta) \mathrm{d}y, \tag{1.21}$$

where $\alpha > 0$ and $\theta > 0$ are positive constants, such that $0 < 2\theta < \alpha$. This model equation may be viewed as a simplified version of (1.1), by setting $\beta = 0$ and $\xi(c) = \delta(c - \infty)$. Suppose that $u(x,t) = U(x + \nu t)$ is a traveling wave front of (1.21), satisfying the conditions $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$, $U'(0) > 0$ and $U > \theta$ on $(0, \infty)$, where $z = x + \nu t$ stands for the moving coordinate. Then the traveling wave equation becomes a linear ordinary differential equation

$$\nu U' + U = \alpha \int_{\mathbb{R}} K(z-y) H(U(y) - \theta) \mathrm{d}y = \alpha \int_{-\infty}^{z} K(x) \mathrm{d}x.$$

There exists a unique traveling wave front $U = U(x + \nu_0 t)$ to this equation:

$$U(z) = \alpha \int_{-\infty}^{z} K(x)\mathrm{d}x - \alpha \int_{-\infty}^{z} \exp\left(\frac{x-z}{\nu_0}\right) K(x)\mathrm{d}x,$$
$$U'(z) = \frac{\alpha}{\nu_0} \int_{-\infty}^{z} \exp\left(\frac{x-z}{\nu_0}\right) K(x)\mathrm{d}x,$$

where ν_0 represents the wave speed, which is the unique positive solution of the speed equation $U(0) = \theta$, that is

$$\alpha \int_{-\infty}^{0} \exp\left(\frac{x}{\nu}\right) K(x)\mathrm{d}x = \frac{\alpha}{2} - \theta.$$

The speed index function for (1.21) is defined by

$$\phi(\nu) = \alpha \int_{-\infty}^{0} \exp\left(\frac{x}{\nu}\right) K(x)\mathrm{d}x, \tag{1.22}$$

for all $\nu > 0$.

Following this idea, we have developed a general method to construct the speed index functions for nonlinear scalar integral differential equations in [77–80]. For equation (1.1), we will construct several speed index functions. By using these speed index functions, we will establish the existence and uniqueness of a wave speed. The speed index functions are very interesting and important for the following reasons. There exists a unique solution to the speed equation $\phi(\nu) = \frac{\alpha+\beta}{2} - \theta$ or a system of speed equations involving the speed index functions and the intrinsic functions (K, W) and (ξ, η) and the positive constants (α, β) and (θ, Θ). This unique solution is precisely the wave speed of a traveling wave front. Through the speed index functions, we will be able to investigate how the wave speeds depend on the synaptic couplings (K, W), the probability density functions (ξ, η), the synaptic rate constants (α, β) and the thresholds (θ, Θ). Many important results such as the monotonicity of the wave speeds and the asymptotic behaviors of the wave speeds as the constants approach certain critical numbers can be investigated very clearly. More appropriately, the speed index functions should be called neurobiological mechanism index functions because they involve very important neurobiological mechanisms, such as (K, W), (ξ, η), (α, β) and (θ, Θ). By using the properties of the speed index functions, we are able to prove the simple but elegant identity: $\frac{1}{\mu} = \frac{1}{c} + \frac{1}{\nu}$ in [77], which connects the wave speed μ of the traveling wave front of the nonlinear scalar integral differential equation $u_t + u = \alpha \int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) \mathrm{d}y$ (where there exists a spatial temporal delay) to the wave speed ν of the traveling wave front of the nonlinear scalar integral differential equation $u_t + u = \alpha \int_{\mathbb{R}} K(x-y) H(u(y,t) - \theta)\mathrm{d}y$ (where there exists no delay).

1.6. The Stability and the Stability Index Functions

To establish the exponential stability of a traveling wave solution, very often we have to study the eigenvalues and eigenfunctions of an associated eigenvalue problem. It turns out that a complex number λ_0 is an eigenvalue if and only if λ_0 is a zero of a stability index function. The stability index functions, also called Evans functions, are complex analytic functions defined in some right half complex plane Ω, usually the left boundary is a vertical straight line located to the left of the imaginary axis. For the nonlinear scalar integral differential equation (1.21) , the stability index function is defined by [77]

$$\mathcal{E}(\lambda) = 1 - \left[\int_{-\infty}^{0} \exp\left(\frac{\lambda+1}{\nu_0}\right) K(x)\mathrm{d}x\right] / \left[\int_{-\infty}^{0} \exp\left(\frac{x}{\nu_0}\right) K(x)\mathrm{d}x\right]. \qquad (1.23)$$

There exists a very important relationship between the speed index function and the stability index function:

$$\mathcal{E}(\lambda) = 1 - \frac{1}{\phi(\nu_0)}\phi\left(\frac{\nu_0}{\lambda+1}\right). \qquad (1.24)$$

This relationship implies that there exists a close relationship between the existence and the stability of the traveling wave solution.

The speed index functions and the stability index functions introduced in this paper are very interesting and important concepts in mathematical neuroscience. They have potential applications and impacts in applied mathematics. With the introduction of the speed index functions and the stability index functions, we can do much more mathematical analysis on the traveling wave fronts than before. One interesting point is that we can build valuable relationships between the stability index functions and the speed index functions. By using these relationships and by proving a global strong maximum principle for the stability index functions (the Evans functions, that is, the complex analytic functions), the stability of the traveling wave solutions can be analyzed easily. The speed index functions and the stability index functions may play very important roles in rigorous mathematical analysis of traveling pulse solutions of nonlinear singularly perturbed systems of integral differential equations. See [8, 11, 16, 21, 27, 28, 44, 49, 78] for such systems. Moreover, the rigorous mathematical analysis and the results on the wave speeds, the speed index functions and the stability index functions can be applied to mathematical/computational neuroscience.

1.7. Related Results

In [39], Hutt and Zhang presented the existence and stability result of a traveling wave front of a simpler nonlinear scalar integral differential equation (it is a simpler equation because $\theta = \Theta$) with minimum amount of rigorous mathematical analysis on the existence and stability of the traveling wave front. Many results of that paper depend heavily on the rigorous mathematical analysis of this paper.

To keep the Introduction from too long, we place other related results in the Appendix.

2. The Existence Analysis

In this section, we will accomplish the existence of the traveling wave solutions of the nonlinear scalar integral differential equations (1.1) and (1.2). We will construct speed index functions and couple together implicit function theorem, intermediate value theorem, mean value theorem and many important techniques in dynamical systems to establish the existence. We will focus on the rigorous mathematical proof of the existence of the large traveling wave front of (1.1) and the large traveling wave back of (1.2). The existence of the first two small traveling wave fronts of (1.1) as well as the existence of the first two small traveling wave backs of (1.2) may be proved very similarly.

2.1. The Formal Representation of the Traveling Wave Solutions

The main purpose of this subsection is to derive the formal representation of the traveling wave fronts of equation (1.1). The main strategy is to make use of the shape of the fronts and a series of change of variables to reduce the nonlinear scalar integral differential equation to a first order nonhomogeneous linear differential equation and then use integrating factor idea to solve it.

There exist three constant solutions $U_0 = 0$, $U_1 = \alpha$ and $U_2 = \alpha+\beta$ to equation (1.1) if $0 < \theta < \alpha < \Theta < \alpha + \beta$; and there exist two constant solutions $U_0 = 0$ and $U_2 = \alpha + \beta$ if $\theta = \Theta$. Suppose that $u(x,t) = U(x + \mu t)$ is a traveling wave front of the nonlinear scalar integral differential equation (1.1), where $\mu > 0$ represents the wave speed and $z = x + \mu t$ represents a moving coordinate. Then the traveling wave front $u(x,t) = U(x + \mu t)$ and its wave speed μ satisfy

$$\begin{aligned}\mu U' + U = \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu}{c}|z-y|\right) - \theta\right) \mathrm{d}y\right] \mathrm{d}c \\ + \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z-y) H(U(y-\mu\tau) - \Theta) \mathrm{d}y\right] \mathrm{d}\tau.\end{aligned}$$

Let $0 < \mu < c$ and let

$$\omega = y - \frac{\mu}{c}|z-y|.$$

Then

$$\begin{aligned}z - \omega =& z - y + \frac{\mu}{c}|z-y| \\ =& (z-y)\left[1 + \frac{\mu}{c} s(z-y)\right] \\ =& (z-y)\left[1 + \frac{\mu}{c} s(z-\omega)\right].\end{aligned}$$

Therefore

$$z - y = \frac{c}{c + s(z-\omega)\mu}(z-\omega),$$

and

$$\mathrm{d}y = \frac{c}{c + s(z-\omega)\mu}\mathrm{d}\omega - \frac{c\mu}{[c + s(z-\omega)\mu]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.$$

Then the traveling wave front $U = U(z)$ and its wave speed μ satisfy

$$\begin{aligned}\mu U' + U = \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} \frac{c}{c + s(z-\omega)\mu} K\left(\frac{c(z-\omega)}{c + s(z-\omega)\mu}\right) H(U(\omega) - \theta)\mathrm{d}\omega\right] \mathrm{d}c \\ + \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z - \mu\tau - y) H(U(y) - \Theta)\mathrm{d}y\right] \mathrm{d}\tau.\end{aligned}$$

Suppose that the large traveling wave front satisfies the conditions $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$, $U'(0) > 0$ and $U > \theta$ on $(0, \infty)$. Similarly, suppose that $U < \Theta$ on $(-\infty, Z)$, $U(Z) = \Theta$, $U'(Z) > 0$ and $U > \Theta$ on (Z, ∞), for some non-negative constant $Z \geq 0$, to be determined later. Intuitively, $Z = 0$ if $\theta = \Theta$ and $Z > 0$ if $\theta < \Theta$. This assumption is made based on the translation invariance of the traveling wave front. Then the traveling wave equation becomes

$$\begin{aligned}\mu U' + U = {} & \alpha \int_0^\infty \xi(c) \left[\int_0^\infty \frac{c}{c + s(z-\omega)\mu} K\left(\frac{c(z-\omega)}{c + s(z-\omega)\mu} \right) \mathrm{d}\omega \right] \mathrm{d}c \\ & + \beta \int_0^\infty \eta(\tau) \left[\int_Z^\infty W(z - \mu\tau - y) \mathrm{d}y \right] \mathrm{d}\tau.\end{aligned}$$

Let

$$x = \frac{c}{c + s(z-\omega)\mu}(z - \omega).$$

Then

$$\mathrm{d}x = -\frac{c}{c + s(z-\omega)\mu}\mathrm{d}\omega + \frac{c\mu}{[c + s(z-\omega)\mu]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.$$

Therefore, the traveling wave equation for the large front becomes

$$\begin{aligned}\mu U' + U = {} & \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x \right] \mathrm{d}c \\ & + \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu\tau-Z} W(x)\mathrm{d}x \right] \mathrm{d}\tau.\end{aligned}$$

This is a first order nonhomogeneous linear differential equation. Solving this equation by using integrating factor idea, we obtain the solution

$$\begin{aligned}U(z) = {} & \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x \right] \mathrm{d}c \\ & - \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu} \right) \frac{c}{c+s(x)\mu} K\left(\frac{cx}{c+s(x)\mu} \right) \mathrm{d}x \right] \mathrm{d}c \\ & + \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu\tau-Z} W(x)\mathrm{d}x \right] \mathrm{d}\tau \\ & - \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu} \right) W(x - \mu\tau - Z)\mathrm{d}x \right] \mathrm{d}\tau, \\ U'(z) = {} & \frac{\alpha}{\mu} \int_0^\infty \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu} \right) \frac{c}{c+s(x)\mu} K\left(\frac{cx}{c+s(x)\mu} \right) \mathrm{d}x \right] \mathrm{d}c \\ & + \frac{\beta}{\mu} \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu} \right) W(x - \mu\tau - Z)\mathrm{d}x \right] \mathrm{d}\tau.\end{aligned}$$

There hold the following limits

$$\lim_{z\to-\infty} U(z) = 0, \qquad \lim_{z\to\infty} U(z) = \alpha + \beta, \qquad \lim_{z\to\pm\infty} U'(z) = 0.$$

Making a simple change of variable, we obtain the formal representation of the large traveling wave front

$$
\begin{aligned}
U(z) = {} & \alpha \int_0^{\infty} \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu)} K(x)\mathrm{d}x \right] \mathrm{d}c \\
& - \alpha \int_0^{\infty} \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu}\right) \frac{c}{c+s(x)\mu} K\left(\frac{cx}{c+s(x)\mu}\right) \mathrm{d}x \right] \mathrm{d}c \\
& + \beta \int_0^{\infty} \eta(\tau) \left[\int_{-\infty}^{z-\mu\tau-Z} W(x)\mathrm{d}x \right] \mathrm{d}\tau \\
& - \beta \exp\left(\frac{Z}{\mu}\right) \int_0^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{z-\mu\tau-Z} \exp\left(\frac{x-z}{\mu}\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
U'(z) = {} & \frac{\alpha}{\mu} \int_0^{\infty} \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu}\right) \frac{c}{c+s(x)\mu} K\left(\frac{cx}{c+s(x)\mu}\right) \mathrm{d}x \right] \mathrm{d}c \\
& + \frac{\beta}{\mu} \exp\left(\frac{Z}{\mu}\right) \int_0^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{z-\mu\tau-Z} \exp\left(\frac{x-z}{\mu}\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
U''(z) = {} & \frac{\alpha}{\mu} \int_0^{\infty} \xi(c) \left[\frac{c}{c+s(z)\mu} K\left(\frac{cz}{c+s(z)\mu}\right) \right] \mathrm{d}c \\
& + \frac{\beta}{\mu} \int_0^{\infty} \eta(\tau) W(z-\mu\tau-Z)\mathrm{d}\tau \\
& - \frac{\alpha}{\mu^2} \int_0^{\infty} \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu}\right) \frac{c}{c+s(x)\mu} K\left(\frac{cx}{c+s(x)\mu}\right) \mathrm{d}x \right] \mathrm{d}c \\
& - \frac{\beta}{\mu^2} \exp\left(\frac{Z}{\mu}\right) \int_0^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{z-\mu\tau-Z} \exp\left(\frac{x-z}{\mu}\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau.
\end{aligned}
$$

We call the system of equations $U(0) = \theta$ and $U(Z) = \Theta$ the speed equations. We will derive the speed index functions by using this system of equations.

Let $U(0) = \theta$, that is

$$
\begin{aligned}
& \frac{\alpha}{2} - \alpha \int_0^{\infty} \xi(c) \left[\int_{-\infty}^{0} \exp\left(\frac{x}{\mu}\right) \frac{c}{c+s(x)\mu} K\left(\frac{cx}{c+s(x)\mu}\right) \mathrm{d}x \right] \mathrm{d}c \\
& + \beta \int_0^{\infty} \eta(\tau) \left[\int_{-\infty}^{-\mu\tau-Z} W(x)\mathrm{d}x \right] \mathrm{d}\tau \\
& - \beta \exp\left(\frac{Z}{\mu}\right) \int_0^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{-\mu\tau-Z} \exp\left(\frac{x}{\mu}\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau \\
= {} & \frac{\alpha+\beta}{2} - \alpha \int_0^{\infty} \xi(c) \left[\int_{-\infty}^{0} \exp\left(\frac{c-\mu}{c\mu} x\right) K(x)\mathrm{d}x \right] \mathrm{d}c
\end{aligned}
$$

$$
\begin{aligned}
&-\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu\tau-Z}^{0} W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&-\beta\exp\left(\frac{Z}{\mu}\right)\int_0^\infty \eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu\tau-Z}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau=\theta.
\end{aligned}
$$

Let $U(Z)=\Theta$, that is

$$
\begin{aligned}
&\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^{cZ/(c+s(Z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c\\
&-\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^{Z}\exp\left(\frac{x-Z}{\mu}\right)\frac{c}{c+s(x)\mu}K\left(\frac{cx}{c+s(x)\mu}\right)\mathrm{d}x\right]\mathrm{d}c\\
&+\beta\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{-\mu\tau} W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&-\beta\exp\left(\frac{Z}{\mu}\right)\int_0^\infty \eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu\tau}\exp\left(\frac{x-Z}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{\alpha+\beta}{2}+\alpha\int_0^\infty \xi(c)\left[\int_{0}^{cZ/(c+s(Z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c\\
&-\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^{0}\exp\left(\frac{x}{\mu}\right)\frac{c}{c+s(x+Z)\mu}K\left(\frac{c(x+Z)}{c+s(x+Z)\mu}\right)\mathrm{d}x\right]\mathrm{d}c\\
&-\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu\tau}^{0} W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&-\beta\int_0^\infty \eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu\tau}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau=\Theta.
\end{aligned}
$$

If $\theta=\Theta$, then $Z=0$ and the speed equation becomes

$$
\begin{aligned}
&\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^{0}\exp\left(\frac{c-\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu\tau}^{0} W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&+\beta\int_0^\infty \eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu\tau}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau=\frac{\alpha+\beta}{2}-\theta.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
U'(0)=&\frac{\alpha}{\mu}\int_0^\infty \xi(c)\left[\int_{-\infty}^{0}\exp\left(\frac{x}{\mu}\right)\frac{c}{c+s(x)\mu}K\left(\frac{cx}{c+s(x)\mu}\right)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu}\exp\left(\frac{Z}{\mu}\right)\int_0^\infty \eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu\tau-Z}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{1}{\mu}\left\{\frac{\alpha+\beta}{2}-\theta-\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu\tau-Z}^{0} W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\},
\end{aligned}
$$

$$
\begin{aligned}
U'(Z) =& \frac{\alpha}{\mu}\int_0^{\infty}\xi(c)\left[\int_{-\infty}^{Z}\exp\left(\frac{x-Z}{\mu}\right)\frac{c}{c+s(x)\mu}K\left(\frac{cx}{c+s(x)\mu}\right)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu}\exp\left(\frac{Z}{\mu}\right)\int_0^{\infty}\eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu\tau}\exp\left(\frac{x-Z}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{1}{\mu}\left\{\frac{\alpha+\beta}{2}-\Theta+\alpha\int_0^{\infty}\xi(c)\left[\int_0^{cZ/(c+s(Z)\mu)}K(x)\mathrm{d}x\right]\mathrm{d}c\right\}\\
&-\frac{\beta}{\mu}\int_0^{\infty}\eta(\tau)\left[\int_{-\mu\tau}^{0}W(x)\mathrm{d}x\right]\mathrm{d}\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
U''(0) =& \frac{\alpha}{\mu}\int_0^{\infty}\xi(c)K(0)\mathrm{d}c+\frac{\beta}{\mu}\int_0^{\infty}\eta(\tau)W(-\mu\tau-Z)\mathrm{d}\tau\\
&-\frac{\alpha}{\mu^2}\int_0^{\infty}\xi(c)\left[\int_{-\infty}^{0}\exp\left(\frac{x}{\mu}\right)\frac{c}{c+s(x)\mu}K\left(\frac{cx}{c+s(x)\mu}\right)\mathrm{d}x\right]\mathrm{d}c\\
&-\frac{\beta}{\mu^2}\exp\left(\frac{Z}{\mu}\right)\int_0^{\infty}\eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu\tau-Z}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{\alpha}{\mu}K(0)+\frac{\beta}{\mu}\int_0^{\infty}\eta(\tau)W(-\mu\tau-Z)\mathrm{d}\tau\\
&-\frac{1}{\mu^2}\left\{\frac{\alpha+\beta}{2}-\theta-\beta\int_0^{\infty}\eta(\tau)\left[\int_{-\mu\tau-Z}^{0}W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\},\\
U''(Z) =& \frac{\alpha}{\mu}\int_0^{\infty}\xi(c)\left[\frac{c}{c+s(Z)\mu}K\left(\frac{cZ}{c+s(Z)\mu}\right)\right]\mathrm{d}c+\frac{\beta}{\mu}\int_0^{\infty}\eta(\tau)W(-\mu\tau)\mathrm{d}\tau\\
&-\frac{\alpha}{\mu^2}\int_0^{\infty}\xi(c)\left[\int_{-\infty}^{Z}\exp\left(\frac{x-Z}{\mu}\right)\frac{c}{c+s(x)\mu}K\left(\frac{cx}{c+s(x)\mu}\right)\mathrm{d}x\right]\mathrm{d}c\\
&-\frac{\beta}{\mu^2}\exp\left(\frac{Z}{\mu}\right)\int_0^{\infty}\eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu\tau}\exp\left(\frac{x-Z}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{\alpha}{\mu}\int_0^{\infty}\xi(c)\left[\frac{c}{c+s(Z)\mu}K\left(\frac{cZ}{c+s(Z)\mu}\right)\right]\mathrm{d}c+\frac{\beta}{\mu}\int_0^{\infty}\eta(\tau)W(-\mu\tau)\mathrm{d}\tau\\
&-\frac{1}{\mu^2}\left\{\frac{\alpha+\beta}{2}-\Theta+\alpha\int_0^{\infty}\xi(c)\left[\int_0^{cZ/(c+s(Z)\mu)}K(x)\mathrm{d}x\right]\mathrm{d}c\right\}\\
&+\frac{\beta}{\mu^2}\int_0^{\infty}\eta(\tau)\left[\int_{-\mu\tau}^{0}W(x)\mathrm{d}x\right]\mathrm{d}\tau.
\end{aligned}
$$

2.2. The Speed Index Functions and the Speed Equations

The main purpose of this subsection is to define the speed index functions and to derive the speed equations.

Definition 2.1. Define the speed index functions $\phi_1 = \phi_1(\alpha, \xi, K, \mu)$, $\phi_2 = \phi_2(\beta, \eta, W, \mu)$, $\phi = \phi(\alpha, \beta, \xi, \eta, K, W, \mu)$, $\Phi_1 = \Phi_1(\alpha, \xi, K, \mu, Z)$ and $\Phi_2 = \Phi_2(\beta, \eta, W, \mu, Z)$

by

$$
\begin{aligned}
\phi_1 =& \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c, \\
\phi_2 =& \beta \int_0^\infty \eta(\tau) \left[\int_{-\mu\tau}^0 W(x) \mathrm{d}x \right] \mathrm{d}\tau \\
&+ \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu\tau} \exp\left(\frac{x}{\mu} \right) W(x) \mathrm{d}x \right] \mathrm{d}\tau, \\
\phi =& \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c \\
&+ \beta \int_0^\infty \eta(\tau) \left[\int_{-\mu\tau}^0 W(x) \mathrm{d}x \right] \mathrm{d}\tau \\
&+ \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu\tau} \exp\left(\frac{x}{\mu} \right) W(x) \mathrm{d}x \right] \mathrm{d}\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_1 =& \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{x}{\mu} \right) \frac{c}{c+s(x+Z)\mu} K\left(\frac{c(x+Z)}{c+s(x+Z)\mu} \right) \mathrm{d}x \right] \mathrm{d}c \\
&- \alpha \int_0^\infty \xi(c) \left[\int_0^{cZ/(c+s(Z)\mu)} K(x) \mathrm{d}x \right] \mathrm{d}c, \\
\Phi_2 =& \beta \int_0^\infty \eta(\tau) \left[\int_{-\mu\tau-Z}^0 W(x) \mathrm{d}x \right] \mathrm{d}\tau \\
&+ \beta \exp\left(\frac{Z}{\mu} \right) \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu\tau-Z} \exp\left(\frac{x}{\mu} \right) W(x) \mathrm{d}x \right] \mathrm{d}\tau,
\end{aligned}
$$

for all (α, β), (ξ, η), (K, W) and $(\mu, Z) \in (0, c_0) \times (0, \infty)$, where we recall that

$$c_0 = \sup\{c > 0 : \xi = 0 \text{ on } (0, c) \text{ and } \xi \geq 0 \text{ on } (c, \infty)\}.$$

Remark 2.1. With the definitions of the speed index functions, the speed equation becomes

$$\phi(\alpha, \beta, \xi, \eta, K, W, \mu) = \frac{\alpha+\beta}{2} - \theta,$$

if $\theta = \Theta$; and the system of speed equations becomes

$$
\begin{aligned}
\Phi_1(\alpha, \xi, K, \mu, 0) + \Phi_2(\beta, \eta, W, \mu, Z) &= \frac{\alpha+\beta}{2} - \theta, \\
\Phi_1(\alpha, \xi, K, \mu, Z) + \Phi_2(\beta, \eta, W, \mu, 0) &= \frac{\alpha+\beta}{2} - \Theta,
\end{aligned}
$$

if $\theta < \Theta$. Note that

$$
\begin{aligned}
\Phi_1(\alpha, \xi, K, \mu, 0) &= \phi_1(\alpha, \xi, K, \mu), \\
\Phi_2(\beta, \eta, W, \mu, 0) &= \phi_2(\beta, \eta, W, \mu),
\end{aligned}
$$

$$\phi(\alpha,\beta,\xi,\eta,K,W,\mu) = \phi_1(\alpha,\xi,K,\mu) + \phi_2(\beta,\eta,W,\mu),$$
$$\Phi_1(\alpha,\xi,K,\mu,0) + \Phi_2(\beta,\eta,W,\mu,0) = \phi(\alpha,\beta,\xi,\eta,K,W,\mu),$$

and

$$\begin{aligned}
&\Phi_1(\alpha,\xi,K,\mu,Z)\\
=&\alpha \exp\left(-\frac{Z}{\mu}\right)\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\alpha \exp\left(-\frac{Z}{\mu}\right)\int_0^\infty \xi(c)\left[\int_0^{cZ/(c+s(Z)\mu)} \exp\left(\frac{c+\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&-\alpha\int_0^\infty \xi(c)\left[\int_0^{cZ/(c+s(Z)\mu)} K(x)\mathrm{d}x\right]\mathrm{d}c.
\end{aligned}$$

We compute the first order partial derivatives of the speed index functions:

$$\begin{aligned}
{\phi_1}'(\mu) =&\frac{\alpha}{\mu^2}\int_0^\infty \xi(c)\left[\int_{-\infty}^0 |x|\exp\left(\frac{c-\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c,\\
{\phi_2}'(\mu) =&\frac{\beta}{\mu^2}\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu\tau} |x|\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau,\\
\phi'(\mu) =&\frac{\alpha}{\mu^2}\int_0^\infty \xi(c)\left[\int_{-\infty}^0 |x|\exp\left(\frac{c-\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu^2}\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu\tau} |x|\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau,\\
\frac{\partial\Phi_1}{\partial\mu} =&\frac{\alpha}{\mu^2}\exp\left(-\frac{Z}{\mu}\right)\int_0^\infty \xi(c)\left[\int_{-\infty}^0 (Z-x)\exp\left(\frac{c-\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\alpha}{\mu^2}\exp\Big(-\frac{Z}{\mu}\Big)\int_0^\infty \xi(c)\Big[\int_0^{cZ/(c+s(Z)\mu)} (Z-x)\exp\Big(\frac{c+\mu}{c\mu}x\Big)K(x)\mathrm{d}x\Big]\mathrm{d}c\\
=&\alpha\int_0^\infty \xi(c)\left[\int_{-Z/\mu}^{-Z/(c+s(Z)\mu)} (-x)\exp\left(x+\frac{\mu x+Z}{c}\right)K(\mu x+Z)\mathrm{d}x\right]\mathrm{d}c,\\
\frac{\partial\Phi_1}{\partial Z} =&-\frac{\alpha}{\mu}\exp\left(-\frac{Z}{\mu}\right)\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&-\frac{\alpha}{\mu}\exp\left(-\frac{Z}{\mu}\right)\int_0^\infty \xi(c)\left[\int_0^{cZ/(c+s(Z)\mu)} \exp\left(\frac{c+\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
=&-\alpha\int_0^\infty \xi(c)\left[\int_{-Z/\mu}^{-Z/(c+s(Z)\mu)} \exp\left(x+\frac{\mu x+Z}{c}\right)K(\mu x+Z)\mathrm{d}x\right]\mathrm{d}c,\\
\frac{\partial\Phi_2}{\partial\mu} =&\frac{\beta}{\mu^2}\exp\left(\frac{Z}{\mu}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu\tau-Z} |x+Z|\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\beta\int_0^\infty \eta(\tau)\left[\int_{-\infty}^0 |x|\exp(x)W(\mu x-\mu\tau-Z)\mathrm{d}x\right]\mathrm{d}\tau,
\end{aligned}$$

$$\begin{aligned}\frac{\partial \Phi_2}{\partial Z} =&\frac{\beta}{\mu}\exp\left(\frac{Z}{\mu}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu\tau-Z}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\ =&\beta\int_0^\infty \eta(\tau)\left[\int_{-\infty}^0 \exp(x)W(\mu x-\mu\tau-Z)\mathrm{d}x\right]\mathrm{d}\tau,\end{aligned}$$

for all $(\mu, Z) \in (0, c_0) \times (0, \infty)$.

2.3. The Existence and Uniqueness of the Wave Speed μ_3 of the Large Traveling Wave Front for the Case $\theta = \Theta$

The main purpose of this subsection is to accomplish the existence and uniqueness of the wave speed μ_3 of the large traveling wave front of (1.1). That is, we will prove that the wave speed μ_3 of the large traveling wave front is the unique solution of the speed equation

$$\phi(\alpha,\beta,\xi,\eta,K,W,\mu) = \frac{\alpha+\beta}{2} - \theta,$$

under the assumptions (1.5)-(1.16). Let $\theta = \Theta$. First of all, it is easy to find the following limits

$$\begin{aligned}&\lim_{\mu\to 0^+}\phi_1(\alpha,\xi,K,\mu) = 0, \qquad \lim_{\mu\to 0^+}\phi_2(\beta,\eta,W,\mu) = 0,\\ &\lim_{\mu\to c_0}\phi_1(\alpha,\beta,K,\mu) = \alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-c_0}{cc_0}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c,\\ &\lim_{\mu\to c_0}\phi_2(\beta,\eta,W,\mu) = \beta\int_0^\infty \eta(\tau)\left[\int_{-c_0\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau\\ &\qquad\qquad + \beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-c_0\tau}\exp\left(\frac{x}{c_0}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau.\end{aligned}$$

Moreover, it is easy to see that

$$\begin{aligned}&\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-c_0}{cc_0}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\ &+\beta\int_0^\infty \eta(\tau)\left[\int_{-c_0\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau\\ &+\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-c_0\tau}\exp\left(\frac{x}{c_0}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\ >&\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{x}{c_0}\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\ &+\beta\int_0^\infty \eta(\tau)\left[\int_{-c_0\tau}^0 \exp\left(\frac{x}{c_0}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\ &+\beta\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{-c_0\tau}\exp\left(\frac{x}{c_0}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
=&\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{x}{c_0}\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\beta\int_0^\infty \eta(\tau)\left[\int_{-\infty}^0 \exp\left(\frac{x}{c_0}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\alpha\int_{-\infty}^0 \exp\left(\frac{x}{c_0}\right)K(x)\mathrm{d}x+\beta\int_{-\infty}^0 \exp\left(\frac{x}{c_0}\right)W(x)\mathrm{d}x\\
\geq&\frac{\alpha+\beta}{2}-\theta.
\end{aligned}
$$

Furthermore, it is easy to find that

$$
\lim_{\mu\to 0^+}\phi(\alpha,\beta,\xi,\eta,K,W,\mu)=0<\frac{\alpha+\beta}{2}-\theta<\lim_{\mu\to c_0}\phi(\alpha,\beta,\xi,\eta,K,W,\mu).
$$

Therefore, the existence of a wave speed μ_3 to the speed equation $\phi(\alpha,\beta,\xi,\eta,K,W,\mu) = \frac{\alpha+\beta}{2}-\theta$ in $(0,c_0)$ is obviously true by using an intermediate value theorem. It suffices to establish the uniqueness of the wave speed. As $\mu\to 0^+$, we obtain the following limits

$$
\begin{aligned}
{\phi_1}'(\mu)&=\frac{\alpha}{\mu^2}\int_0^\infty \xi(c)\left[\int_{-\infty}^0 |x|\exp\left(\frac{c-\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&=\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 |x|\exp\left(\frac{c-\mu}{c}x\right)K(\mu x)\mathrm{d}x\right]\mathrm{d}c\\
&\to\alpha\left\{\int_0^\infty \xi(c)\left[\int_{-\infty}^0 |x|\exp(x)K(0)\mathrm{d}x\right]\mathrm{d}c\right\}=\alpha K(0),
\end{aligned}
$$

and

$$
\begin{aligned}
{\phi_2}'(\mu)&=\frac{\beta}{\mu^2}\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu\tau} |x|\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&=\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\tau} |x|\exp(x)W(\mu x)\mathrm{d}x\right]\mathrm{d}\tau\\
&\to\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\tau} |x|\exp(x)W(0)\mathrm{d}x\right]\mathrm{d}\tau\\
&=\beta\left\{\int_0^\infty (1+\tau)\eta(\tau)W(0)\mathrm{d}\tau\right\}.
\end{aligned}
$$

We will show that $\phi'(\mu)>0$ for all synaptic couplings in classes (A) and (B). Also we will demonstrate that $\phi'(\mu)<0$ on $(0,\mu_\#)$, $\phi'(\mu_\#)=0$ and $\phi'(\mu)>0$ on $(\mu_\#,c_0)$, where the positive constant $\mu_\#$ depends on the kernel functions K and W, for all kernel functions in class (C). Overall, if $0<2\theta<\alpha+\beta$, then as a function of μ, the graph of ϕ crosses the line $\phi=\frac{\alpha+\beta}{2}-\theta$ exactly once.

The rest of the proof of the uniqueness of the wave speed μ_3 is divided into three parts.

(A) For all nonnegative kernel functions K and W, because

$$
\int_{-\infty}^0 K(x)\mathrm{d}x=\int_{-\infty}^0 W(x)\mathrm{d}x=\frac{1}{2},
$$

so we must have that $\phi'(\mu) > 0$ on $(0, c_0)$.

(B1) For each Mexican hat kernel function K, there exists a positive constant $M > 0$, such that $K \geq 0$ on $(-M, 0)$ and $K \leq 0$ on $(-\infty, -M)$. Therefore, there hold the estimates

$$|x| \exp\left(\frac{c-\mu}{c\mu}x\right) K(x) \geq \exp\left(-\frac{c-\mu}{c\mu}M\right) |x| K(x) \geq 0,$$

on $(-M, 0)$, and

$$0 \geq |x| \exp\left(\frac{c-\mu}{c\mu}x\right) K(x) \geq \exp\left(-\frac{c-\mu}{c\mu}M\right) |x| K(x),$$

on $(-\infty, -M)$. Hence for all $\mu \in (0, c_0)$, we have

$$\phi_1{}'(\mu) > \frac{\alpha}{\mu^2} \int_0^\infty \xi(c) \left[\exp\left(-\frac{c-\mu}{c\mu}M\right) \int_{-\infty}^0 |x| K(x) \mathrm{d}x\right] \mathrm{d}c \geq 0.$$

(B2) For each Mexican hat kernel function W, there exists a positive constant $M > 0$, such that $W \geq 0$ on $(-M, 0)$ and $W \leq 0$ on $(-\infty, -M)$. Define two sequences of functions $\{\phi_{1,n}(\mu) : n = 1, 2, 3, \cdots\}$ and $\{\phi_{2,n}(\mu) : n = 1, 2, 3, \cdots\}$ on $(0, \infty)$ by

$$\begin{aligned}
\phi_{1,n}(\mu) =& \beta \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu\tau} |x|^n \exp\left(\frac{x}{\mu}\right) W(x) \mathrm{d}x\right] \mathrm{d}\tau, \\
\phi_{2,n}(\mu) =& \beta\mu^n \int_0^\infty \eta(\tau) \tau^{n+1} W(\mu\tau) \mathrm{d}\tau.
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{\mu\to 0^+} \phi_{1,n}(\mu) = 0, \lim_{\mu\to\infty} \phi_{1,n}(\mu) = 0, \\
\lim_{\mu\to 0^+} \phi_{2,n}(\mu) = 0, \lim_{\mu\to\infty} \phi_{2,n}(\mu) = 0.
\end{aligned}$$

Also, as $\mu \to 0^+$, we have the limit

$$\begin{aligned}
\frac{\phi_{1,n}(\mu)}{\mu^{n+1}} &= \frac{\beta}{\mu^{n+1}} \left\{\int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu\tau} |x|^n \exp\left(\frac{x}{\mu}\right) W(x) \mathrm{d}x\right] \mathrm{d}\tau\right\} \\
&= \beta \left\{\int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\tau} |x|^n \exp(x) W(\mu x) \mathrm{d}x\right] \mathrm{d}\tau\right\} \\
&\to \beta \left\{\int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\tau} |x|^n \exp(x) W(0) \mathrm{d}x\right] \mathrm{d}\tau\right\}.
\end{aligned}$$

For the function $\phi_{2,n}$, we have the following simple estimate

$$\phi_{2,n}(\mu) = \frac{\beta}{\mu^2} \int_0^\infty \eta\left(\frac{\tau}{\mu}\right) \tau^{n+1} W(\tau) \mathrm{d}\tau \leq \frac{C_n}{\mu^2},$$

for some positive constant $C_n > 0$. It is easy to derive the differential equation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\mu}\phi_{1,n}(\mu) =& \frac{\beta}{\mu^2} \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu\tau} |x|^{n+1} \exp\left(\frac{x}{\mu}\right) W(x) \mathrm{d}x\right] \mathrm{d}\tau \\
&- \beta\mu^n \int_0^\infty \eta(\tau) \tau^{n+1} W(-\mu\tau) \mathrm{d}\tau.
\end{aligned}$$

That is

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\phi_{1,n}(\mu) = \frac{1}{\mu^2}\phi_{1,n+1}(\mu) - \phi_{2,n}(\mu).$$

For all integers $n \geq 1$ and $k \geq 1$, we have the estimate

$$\phi_{1,n+k}(\mu) \leq M^k \phi_{1,n}(\mu).$$

Making use of this estimate, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\mu}\phi_{1,n}(\mu) \leq \frac{M}{\mu^2}\phi_{1,n}(\mu) - \phi_{2,n}(\mu), \\
&\frac{\mathrm{d}}{\mathrm{d}\mu}\left\{\exp\left(\frac{M}{\mu}\right)\phi_{1,n}(\mu)\right\} \leq -\exp\left(\frac{M}{\mu}\right)\phi_{2,n}(\mu).
\end{aligned}$$

Integrating the last differential inequality with respect to μ on the interval (μ, ∞), we get

$$-\exp\left(\frac{M}{\mu}\right)\phi_{1,n}(\mu) \leq -\int_\mu^\infty \exp\left(\frac{M}{\nu}\right)\phi_{2,n}(\nu)\mathrm{d}\nu.$$

That is

$$\phi_{1,n}(\mu) \geq \exp\left(-\frac{M}{\mu}\right)\int_\mu^\infty \exp\left(\frac{M}{\nu}\right)\phi_{2,n}(\nu)\mathrm{d}\nu.$$

Because

$$\begin{aligned}
&\int_\mu^\infty \exp\left(\frac{M}{\nu}\right)\phi_{2,n}(\nu)\mathrm{d}\nu \\
=&\beta\int_\mu^\infty \exp\left(\frac{M}{\nu}\right)\nu^n\left[\int_0^\infty \eta(\tau)\tau^{n+1}W(\nu\tau)\mathrm{d}\tau\right]\mathrm{d}\nu \\
=&\beta\int_0^\infty \tau^{n+1}W(\tau)\left[\int_\mu^\infty \frac{1}{\nu^2}\exp\left(\frac{M}{\nu}\right)\eta\left(\frac{\tau}{\nu}\right)\mathrm{d}\nu\right]\mathrm{d}\tau > 0,
\end{aligned}$$

and

$$M^n\phi_{1,1}(\mu) > \phi_{1,n+1}(\mu) > 0,$$

we obtain the estimate

$$\phi_{1,1}(\mu) > 0.$$

Therefore, we obtain the desired estimate

$${\phi_2}'(\mu) = \frac{1}{\mu^2}\phi_{1,1}(\mu) > 0.$$

(C1) For each upside down Mexican hat kernel function K, there exists a positive constant $M > 0$, such that $K \leq 0$ on $(-M, 0)$ and $K \geq 0$ on $(-\infty, -M)$. Therefore, there exist two positive constants C_1 and C_2, with $C_2 > C_1 > 0$, such that

$$\int_{-M}^0 K(x)\mathrm{d}x + \int_{-(1+C_2)M}^{-(1+C_1)M} K(x)\mathrm{d}x > 0.$$

First of all, for all integers $n \geq 1$, we have the following estimates

$$
\begin{aligned}
&\alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^n K(x) \mathrm{d}x \right] \mathrm{d}c \\
=&\alpha M^{n+1} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^n K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
=&\alpha M^{n+1} \int_0^\infty \xi(c) \left[\int_{-1}^0 |x|^n K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha M^{n+1} \int_0^\infty \xi(c) \left[\int_{-\infty}^{-1} |x|^n K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
\geq&\alpha M^{n+1} \int_0^\infty \xi(c) \left[\int_{-1}^0 |x|^n K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha M^{n+1} \int_0^\infty \xi(c) \left[\int_{-1-C_2}^{-1-C_1} |x|^n K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
\geq&\alpha M^{n+1} \int_0^\infty \xi(c) \left[\int_{-1}^0 K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha M^{n+1} \int_0^\infty \xi(c) \left[\int_{-1-C_2}^{-1-C_1} K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
=&\alpha M^{n} \int_0^\infty \xi(c) \left[\int_{-M}^0 K(x) \mathrm{d}x \right] \mathrm{d}c + \alpha M^{n} \int_0^\infty \xi(c) \left[\int_{-(1+C_2)M}^{-(1+C_1)M} K(x) \mathrm{d}x \right] \mathrm{d}c \\
=&\alpha M^{n} \int_0^\infty \xi(c) \left[\int_{-M}^0 K(x) \mathrm{d}x + \int_{-(1+C_2)M}^{-(1+C_1)M} K(x) \mathrm{d}x \right] \mathrm{d}c > 0.
\end{aligned}
$$

That is

$$
\alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^n K(x) \mathrm{d}x \right] \mathrm{d}c > 0.
$$

Second, for all integers $n \geq 1$ and $k \geq 1$, we have the estimates

$$
\begin{aligned}
&\alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+k} \exp\left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c \\
=&\alpha M^{n+k+1} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+k} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
=&\alpha M^{n+k+1} \int_0^\infty \xi(c) \left[\int_{-1}^0 |x|^{n+k} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha M^{n+k+1} \int_0^\infty \xi(c) \left[\int_{-\infty}^{-1} |x|^{n+k} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
>&\alpha M^{n+k+1} \int_0^\infty \xi(c) \left[\int_{-1}^0 |x|^{n} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha M^{n+k+1} \int_0^\infty \xi(c) \left[\int_{-\infty}^{-1} |x|^{n} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha M^{n+k+1} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^n \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
&= \alpha M^k \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^n \exp\left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c.
\end{aligned}
$$

That is

$$
\begin{aligned}
&\alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+k} \exp\left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c \\
>&\alpha M^k \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^n \left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c.
\end{aligned}
$$

Third, for each fixed constant $\mu_0 \in (0, c_0)$, there exists a sufficiently large integer $N_0 = N_0(\mu_0)$, such that for all integers $n \geq N_0 + 1$ and for all $\mu \in (\mu_0, c_0)$, there holds

$$
\int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c > 0.
$$

In fact, fix $\mu_0 \in (0, c_0)$ and let $\mu_0 < \mu < c_0$. Then we have the following estimates

$$
\begin{aligned}
&\alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} x \right) K(x) \mathrm{d}x \right] \mathrm{d}c \\
=&\alpha M^{n+2} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
=&\alpha M^{n+2} \int_0^\infty \xi(c) \left[\int_{-1}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha M^{n+2} \int_0^\infty \xi(c) \left[\int_{-\infty}^{-1} |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
\geq&\alpha M^{n+2} \int_0^\infty \xi(c) \left[\int_{-1}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha M^{n+2} \int_0^\infty \xi(c) \left[\int_{-1-C_2}^{-1-C_1} |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} Mx \right) K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
\geq&\alpha M^{n+2} \int_0^\infty \xi(c) \left[\int_{-1}^0 K(Mx) \mathrm{d}x + \int_{-1-C_2}^{-1-C_1} K(Mx) \mathrm{d}x \right] \mathrm{d}c \\
=&\alpha M^{n+1} \int_0^\infty \xi(c) \left[\int_{-M}^0 K(x) \mathrm{d}x + \int_{-(1+C_2)M}^{-(1+C_1)M} K(x) \mathrm{d}x \right] \mathrm{d}c > 0,
\end{aligned}
$$

where we have applied the following elementary estimates

$$
\begin{aligned}
|x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} Mx \right) &\leq 1 \text{ and } K(Mx) \leq 0, \text{ on } (-1, 0), \text{ for all } n \geq 0, \\
|x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} Mx \right) &\geq 1 \text{ and } K(Mx) \geq 0, \text{ on } (-1-C_2, -1-C_1),
\end{aligned}
$$

provided that the integer n is sufficiently large, namely, $n \geq N_0 + 1$, where

$$
N_0 = 1 + \left[\left[\frac{1}{\mu_0} \frac{(1+C_2)M}{\ln(1+C_1)} \right] \right].
$$

Here $[[x]]$ represents the greatest integer function of x.

Fourth, define a sequence of nonlinear smooth functions $\{\phi_{3,n}(\mu)\}$ on $(0, c_0)$ by

$$\phi_{3,n}(\mu) = \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^n \exp\left(\frac{c-\mu}{c\mu} x \right) K(x)\mathrm{d}x \right] \mathrm{d}c,$$

for all integers $n \geq 1$. Then

$$\begin{aligned} &\lim_{\mu \to 0^+} \phi_{3,n}(\mu) = 0, \\ &\lim_{\mu \to c_0} \phi_{3,n}(\mu) = \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^n \exp\left(\frac{c-c_0}{cc_0} x \right) K(x)\mathrm{d}x \right] \mathrm{d}c > 0, \end{aligned}$$

for all sufficiently large integer $n \geq N_0 + 1$. Moreover

$$\phi_{3,n}{}'(\mu) = \frac{\alpha}{\mu^2} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} x \right) K(x)\mathrm{d}x \right] \mathrm{d}c = \frac{1}{\mu^2} \phi_{3,n+1}(\mu).$$

Therefore, for all $n \geq N_0 + 1$ and for all $\mu \in (\mu_0, c_0)$, we obtain the estimates

$$\begin{aligned} &\phi_{3,n}{}'(\mu) = \frac{1}{\mu^2} \phi_{3,n+1}(\mu) \\ > &\frac{\alpha M^{n+1}}{\mu^2} \int_0^\infty \xi(c) \left[\int_{-M}^0 K(x)\mathrm{d}x + \int_{-(1+C_2)M}^{-(1+C_1)M} K(x)\mathrm{d}x \right] \mathrm{d}c > 0. \end{aligned}$$

If $\phi_{3,n}(\mu_\#) = 0$, at some number $\mu_\# \in (0, c_0)$, then for the same number $\mu_\#$, we have $\phi_{3,n+k}(\mu_\#) > 0$, because $\phi_{3,n+k}(\mu_\#) > M^k \phi_{3,n}(\mu_\#) = 0$.

Fifth, by fixing the integer $n = N_0 + 1$ and by making the change of variable $x' = \frac{x}{\mu}$, where $\mu \in (0, c_0)$, we have

$$\begin{aligned} &\int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} x \right) K(x)\mathrm{d}x \right] \mathrm{d}c \\ = &\mu^{n+2} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c} x \right) K(\mu x)\mathrm{d}x \right] \mathrm{d}c, \end{aligned}$$

and

$$\begin{aligned} &\lim_{\mu \to 0^+} \left\{ \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c} x \right) K(\mu x)\mathrm{d}x \right] \mathrm{d}c \right\} \\ = &\left\{ \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp(x) K(0)\mathrm{d}x \right] \mathrm{d}c \right\} = (n+1)! K(0) < 0, \end{aligned}$$

if $K(0) < 0$. Therefore, there exists a small number $\mu = \mu_{n+1} > 0$, such that

$$\begin{aligned} &\phi_{3,n+1}(\mu) = \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c\mu} x \right) K(x)\mathrm{d}x \right] \mathrm{d}c \\ = &\alpha\mu^{n+2} \int_0^\infty \xi(c) \left[\int_{-\infty}^0 |x|^{n+1} \exp\left(\frac{c-\mu}{c} x \right) K(\mu x)\mathrm{d}x \right] \mathrm{d}c < 0, \end{aligned}$$

for all μ with $0 < \mu < \mu_{n+1}$. This result is also true if $K(0) = 0$, $K \geq 0$ on $(-\infty, -M)$ and $K \leq 0$ on $(-M, 0)$, for some positive constant $M > 0$.

Sixth, set $A^*_{n+1} = \inf\{\mu \le c_0 : \phi_{3,n+1}(\mu) > 0 \text{ on } (\mu, c_0)\}$ and $A^{**}_{n+1} = \sup\{\mu \ge 0 : \phi_{3,n+1}(\mu) < 0 \text{ on } (0,\mu)\}$. We will show that $A^*_{n+1} = A^{**}_{n+1}$.

In fact, if $\phi_{3,n+1}(\mu_\#) = 0$ at some constant $\mu_\# \in (0, c_0)$, then $\frac{\mathrm{d}}{\mathrm{d}\mu}\phi_{3,n+1}(\mu_\#) = \frac{1}{{\mu_\#}^2}\phi_{3,n+2}(\mu_\#) > 0$. Thus the graph of the smooth function $\phi_{3,n+1}(\mu)$ crosses the μ-axis exactly once. Without any difficulty, we may conclude that $A_{n+1} = A^*_{n+1} = A^{**}_{n+1}$. Now $\phi_{3,n+1}(A^*_{n+1}) = 0$. Below we will use mathematical induction method backward. Note that

$$\lim_{\mu\to 0^+} \phi_{3,n}(\mu) = 0, \qquad \phi_{3,n}(\mu) = \int_0^\mu \frac{1}{\nu^2}\phi_{3,n+1}(\nu)\mathrm{d}\nu.$$

Now it is easy to find that $\phi_{3,n}(\mu) < 0$ on $(0, \mu_n)$ and $\phi_{3,n}(\mu) > 0$ on (μ_n, c_0), for some constant $\mu_n \in (0, c_0)$. Recall that we have

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\phi_{3,n-1}(\mu) = \frac{\phi_{3,n}(\mu)}{\mu^2} \qquad \text{and} \qquad \lim_{\mu\to 0^+} \phi_{3,n-1}(\mu) = 0.$$

Therefore, it is easy to conclude that $\phi_{3,n-1}(\mu) < 0$ on $(0, \mu_{n-1})$ and $\phi_{3,n-1}(\mu) > 0$ on (μ_{n-1}, c_0), for some constant $\mu_{n-1} \in (0, c_0)$, and so on. Finally, we get $\lim_{\mu\to 0^+} \phi_{3,1}(\mu) = 0$, $\phi_{3,1}(\mu) < 0$ on $(0, \mu_1)$ and $\phi_{3,1}(\mu) > 0$ on (μ_1, c_0), for some constant $\mu_1 \in (0, c_0)$. By mathematical induction method backward, we find that there exists some constant $\mu_m = \mu(m)$, such that $\phi_{3,m}(\mu) < 0$ if $0 < \mu < \mu_m$, $\phi_{3,m}(\mu_m) = 0$ and $\phi_{3,m}(\mu) > 0$ if $\mu_m < \mu < c_0$. Note that $0 < \mu_{m+1} < \mu_m < c_0$.

(C2) For each upside down Mexican hat kernel function W, there exists a positive constant $M > 0$, such that $W \le 0$ on $(-M, 0)$ and $W \ge 0$ on $(-\infty, -M)$. Therefore, there exist two positive constants C_1 and C_2, with $C_2 > C_1 > 0$, such that

$$\int_{-M}^0 W(x)\mathrm{d}x + \int_{-(1+C_2)M}^{-(1+C_1)M} W(x)\mathrm{d}x > 0.$$

There exists a positive constant $\mu_1 > 0$, such that ${\phi_2}'(\mu) < 0$ on $(0, \mu_1)$. Note that

$$\begin{aligned}
{\phi_2}'(\mu) &= \frac{\beta}{\mu^2}\int_0^\infty \eta(\tau)e^\tau \left[\int_{-\infty}^{-\mu\tau} |x| \exp\left(\frac{x}{\mu}\right) W(x)\mathrm{d}x\right]\mathrm{d}\tau \\
&= \beta\int_0^\infty \eta(\tau)e^\tau \left[\int_{-\infty}^{-\tau} |x| \exp(x) W(\mu x)\mathrm{d}x\right]\mathrm{d}\tau \\
&= \beta\int_{-\infty}^0 |x| \exp(x) W(\mu x)\left[\int_0^{-x} \eta(\tau)e^\tau \mathrm{d}\tau\right]\mathrm{d}x \\
&= \beta\int_{-\infty}^0 |x| \exp(x)\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_{-\infty}^x W(\mu y)\left[\int_0^{-y} \eta(\tau)e^\tau \mathrm{d}\tau\right]\mathrm{d}y\right\}\mathrm{d}x \\
&= \beta\int_{-\infty}^0 (1+x)\exp(x)\left\{\int_{-\infty}^x W(\mu y)\left[\int_0^{-y} \eta(\tau)e^\tau \mathrm{d}\tau\right]\mathrm{d}y\right\}\mathrm{d}x.
\end{aligned}$$

This means that there exists a unique positive constant $\mu_\# > 0$, such that ${\phi_2}'(\mu) < 0$ on $(0, \mu_\#)$, ${\mu_2}'(\mu_\#) = 0$ and ${\phi_2}'(\mu) > 0$ on $(\mu_\#, \infty)$.

Now let us consider more general kernel functions which may cross the x-axis for finitely or even infinitely many times.

(D1) If the synaptic coupling K satisfies condition (1.7), then by using integration by parts for M times, we find that

$$
\begin{aligned}
&\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
=&\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mu}{c\mu}x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_x^0 K(y_1)\mathrm{d}y_1\right)\mathrm{d}x\right]\mathrm{d}c\\
=&\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \frac{c-\mu}{c\mu}\exp\left(\frac{c-\mu}{c\mu}x\right)\left(\int_x^0 K(y_1)\mathrm{d}y_1\right)\mathrm{d}x\right]\mathrm{d}c\\
=&\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \frac{c-\mu}{c\mu}\exp\left(\frac{c-\mu}{c\mu}x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_x^0\int_{y_2}^0 K(y_1)\mathrm{d}y_1\mathrm{d}y_2\right)\mathrm{d}x\right]\mathrm{d}c\\
=&\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \left(\frac{c-\mu}{c\mu}\right)^2\exp\left(\frac{c-\mu}{c\mu}x\right)\left(\int_x^0\int_{y_2}^0 K(y_1)\mathrm{d}y_1\mathrm{d}y_2\right)\mathrm{d}x\right]\mathrm{d}c\\
=&\cdots=\\
=&\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \left(\frac{c-\mu}{c\mu}\right)^M\exp\left(\frac{c-\mu}{c\mu}x\right)\right.\\
&\left.\cdot\left(\int_x^0\int_{y_M}^0\cdots\int_{y_3}^0\int_{y_2}^0 K(y_1)\mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_{M-1}\mathrm{d}y_M\right)\mathrm{d}x\right]\mathrm{d}c>0.
\end{aligned}
$$

(D2) If the synaptic coupling W satisfies condition (1.8), then by using integration by parts for N times, we find that

$$
\begin{aligned}
&\int_0^\infty \eta(\tau)\left[\int_{-\mu\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau+\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu\tau}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\int_{-\infty}^0\left[\int_{-x/\mu}^\infty \eta(\tau)\mathrm{d}\tau+\exp\left(\frac{x}{\mu}\right)\int_0^{-x/\mu}\eta(\tau)e^\tau\mathrm{d}\tau\right]W(x)\mathrm{d}x\\
=&\int_{-\infty}^0\left[\int_{-x/\mu}^\infty \eta(\tau)\mathrm{d}\tau+\exp\left(\frac{x}{\mu}\right)\int_0^{-x/\mu}\eta(\tau)e^\tau\mathrm{d}\tau\right]\\
&\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[-\int_x^0 W(y_1)\mathrm{d}y_1\right]\mathrm{d}x\\
=&\frac{1}{\mu}\int_{-\infty}^0\left[\exp\left(\frac{x}{\mu}\right)\int_0^{-x/\mu}\eta(\tau)e^\tau\mathrm{d}\tau\right]\left[\int_x^0 W(y_1)\mathrm{d}y_1\right]\mathrm{d}x\\
=&\frac{1}{\mu}\int_{-\infty}^0\left[\exp\left(\frac{x}{\mu}\right)\int_0^{-x/\mu}\eta(\tau)e^\tau\mathrm{d}\tau\right]\frac{\mathrm{d}}{\mathrm{d}x}\left[-\int_x^0\int_{y_2}^0 W(y_1)\mathrm{d}y_1\mathrm{d}y_2\right]\mathrm{d}x\\
=&\frac{1}{\mu^2}\int_{-\infty}^0\left[\exp\left(\frac{x}{\mu}\right)\int_0^{-x/\mu}\eta(\tau)e^\tau\mathrm{d}\tau-\eta\left(-\frac{x}{\mu}\right)\right]\left[\int_x^0\int_{y_2}^0 W(y_1)\mathrm{d}y_1\mathrm{d}y_2\right]\mathrm{d}x\\
=&\frac{1}{\mu^2}\int_{-\infty}^0\left[\exp\left(\frac{x}{\mu}\right)\int_0^{-x/\mu}\eta(\tau)e^\tau\mathrm{d}\tau-\eta\left(-\frac{x}{\mu}\right)\right]\\
&\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[-\int_x^0\int_{y_3}^0\int_{y_2}^0 W(y_1)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\right]\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{\mu^3}\int_{-\infty}^{0}\left[\exp\left(\frac{x}{\mu}\right)\int_0^{-x/\mu}\eta(\tau)e^{\tau}\mathrm{d}\tau-\eta\left(-\frac{x}{\mu}\right)+\eta'\left(-\frac{x}{\mu}\right)\right]\\
&\cdot\left[\int_x^0\int_{y_3}^0\int_{y_2}^0 W(y_1)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\right]\mathrm{d}x=\cdots\\
=&\frac{1}{\mu^N}\int_{-\infty}^{0}\left[\exp\left(\frac{x}{\mu}\right)\int_0^{-x/\mu}\eta(\tau)e^{\tau}\mathrm{d}\tau-\eta\left(-\frac{x}{\mu}\right)+\eta'\left(-\frac{x}{\mu}\right)-\eta''\left(-\frac{x}{\mu}\right)\right.\\
&\left.+\cdots+(-1)^{N-1}\eta^{(N-2)}\left(-\frac{x}{\mu}\right)\right]\\
&\cdot\left[\int_x^0\int_{y_N}^0\cdots\int_{y_3}^0\int_{y_2}^0 W(y_1)\mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_{N-1}\mathrm{d}y_N\right]\mathrm{d}x>0.
\end{aligned}
$$

Therefore, for all synaptic couplings in classes (A), (B), (C) and for more general synaptic couplings satisfying conditions (1.7) or (1.8), there exists a unique wave speed $\mu_3=\mu_3(\alpha,\beta,\xi,\eta,K,W,\theta)>0$, such that $\mu_3\in(0,c_0)$, $\phi(\alpha,\beta,\xi,\eta,K,W,\mu_3)=\frac{\alpha+\beta}{2}-\theta$ and then $U(0)=\theta$.

2.4. The Existence and Uniqueness of the Wave Speed μ_3 of the Large Traveling Wave Front for the Case $\theta<\Theta$

The main purpose of this subsection is to accomplish the existence and uniqueness of the solution (μ_3,Z_3) of the system of speed equations $U(0)=\theta$ and $U(Z_3)=\Theta$, where $\mu_3=\mu_3(Z_3)>0$ represents the wave speed and $Z_3>0$ is a positive constant. That is, we will establish the existence and uniqueness of the solution of the system of speed equations

$$
\begin{aligned}
\Phi_1(\alpha,\xi,K,\mu,0)+\Phi_2(\beta,\eta,W,\mu,Z)&=\frac{\alpha+\beta}{2}-\theta,\\
\Phi_1(\alpha,\xi,K,\mu,Z)+\Phi_2(\beta,\eta,W,\mu,0)&=\frac{\alpha+\beta}{2}-\Theta.
\end{aligned}
$$

Suppose that the assumptions (1.5)-(1.16) hold. Let $\theta<\Theta$. First of all, we have the following limits:

$$
\begin{aligned}
&\lim_{\mu\to0^+}\Phi_1(\alpha,\xi,K,\mu,Z)=-\alpha\int_0^Z K(x)\mathrm{d}x,\\
&\lim_{\mu\to c_0}\Phi_1(\alpha,\xi,K,\mu,Z)\\
=&\alpha\int_0^{\infty}\xi(c)\left[\int_{-\infty}^0\exp\left(\frac{x}{c_0}\right)\frac{c}{c+s(x+Z)c_0}K\left(\frac{c(x+Z)}{c+s(x+Z)c_0}\right)\mathrm{d}x\right]\mathrm{d}c\\
&-\alpha\int_0^{\infty}\xi(c)\left[\int_0^{cZ/(c+s(Z)c_0)}K(x)\mathrm{d}x\right]\mathrm{d}c,\\
&\lim_{Z\to0^+}\Phi_1(\alpha,\xi,K,\mu,Z)=\phi_1,\qquad\lim_{Z\to\infty}\Phi_1(\alpha,\xi,K,\mu,Z)=-\frac{\alpha}{2},\\
&\lim_{\mu\to0^+}\Phi_2(\beta,\eta,W,\mu,Z)=\beta\int_{-Z}^0 W(x)\mathrm{d}x,\qquad\lim_{\mu\to\infty}\Phi_2(\beta,\eta,W,\mu,Z)=\frac{\beta}{2},\\
&\lim_{Z\to0^+}\Phi_2(\beta,\eta,W,\mu,Z)=\phi_2,\qquad\lim_{Z\to\infty}\Phi_2(\beta,\eta,W,\mu,Z)=\frac{\beta}{2}.
\end{aligned}
$$

Moreover, we have the following limits about the partial derivatives of the speed index functions:

$$
\begin{aligned}
\lim_{\mu\to 0^+}\frac{\partial\Phi_1}{\partial\mu} &= \alpha\left[Z\int_0^\infty \frac{1}{c}\xi(c)\mathrm{d}c-1\right]K(Z),\\
\lim_{\mu\to c_0}\frac{\partial\Phi_1}{\partial\mu} &= \frac{\alpha}{{c_0}^2}\exp\left(-\frac{Z}{c_0}\right)\int_0^\infty \xi(c)\left[\int_{-\infty}^0 (Z-x)\exp\left(\frac{c-c_0}{cc_0}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&\quad+\frac{\alpha}{{c_0}^2}\exp\left(-\frac{Z}{c_0}\right)\\
&\quad\cdot\int_0^\infty \xi(c)\left[\int_0^{cZ/(c+s(Z)c_0)}(Z-x)\exp\left(\frac{c+c_0}{cc_0}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c,\\
\lim_{Z\to 0^+}\frac{\partial\Phi_1}{\partial\mu} &= {\phi_1}'(\mu),\qquad \lim_{Z\to\infty}\frac{\partial\Phi_1}{\partial\mu}=0,\\
\lim_{\mu\to 0^+}\frac{\partial\Phi_1}{\partial Z} &= -\alpha K(Z),\\
\lim_{\mu\to c_0}\frac{\partial\Phi_1}{\partial Z} &= -\frac{\alpha}{c_0}\exp\left(-\frac{Z}{c_0}\right)\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-c_0}{cc_0}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&\quad-\frac{\alpha}{c_0}\exp\left(-\frac{Z}{c_0}\right)\int_0^\infty \xi(c)\left[\int_0^{cZ/(c+s(Z)c_0)}\exp\left(\frac{c+c_0}{cc_0}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c,\\
\lim_{Z\to 0^+}\frac{\partial\Phi_1}{\partial Z} &= -\frac{1}{\mu}\phi_1,\qquad \lim_{Z\to\infty}\frac{\partial\Phi_1}{\partial Z}=0,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{\mu\to 0^+}\frac{\partial\Phi_2}{\partial\mu} &= \beta\left[\int_0^\infty \tau\eta(\tau)\mathrm{d}\tau-1\right]W(-Z), &\lim_{\mu\to\infty}\frac{\partial\Phi_2}{\partial\mu}&=0,\\
\lim_{Z\to 0^+}\frac{\partial\Phi_2}{\partial\mu} &= {\phi_2}'(\mu), &\lim_{Z\to\infty}\frac{\partial\Phi_2}{\partial\mu}&=0,\\
\lim_{\mu\to 0^+}\frac{\partial\Phi_2}{\partial Z} &= \beta W(-Z), &\lim_{\mu\to\infty}\frac{\partial\Phi_2}{\partial Z}&=0,\\
\lim_{Z\to 0^+}\frac{\partial\Phi_2}{\partial Z} &= \frac{\beta}{\mu}\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu\tau}\exp\left(\frac{x}{\mu}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau, &\lim_{Z\to\infty}\frac{\partial\Phi_2}{\partial Z}&=0.
\end{aligned}
$$

To prove the existence and uniqueness of (μ_3, Z_3), we use the speed index functions Φ_1 and Φ_2 to construct the following auxiliary functions in $(0, c_0)\times(0,\infty)$. Define

$$
\begin{aligned}
\Gamma\mu, Z) &= \Phi_1(\alpha,\xi,K,\mu,0)+\Phi_2(\beta,\eta,W,\mu,Z),\\
\Lambda(\mu, Z) &= \Phi_1(\alpha,\xi,K,\mu,Z)+\Phi_2(\beta,\eta,W,\mu,0).
\end{aligned}
$$

Then, we have the partial derivatives

$$
\begin{aligned}
\frac{\partial\Gamma}{\partial\mu} &= \frac{\partial\Phi_1}{\partial\mu}(\alpha,\xi,K,\mu,0)+\frac{\partial\Phi_2}{\partial\mu}(\beta,\eta,W,\mu,Z)>0,\\
\frac{\partial\Gamma}{\partial Z} &= \frac{\partial\Phi_2}{\partial Z}(\beta,\eta,W,\mu,Z)>0,\\
\frac{\partial\Lambda}{\partial\mu} &= \frac{\partial\Phi_1}{\partial\mu}(\alpha,\xi,K,\mu,Z)+\frac{\partial\Phi_2}{\partial\mu}(\beta,\eta,W,\mu,0)>0,\\
\frac{\partial\Lambda}{\partial Z} &= \frac{\partial\Phi_1}{\partial Z}(\alpha,\xi,K,\mu,Z)<0,
\end{aligned}
$$

for all $(\mu, Z) \in (0, c_0)\times(0,\infty)$. By using implicit function theorem and the system of speed equations, we know that there exist two well defined functions $\mu = \mathcal{A}(Z)$ and $\mu = \mathcal{B}(Z)$ on $(0,\infty)$, such that $\Gamma(\mathcal{A}(Z), Z) = \frac{\alpha+\beta}{2} - \theta$ and $\Lambda(\mathcal{B}(Z), Z) = \frac{\alpha+\beta}{2} - \Theta$.

Differentiating the system of speed equations $\Gamma(\mathcal{A}(Z), Z) = \frac{\alpha+\beta}{2} - \theta$ and $\Lambda(\mathcal{B}(Z), Z) = \frac{\alpha+\beta}{2} - \Theta$ with respect to Z, we obtain

$$\frac{\partial\Gamma}{\partial\mu}(\mathcal{A}(Z), Z)\mathcal{A}'(Z) + \frac{\partial\Gamma}{\partial Z}(\mathcal{A}(Z), Z) = 0,$$
$$\frac{\partial\Lambda}{\partial\mu}(\mathcal{B}(Z), Z)\mathcal{B}'(Z) + \frac{\partial\Lambda}{\partial Z}(\mathcal{B}(Z), Z) = 0.$$

Solving these equations, we have

$$\mathcal{A}'(Z) = -\frac{(\partial\Gamma/\partial Z)(\mathcal{A}(Z), Z)}{(\partial\Gamma/\partial\mu)(\mathcal{A}(Z), Z)} < 0,$$
$$\mathcal{B}'(Z) = -\frac{(\partial\Lambda/\partial Z)(\mathcal{B}(Z), Z)}{(\partial\Lambda/\partial\mu)(\mathcal{B}(Z), Z)} > 0.$$

Thus, $\mathcal{A} = \mathcal{A}(Z)$ is a strictly decreasing function and $\mathcal{B} = \mathcal{B}(Z)$ is a strictly increasing function on $(0,\infty)$.

Consider the following equations

$$\Gamma(\mathcal{A}_0, 0) = \Phi_1(\alpha, \xi, K, \mathcal{A}_0, 0) + \Phi_2(\beta, \eta, W, \mathcal{A}_0, 0) = \frac{\alpha+\beta}{2} - \theta,$$
$$\Lambda(\mathcal{B}_0, 0) = \Phi_1(\alpha, \xi, K, \mathcal{B}_0, 0) + \Phi_2(\beta, \eta, W, \mathcal{B}_0, 0) = \frac{\alpha+\beta}{2} - \Theta,$$
$$\Gamma(\mathcal{A}_+, \infty) = \Phi_1(\alpha, \xi, K, \mathcal{A}_+, 0) + \Phi_2(\beta, \eta, W, \mathcal{A}_+, \infty) = \frac{\alpha+\beta}{2} - \theta,$$
$$\Lambda(\mathcal{B}_+, \infty) = \Phi_1(\alpha, \xi, K, \mathcal{B}_+, \infty) + \Phi_2(\beta, \eta, W, \mathcal{B}_+, 0) = \frac{\alpha+\beta}{2} - \Theta.$$

That is, the explicit equations

$$\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mathcal{A}_0}{c\mathcal{A}_0}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c + \beta\int_0^\infty \eta(\tau)\left[\int_{-\mathcal{A}_0\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau$$
$$+\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mathcal{A}_0\tau}\exp\left(\frac{x}{\mathcal{A}_0}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau = \frac{\alpha+\beta}{2} - \theta,$$
$$\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mathcal{B}_0}{c\mathcal{B}_0}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c + \beta\int_0^\infty \eta(\tau)\left[\int_{-\mathcal{B}_0\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau$$
$$+\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mathcal{B}_0\tau}\exp\left(\frac{x}{\mathcal{B}_0}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau = \frac{\alpha+\beta}{2} - \Theta,$$
$$\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mathcal{A}_+}{c\mathcal{A}_+}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c + \frac{\beta}{2} = \frac{\alpha+\beta}{2} - \theta,$$
$$-\frac{\alpha}{2} + \beta\int_0^\infty \eta(\tau)\left[\int_{-\mathcal{B}_+\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau$$
$$+\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mathcal{B}_+\tau}\exp\left(\frac{x}{\mathcal{B}_+}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau = \frac{\alpha+\beta}{2} - \Theta.$$

The existence and uniqueness of each of the solutions $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{A}_+$, $\mathcal{B}_+$ of the above equations in $(0, c_0)$ may be proved by using intermediate value theorem, mean value theorem and the conditions $0 < 2\theta < \alpha < \Theta < \frac{\alpha+\beta}{2}$, respectively. Therefore, there exist the following limits

$$\mathcal{A}_0 = \lim_{Z\to 0^+} \mathcal{A}(Z), \qquad \mathcal{B}_0 = \lim_{Z\to 0^+} \mathcal{B}(Z),$$
$$\mathcal{A}_+ = \lim_{Z\to \infty} \mathcal{A}(Z), \qquad \mathcal{B}_+ = \lim_{Z\to \infty} \mathcal{B}(Z).$$

Note that $\mathcal{A}_0 > \mathcal{B}_0$ and $\mathcal{A}_+ < \mathcal{B}_+$ due to the conditions $0 < \alpha(\Theta - \alpha) < \beta\theta$.

Therefore, we find that the two smooth functions $\mathcal{A} = \mathcal{A}(Z)$ and $\mathcal{B} = \mathcal{B}(Z)$ intersect exactly once at $(\mu_3, Z_3) = (\mathcal{A}(Z_3), Z_3) = (\mathcal{B}(Z_3), Z_3)$, such that $\Gamma(\mathcal{A}(Z_3), Z_3) = \frac{\alpha+\beta}{2} - \theta$ and $\Lambda(\mathcal{B}(Z_3), Z_3) = \frac{\alpha+\beta}{2} - \Theta$. The existence and uniqueness of the wave speed μ_3 of the large traveling wave front of (1.1) for the case $\theta < \Theta$ is proved.

2.5. The Existence and Uniqueness of the Large Traveling Wave Front

The main purpose of this subsection is to finish the proof of the existence and uniqueness of the large traveling wave front of (1.1) .

We have made the assumptions $U(0) = \theta$ and $U(Z) = \Theta$ and have proved that there exists a unique solution (μ_3, Z_3) to the system of speed equations. If there exists another number $z_1 \neq 0$, such that the large traveling wave front crosses the small threshold θ, then Lemma 5.2 (in Section 5) says that $U'(z_1) > 0$. By continuity and intermediate value theorem, there exists another number $z_2 \neq 0$, $0 < |z_2| < |z_1|$, such that $U(z_2) = \theta$ and $U'(z_2) < 0$. But this is a contradiction to Lemma 5.2. Therefore, the large traveling wave front crosses the small threshold θ only once. Similarly, it crosses the large threshold Θ only once, as expected.

Overall, for all synaptic couplings and for all probability density functions satisfying the conditions (1.5)-(1.16), including all synaptic couplings in the three classes (A), (B), (C), the large traveling wave front crosses the small threshold θ only once and it crosses the large threshold Θ only once. Therefore, the large traveling wave front really satisfies the conditions $U(z) < \theta$ for all $z < 0$, $U(0) = \theta$, $U'(0) > 0$ and $U(z) > \theta$ for all $z > 0$. Similarly, $U(z) < \Theta$ for all $z < Z_3$, $U(Z_3) = \Theta$, $U'(Z_3) > 0$ and $U(z) > \Theta$ for all $z > Z_3$. For the wave speed μ_3, the uniqueness of the large traveling wave front is true up to translation invariance. The proof of Theorem 1 is completely finished. □

2.6. The Traveling Wave Backs

The main purpose of this subsection is to establish the existence and uniqueness of each of the three traveling wave solutions of the nonlinear scalar integral differential equation (1.2) with three appropriate constants $w_k > 0$, under the assumptions (1.5)-(1.16). We want the wave speed of the k-th traveling wave front of (1.1) and the k-th traveling wave back of (1.2) to be the same, where $k = 1, 2, 3$. We will make use of the existence and uniqueness of each of the three traveling wave fronts of equation (1.1) to accomplish the existence and uniqueness of each of the three traveling wave backs of equation (1.2). Hence we will have to be very careful to select the constant w_k in (1.2).

(I) Suppose that $U = U(z)$ is a solution of the following nonlinear scalar integral differential equation

$$\mu_1 U' + U = \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_1}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c.$$

Define a function $V = V(z)$ by

$$U(z) - \theta = \theta - V(z).$$

Then

$$U'(z) = -V'(z), \qquad H(U-\theta) = H(\theta - V) = 1 - H(V - \theta).$$

Moreover, V solves the differential equation

$$\mu_1 V' + V + \alpha - 2\theta = \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(V\left(y - \frac{\mu_1}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c.$$

Let $w_1 = \alpha - 2\theta$. Then the first small traveling wave back of (1.2) is given by

$$U_{\mathrm{back}-1}(z) = 2\theta - U_{\mathrm{front}-1}(z).$$

(II) Suppose that $U = U(z)$ is a solution of the following nonlinear scalar integral differential equation

$$\mu_2 U' + U = \alpha + \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z-y) H(U(y - \mu_2\tau) - \Theta) \mathrm{d}y \right] \mathrm{d}\tau.$$

Define a function $V = V(z)$ by

$$U(z) - \Theta = \Theta - V(z).$$

Then

$$U'(z) = -V'(z), \qquad H(U-\Theta) = H(\Theta - V) = 1 - H(V - \Theta).$$

Moreover, V solves the differential equation

$$\mu_2 V' + V + 2\alpha + \beta - 2\Theta = \alpha + \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z-y) H(V(y - \mu_2\tau) - \Theta) \mathrm{d}y \right] \mathrm{d}\tau.$$

Let $w_2 = 2\alpha + \beta - 2\Theta = \beta - 2(\Theta - \alpha)$. Then the second traveling wave back of (1.2) is given by

$$U_{\mathrm{back}-2}(z) = 2\Theta - U_{\mathrm{front}-2}(z).$$

Now, let us establish the existence and uniqueness of the large traveling wave solution $u(x,t) = U_{\mathrm{back}-3}(x + \mu_3 t)$ of equation (1.2) by using two cases.

(III-1) Let $\theta = \Theta$. Suppose that $U = U(z)$ is a solution of the following nonlinear scalar integral differential equation

$$\begin{aligned} \mu_3 U' + U = {} & \alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_3}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ & + \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z-y) H(U(y - \mu_3\tau) - \theta) \mathrm{d}y \right] \mathrm{d}\tau. \end{aligned}$$

Define a function V by

$$U(z) - \theta = \theta - V(z).$$

Then

$$U'(z) = -V'(z), \qquad H(U-\theta) = H(\theta - V) = 1 - H(V-\theta).$$

Moreover, V solves the differential equation

$$\begin{aligned}&\mu_3 V' + V + \alpha + \beta - 2\theta \\ =&\alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(V\left(y - \frac{\mu_3}{c}|z-y|\right) - \theta\right) \mathrm{d}y\right] \mathrm{d}c \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z-y) H(V(y-\mu_3\tau) - \Theta) \mathrm{d}y\right] \mathrm{d}\tau.\end{aligned}$$

Let $w_3 = \alpha + \beta - 2\theta$. Then the third traveling wave back of (1.2) is given by

$$U_{\mathrm{back}-3}(z) = 2\theta - U_{\mathrm{front}-3}(z).$$

(III-2) Let $\theta < \Theta$. The large traveling wave back $u(x,t) = U(x + \mu_3 t)$ of equation (1.2) and the constant w_3 satisfy the equation

$$\begin{aligned}\mu_3 U' + U = \alpha &\int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_3}{c}|z-y|\right) - \theta\right) \mathrm{d}y\right] \mathrm{d}c \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z-y) H(U(y-\mu_3\tau) - \Theta) \mathrm{d}y\right] \mathrm{d}\tau - w_3.\end{aligned}$$

As before, make the following change of variable

$$\omega = y - \frac{\mu_3}{c}|z-y|.$$

Then we have

$$\begin{aligned}&\mu_3 U' + U \\ =&\alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} \frac{c}{c + s(z-\omega)\mu_3} K\left(\frac{c(z-\omega)}{c + s(z-\omega)\mu_3}\right) H(U(\omega) - \theta) \mathrm{d}\omega\right] \mathrm{d}c \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z - \mu_3\tau - y) H(U(y) - \Theta) \mathrm{d}y\right] \mathrm{d}\tau - w_3.\end{aligned}$$

Suppose that $U > \theta$ on $(-\infty, 0)$, $U(0) = \theta$, $U'(0) < 0$ and $U < \theta$ on $(0, \infty)$. Similarly, suppose that $U > \Theta$ on $(-\infty, -Z)$, $U(-Z) = \Theta$, $U'(-Z) < 0$ and $U < \Theta$ on $(-Z, \infty)$, where $Z > 0$ is a positive constant to be determined later. Then

$$\begin{aligned}\mu_3 U' + U = \alpha &\int_0^\infty \xi(c) \left[\int_{-\infty}^0 \frac{c}{c + s(z-\omega)\mu_3} K\left(\frac{c(z-\omega)}{c + s(z-\omega)\mu_3}\right) \mathrm{d}\omega\right] \mathrm{d}c \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{-Z} W(z - \mu_3\tau - y) \mathrm{d}y\right] \mathrm{d}\tau - w_3.\end{aligned}$$

Let

$$x = \frac{c(z-\omega)}{c + s(z-\omega)\mu_3}.$$

We have

$$\begin{aligned}\mu_3 U' + U = \alpha \int_0^\infty \xi(c) \left[\int_{cz/(c+s(z)\mu_3)}^\infty K(x)\mathrm{d}x \right] \mathrm{d}c \\ + \beta \int_0^\infty \eta(\tau) \left[\int_{z+Z-\mu_3\tau}^\infty W(x)\mathrm{d}x \right] \mathrm{d}\tau - w_3.\end{aligned}$$

Solving the differential equation, we find the solution

$$\begin{aligned}U(z) =& \alpha \int_0^\infty \xi(c) \left[\int_{cz/(c+s(z)\mu_3)}^\infty K(x)\mathrm{d}x \right] \mathrm{d}c \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{z+Z-\mu_3\tau}^\infty W(x)\mathrm{d}x \right] \mathrm{d}\tau \\ &+ \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^z \exp\left(\frac{x-z}{\mu_3}\right) \frac{c}{c+s(x)\mu_3} K\left(\frac{cx}{c+s(x)\mu_3}\right) \mathrm{d}x \right] \mathrm{d}c \\ &+ \beta \exp\left(-\frac{Z}{\mu_3}\right) \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{z+Z-\mu_3\tau} \exp\left(\frac{x-z}{\mu_3}\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau - w_3.\end{aligned}$$

The derivative is

$$\begin{aligned}U'(z) =& -\frac{\alpha}{\mu_3} \int_0^\infty \xi(c) \left[\int_{-\infty}^z \exp\left(\frac{x-z}{\mu_3}\right) \frac{c}{c+s(x)\mu_3} K\left(\frac{cx}{c+s(x)\mu_3}\right) \mathrm{d}x \right] \mathrm{d}c \\ &- \frac{\beta}{\mu_3} \exp\left(-\frac{Z}{\mu_3}\right) \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{z+Z-\mu_3\tau} \exp\left(\frac{x-z}{\mu_3}\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau.\end{aligned}$$

The compatible conditions are given by $U(0) = \theta$ and $U(-Z) = \Theta$. That is

$$\begin{aligned}&\alpha \int_0^\infty \xi(c) \left[\int_0^\infty K(x)\mathrm{d}x \right] \mathrm{d}c + \beta \int_0^\infty \eta(\tau) \left[\int_{Z-\mu_3\tau}^\infty W(x)\mathrm{d}x \right] \mathrm{d}\tau \\ &+ \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left(\frac{x}{\mu_3}\right) \frac{c}{c+s(x)\mu_3} K\left(\frac{cx}{c+s(x)\mu_3}\right) \mathrm{d}x \right] \mathrm{d}c \\ &+ \beta \exp\left(-\frac{Z}{\mu_3}\right) \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{Z-\mu_3\tau} \exp\left(\frac{x}{\mu_3}\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau - w_3 = \theta, \\ &\alpha \int_0^\infty \xi(c) \left[\int_{-cZ/(c+s(-Z)\mu_3)}^\infty K(x)\mathrm{d}x \right] \mathrm{d}c + \beta \int_0^\infty \eta(\tau) \left[\int_{-\mu_3\tau}^\infty W(x)\mathrm{d}x \right] \mathrm{d}\tau \\ &+ \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{-Z} \exp\left(\frac{x+Z}{\mu_3}\right) \frac{c}{c+s(x)\mu_3} K\left(\frac{cx}{c+s(x)\mu_3}\right) \mathrm{d}x \right] \mathrm{d}c \\ &+ \beta \exp\left(-\frac{Z}{\mu_3}\right) \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu_3\tau} \exp\left(\frac{x+Z}{\mu_3}\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau - w_3 = \Theta.\end{aligned}$$

To establish the existence and uniqueness of (w_3, Z), let us define the following auxiliary functions in $(0,\infty)\times(0,\infty)$. Define

$$\begin{aligned}
&\Gamma(W,Z)\\
=&\alpha\int_0^\infty \xi(c)\left[\int_0^\infty K(x)\mathrm{d}x\right]\mathrm{d}c+\beta\int_0^\infty \eta(\tau)\left[\int_{Z-\mu_3\tau}^\infty W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&+\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{x}{\mu_3}\right)\frac{c}{c+s(x)\mu_3}K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\\
&+\beta\exp\left(-\frac{Z}{\mu_3}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{Z-\mu_3\tau}\exp\left(\frac{x}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau-W,
\end{aligned}$$

and

$$\begin{aligned}
&\Lambda(W,Z)\\
=&\alpha\int_0^\infty \xi(c)\left[\int_{-cZ/(c+s(-Z)\mu_3)}^\infty K(x)\mathrm{d}x\right]\mathrm{d}c+\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu_3\tau}^\infty W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&+\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{x}{\mu_3}\right)\frac{c}{c+s(x-Z)\mu_3}K\left(\frac{c(x-Z)}{c+s(x-Z)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\\
&+\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu_3\tau}\exp\left(\frac{x}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau-W.
\end{aligned}$$

By using the system of compatible equations $\Gamma(W,Z)=\theta$ and $\Lambda(W,Z)=\Theta$, we obtain two well defined functions $W\equiv\mathcal{A}(Z)$ and $W\equiv\mathcal{B}(Z)$ on $(0,\infty)$, such that $\Gamma(\mathcal{A}(Z),Z)=\theta$ and $\Lambda(\mathcal{B}(Z),Z)=\Theta$. Differentiating the system of compatible equations $\Gamma(\mathcal{A}(Z),Z)=\theta$ and $\Lambda(\mathcal{B}(Z),Z)=\Theta$ with respect to Z, we obtain

$$\begin{aligned}
\frac{\partial\Gamma}{\partial W}(\mathcal{A}(Z),Z)\mathcal{A}'(Z)+\frac{\partial\Gamma}{\partial Z}(\mathcal{A}(Z),Z)=0,\\
\frac{\partial\Lambda}{\partial W}(\mathcal{B}(Z),Z)\mathcal{B}'(Z)+\frac{\partial\Lambda}{\partial Z}(\mathcal{B}(Z),Z)=0.
\end{aligned}$$

Therefore

$$\mathcal{A}'(Z)=-\frac{(\partial\Gamma/\partial Z)(\mathcal{A}(Z),Z)}{(\partial\Gamma/\partial W)(\mathcal{A}(Z),Z)}<0,\qquad \mathcal{B}'(Z)=-\frac{(\partial\Lambda/\partial Z)(\mathcal{B}(Z),Z)}{(\partial\Lambda/\partial W)(\mathcal{B}(Z),Z)}>0.$$

It is easy to see that $\mathcal{A}=\mathcal{A}(Z)$ is a strictly decreasing function of Z on $(0,\infty)$ and $\mathcal{B}=\mathcal{B}(Z)$ is a strictly increasing function of Z on $(0,\infty)$. There exist the following limits

$$\begin{aligned}
\mathcal{A}_0&=\lim_{Z\to0^+}\mathcal{A}(Z), & \mathcal{B}_0&=\lim_{Z\to0^+}\mathcal{B}(Z),\\
\mathcal{A}_+&=\lim_{Z\to\infty}\mathcal{A}(Z), & \mathcal{B}_+&=\lim_{Z\to\infty}\mathcal{B}(Z).
\end{aligned}$$

The limits $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{A}_+$, $\mathcal{B}_+$ satisfy the following equations, respectively

$$\begin{aligned}
&\frac{\alpha+\beta}{2}+\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu_3\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
&+\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mu_3}{c\mu_3}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu_3\tau}\exp\left(\frac{x}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau-\mathcal{A}_0=\theta,
\end{aligned}$$

$$\frac{\alpha+\beta}{2}+\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu_3\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau$$
$$+\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mu_3}{c\mu_3}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c$$
$$+\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu_3\tau} \exp\left(\frac{x}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau-\mathcal{B}_0=\Theta,$$
$$\frac{\alpha}{2}+\alpha\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mu_3}{c\mu_3}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c-\mathcal{A}_+=\theta,$$
$$\alpha+\frac{\beta}{2}+\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu_3\tau}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau$$
$$+\beta\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu_3\tau} \exp\left(\frac{x}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau-\mathcal{B}_+=\Theta.$$

It is easy to find that $\mathcal{A}_+ < \mathcal{B}_+$ and $\mathcal{A}_0 > \mathcal{B}_0$. Therefore, the two functions $\mathcal{A} = \mathcal{A}(Z)$ and $\mathcal{B} = \mathcal{B}(Z)$ intersect exactly once. Finally, we obtain a unique solution (W, Z) to the system of equations $\Gamma(W, Z) = \theta$ and $\Lambda(W, Z) = \Theta$. The existence and uniqueness of the large traveling wave back of the nonlinear scalar integral differential equation (1.2) are proved for the case $\theta < \Theta$. The proof of Theorem 1.2 is finished. □

3. The Stability Analysis

The main purpose of this section is to accomplish the stability of the traveling wave solutions of the nonlinear scalar integral differential equations (1.1) and (1.2). We will focus on the stability analysis of the large traveling wave front of equation (1.1). The stability analysis of the first two small traveling wave fronts of (1.1) and the stability analysis of the three traveling wave backs of (1.2) may be established very similarly. We are going to use the same assumptions (1.5)-(1.16) as we did for the existence analysis.

First of all, we will use linearization technique and the method of separation of variables to derive the associated eigenvalue problems corresponding to a family of linear differential operators $\mathcal{L}(\lambda)$ from $C^1(\mathbb{R})$ to $C^0(\mathbb{R})$. Then, we will study the solutions of the general eigenvalue problem $\mathcal{L}(\lambda)\psi + \kappa = \lambda\psi$. Next, we will construct the stability index functions (that is, the Evans functions) by using the solutions of the general eigenvalue problem. One very important point is that the complex number λ_0 is an eigenvalue of the eigenvalue problem if and only if λ_0 is a zero of the Evans function. We will prove a global strong maximum principle for the Evans functions to analyze the zeros of the eigenvalue problems. The stability of the traveling wave fronts is determined by the zeros of the Evans functions.

3.1. The Eigenvalue Problems and Solutions

The main purpose of this subsection is to derive an associated eigenvalue problem and to define a family of linear differential operators.

Set $P(z, t) = u(x, t)$, where $z = x + \mu_3 t$. Then the nonlinear scalar integral differential equation (1.1), that is

$$\frac{\partial u}{\partial t}+u=\alpha\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c$$
$$+\beta\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau$$

becomes

$$\frac{\partial P}{\partial t}+\mu_3\frac{\partial P}{\partial z}+P$$
$$=\alpha\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(z-y)H\left(P\left(y-\frac{\mu_3}{c}|z-y|,t-\frac{1}{c}|z-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c$$
$$+\beta\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-y)H(P(y-\mu_3\tau,t-\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau.$$

Let

$$\omega=y-\frac{\mu_3}{c}|z-y|.$$

Then

$$z-\omega=(z-y)\left[1+\frac{\mu_3}{c}s(z-y)\right]=(z-y)\left[1+\frac{\mu_3}{c}s(z-\omega)\right],$$

and

$$z-y=\frac{c}{c+s(z-\omega)\mu_3}(z-\omega),$$

and

$$\mathrm{d}y=\frac{c}{c+s(z-\omega)\mu_3}\mathrm{d}\omega-\frac{c\mu_3}{[c+s(z-\omega)\mu_3]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.$$

Therefore, the above equation becomes

$$\frac{\partial P}{\partial t}+\mu_3\frac{\partial P}{\partial z}+P$$
$$=\alpha\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} \frac{c}{c+s(z-\omega)\mu_3}K\left(\frac{c(z-\omega)}{c+s(z-\omega)\mu_3}\right)\right.$$
$$\left.\cdot\, H\left(P\left(\omega,t-\frac{|z-\omega|}{c+s(z-\omega)\mu_3}\right)-\theta\right)\mathrm{d}\omega\right]\mathrm{d}c$$
$$+\beta\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-\mu_3\tau-y)H(P(y,t-\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau.$$

The large traveling wave front is a stationary solution of this equation, that is

$$\mu_3U'+U=\alpha\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} \frac{c}{c+s(z-\omega)\mu_3}K\left(\frac{c(z-\omega)}{c+s(z-\omega)\mu_3}\right)H(U(\omega)-\theta)\mathrm{d}\omega\right]\mathrm{d}c$$
$$+\beta\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-\mu_3\tau-y)H(U(y)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau.$$

The linearization of this nonlinear scalar integral differential equation with respect to the large traveling wave front $U = U_{\text{front}-3}(z)$ is given by

$$\begin{aligned}&\frac{\partial p}{\partial t} + \mu_3 \frac{\partial p}{\partial z} + p \\ =&\frac{\alpha}{U'(0)} \int_0^\infty \xi(c) \left[\frac{c}{c+s(z)\mu_3} K\left(\frac{cz}{c+s(z)\mu_3} \right) p\left(0, t - \frac{|z|}{c+s(z)\mu_3}\right)\right] \mathrm{d}c \\ &+ \frac{\beta}{U'(Z_3)} \int_0^\infty \eta(\tau) \big[W(z-\mu_3\tau - Z_3) p(Z_3, t-\tau) \big] \mathrm{d}\tau.\end{aligned}$$

Suppose that $p(z,t) = \exp(\lambda t)\psi(z)$ is a solution of this linear differential equation, where λ is a complex number and ψ is a complex bounded continuously differentiable function defined on $\mathbb{R}$. Then we obtain the eigenvalue problem

$$\begin{aligned}&\mu_3\psi' + (\lambda+1)\psi \\ =&\frac{\alpha}{U'(0)} \left\{ \int_0^\infty \xi(c) \left[\frac{c}{c+s(z)\mu_3} K\left(\frac{cz}{c+s(z)\mu_3} \right)\right] \exp\left[- \frac{\lambda|z|}{c+s(z)\mu_3}\right] \mathrm{d}c \right\} \psi(\lambda, 0) \\ &+ \frac{\beta}{U'(Z_3)} \left\{ \int_0^\infty \eta(\tau) \big[W(z-\mu_3\tau - Z_3) \exp(-\lambda\tau) \big] \mathrm{d}\tau \right\} \psi(\lambda, Z_3).\end{aligned}$$

We define a family of linear differential operators $\mathcal{L}(\lambda) : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ by

$$\begin{aligned}&\mathcal{L}(\lambda)\psi = -\mu_3\psi' - \psi \\ &+ \frac{\alpha}{U'(0)} \left\{ \int_0^\infty \xi(c) \left[\frac{c}{c+s(z)\mu_3} K\left(\frac{cz}{c+s(z)\mu_3} \right)\right] \exp\left[- \frac{\lambda|z|}{c+s(z)\mu_3}\right] \mathrm{d}c \right\} \psi(\lambda, 0) \\ &+ \frac{\beta}{U'(Z_3)} \left\{ \int_0^\infty \eta(\tau) \big[W(z-\mu_3\tau - Z_3) \exp(-\lambda\tau) \big] \mathrm{d}\tau \right\} \psi(\lambda, Z_3).\end{aligned}$$

For the first two small traveling wave fronts of (1.1) , the associated linear differential operators $\mathcal{L}_1(\lambda) : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ and $\mathcal{L}_2(\lambda) : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ are defined by

$$\begin{aligned}&\mathcal{L}_1(\lambda)\psi = -\mu_1\psi' - \psi \\ &+ \frac{\alpha}{U'(0)} \left\{ \int_0^\infty \xi(c) \left[\frac{c}{c+s(z)\mu_1} K\left(\frac{cz}{c+s(z)\mu_1} \right)\right] \exp\left[- \frac{\lambda|z|}{c+s(z)\mu_1}\right] \mathrm{d}c \right\} \psi(\lambda, 0),\end{aligned}$$

and

$$\mathcal{L}_2(\lambda)\psi = -\mu_2\psi' - \psi + \frac{\beta}{U'(0)} \left\{ \int_0^\infty \eta(\tau) \big[W(z-\mu_2\tau) \exp(-\lambda\tau) \big] \mathrm{d}\tau \right\} \psi(\lambda, 0),$$

respectively.

The associated linear differential operators $\mathcal{L}_k(\lambda) : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ for the stability analysis of the three traveling wave backs of (1.2) may be defined very similarly. The main difference is to replace $U'(0)$ and $U'(Z_3)$ for the front of (1.1) with $|U'(0)|$ and $|U'(-Z_3)|$ for the back of (1.2), respectively.

The essential spectrum of the linear differential operator $\mathcal{L}_k(\lambda)$ is easy to find by using ideas in John Evans [25] and it is given by

$$\sigma_{\text{essential}}(\mathcal{L}_k(\lambda)) = \{\lambda \in \mathbb{C} : \text{Re}\lambda = -1\}, \qquad k = 1, 2, 3.$$

Define the open, simply connected domain $\Omega = \{\lambda \in \mathbb{C} : \text{Re}\lambda > -1\}$.

If there exists a complex number λ_0 and there exists a complex valued bounded continuous function $\psi_0 = \psi_0(\lambda_0, z)$ on $\mathbb{R}$, such that $\mathcal{L}(\lambda_0)\psi_0 = \lambda_0\psi_0$, then λ_0 is called an eigenvalue and $\psi_0 = \psi_0(\lambda_0, z)$ is called an eigenfunction of the eigenvalue problem.

Differentiating the following traveling wave equation

$$\begin{aligned}\mu_3 U' + U =& \alpha \int_0^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_3)} K(x)\mathrm{d}x\right] \mathrm{d}c \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu_3\tau-Z_3} W(x)\mathrm{d}x\right] \mathrm{d}\tau,\end{aligned}$$

with respect to z, we have

$$\begin{aligned}\mu_3 U'' + U' =& \alpha \int_0^\infty \xi(c) \left[\frac{c}{c+s(z)\mu_3} K\left(\frac{cz}{c+s(z)\mu_3}\right)\right] \mathrm{d}c \\ &+ \beta \int_0^\infty \eta(\tau) W(z-\mu_3\tau - Z_3)\mathrm{d}\tau.\end{aligned}$$

This means that $\lambda_0 = 0$ is an eigenvalue and $\psi_0 = U'(z)$ is an eigenfunction of the eigenvalue problem.

3.2. The Linearized Stability Criterion

Theorem 3.1. (I) *The nonlinear stability of a traveling wave front $U = U_k(z)$ of the nonlinear scalar integral differential equation (1.1) is equivalent to the linear stability.*

(II) *The traveling wave front $U = U_k(z)$ of the nonlinear scalar integral differential equation (1.1) is exponentially stable, if* $\max\{\mathrm{Re}\lambda\colon \lambda \in \sigma(\mathcal{L}_k(\lambda)),\ \lambda \neq 0\} \leq -C_k$ *and $\lambda_0 = 0$ is algebraically simple, where $\sigma(\mathcal{L}_k(\lambda))$ denotes the spectrum of $\mathcal{L}_k(\lambda)$ and $C_k > 0$ is a positive constant, for all $k = 1, 2, 3$.*

(III) *The traveling wave front $U = U_k(z)$ of (1.1) is unstable, if there exists an eigenvalue λ_0 with positive real part or if the neutral eigenvalue $\lambda = 0$ is not simple.*

Proof. It is standard and it is omitted. Please see [77]. □

We will not only study the eigenvalues and eigenfunctions of the eigenvalue problem $\mathcal{L}(\lambda)\psi = \lambda\psi$, but also we will study the simplicity of the neutral eigenvalue $\lambda_0 = 0$. Hence, let us study the solutions of a general eigenvalue problem $\mathcal{L}(\lambda)\psi + \kappa = \lambda\psi$, that is,

$$\begin{aligned}&\mu_3\psi' + (\lambda+1)\psi \\ =& \frac{\alpha}{U'(0)} \left\{\int_0^\infty \xi(c) \left[\frac{c}{c+s(z)\mu_3} K\left(\frac{cz}{c+s(z)\mu_3}\right)\right] \exp\left[-\frac{\lambda|z|}{c+s(z)\mu_3}\right] \mathrm{d}c\right\} \psi(\lambda, 0) \\ &+ \frac{\beta}{U'(Z_3)} \left\{\int_0^\infty \eta(\tau)\big[W(z-\mu_3\tau - Z_3)\exp(-\lambda\tau)\big]\mathrm{d}\tau\right\} \psi(\lambda, Z_3) + \kappa(z),\end{aligned}$$

where $\kappa = \kappa(z)$ is a complex bounded continuous function defined on $\mathbb{R}$.

3.3. The Solutions of the General Eigenvalue Problem

The main purpose of this subsection is to solve the general eigenvalue problem $\mathcal{L}(\lambda)\psi + \kappa = \lambda\psi$ for formal solutions and prepare to define the stability index functions.

Theorem 3.2. **(I)** *The solutions of the general eigenvalue problem* $\mathcal{L}(\lambda)\psi+\kappa = \lambda\psi$ *are given by*

$$\begin{aligned}
&\psi(\lambda, z) = C(\lambda)\exp\left(-\frac{\lambda+1}{\mu_3}z\right)\\
&+\frac{\alpha}{\mu_3 U'(0)}\left\{\int_0^\infty \xi(c)\left[\int_{-\infty}^z \frac{c}{c+s(x)\mu_3}\exp\left(\frac{\lambda+1}{\mu_3}(x-z)\right)\right.\right.\\
&\left.\left.\cdot\ \exp\left(-\frac{\lambda|x|}{c+s(x)\mu_3}\right)K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\right\}\psi(\lambda,0)\\
&+\frac{\beta}{\mu_3 U'(Z_3)}\left\{\int_0^\infty \eta(\tau)\left[\int_{-\infty}^z \exp\left(\frac{\lambda+1}{\mu_3}(x-z)\right)\right.\right.\\
&\left.\left.\cdot\exp(-\lambda\tau)W(x-\mu_3\tau-Z_3)\mathrm{d}x\right]\mathrm{d}\tau\right\}\psi(\lambda,Z_3)\\
&+\frac{1}{\mu_3}\int_{-\infty}^z \exp\left[\frac{\lambda+1}{\mu_3}(x-z)\right]\kappa(x)\mathrm{d}x,
\end{aligned}$$

where $C(\lambda)$ *is an appropriate complex constant to be determined later.*

(II) *The solution* $\psi = \psi(\lambda, z)$ *of the general eigenvalue problem* $\mathcal{L}(\lambda)\psi + \kappa = \lambda\psi$ *is bounded on* $\mathbb{R}$ *if and only if* $C(\lambda) = 0$.

Proof. (I) The representation of the solutions follows from Lemma 5.1 (in Subsection 4). (II) It is obviously true and the proof is omitted. □

Now, let us appropriately determine the complex constant $C(\lambda)$, which is closely related to the construction of the stability index functions of the nonlinear scalar integral differential equation (1.1) .

For the case $\theta = \Theta$, $Z_3 = 0$, letting $z = 0$ in the formal solution of the general eigenvalue problem $\mathcal{L}(\lambda)\psi + \kappa = \lambda\psi$, it is not difficult to find that

$$\begin{aligned}
C(\lambda) = &\left\{1-\frac{\alpha}{\mu_3 U'(0)}\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \frac{c}{c+s(x)\mu_3}\exp\left(\frac{\lambda+1}{\mu_3}x\right)\right.\right.\\
&\left.\cdot\exp\left(-\frac{\lambda|x|}{c+s(x)\mu_3}\right)K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\\
&\left.-\frac{\beta}{\mu_3 U'(0)}\int_0^\infty \eta(\tau)\left[\int_{-\infty}^0 \exp\left(\frac{\lambda+1}{\mu_3}x\right)\exp(-\lambda\tau)W(x-\mu_3\tau)\mathrm{d}x\right]\mathrm{d}\tau\right\}\psi(\lambda,0)\\
&-\frac{1}{\mu_3}\int_{-\infty}^0 \exp\left(\frac{\lambda+1}{\mu_3}x\right)\kappa(x)\mathrm{d}x\\
=&\left\{1-\frac{\alpha}{\mu_3 U'(0)}\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left((\lambda+1)\frac{c-\mu_3}{c\mu_3}x\right)\exp\left(\frac{\lambda}{c}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\beta}{\mu_3 U'(0)} \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu_3\tau} \exp\left(\frac{\lambda+1}{\mu_3}x\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau \Bigg\} \psi(\lambda,0) \\
&- \frac{1}{\mu_3} \int_{-\infty}^0 \exp\left(\frac{\lambda+1}{\mu_3}x\right) \kappa(x)\mathrm{d}x.
\end{aligned}
$$

For the case $\theta < \Theta$, $Z_3 > 0$, letting $z = 0$ and $z = Z_3$ in the formal solution of the general eigenvalue problem $\mathcal{L}(\lambda)\psi + \kappa = \lambda\psi$, respectively, we get

$$
\begin{aligned}
&\psi(\lambda,0) = C(\lambda) + \frac{\alpha}{\mu_3 U'(0)} \left\{ \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \frac{c}{c+s(x)\mu_3} \exp\left(\frac{\lambda+1}{\mu_3}x\right) \right.\right. \\
&\left.\left. \cdot\ \exp\left(-\frac{\lambda|x|}{c+s(x)\mu_3}\right) K\left(\frac{cx}{c+s(x)\mu_3}\right) \mathrm{d}x \right] \mathrm{d}c \right\} \psi(\lambda,0) \\
&+ \frac{\beta}{\mu_3 U'(Z_3)} \left\{ \int_0^\infty \eta(\tau) \left[\int_{-\infty}^0 \exp\left(\frac{\lambda+1}{\mu_3}x\right) \exp(-\lambda\tau) W(x-\mu_3\tau-Z_3)\mathrm{d}x \right] \mathrm{d}\tau \right\} \\
&\cdot \psi(\lambda,Z_3) + \frac{1}{\mu_3} \int_{-\infty}^0 \exp\left(\frac{\lambda+1}{\mu_3}x\right) \kappa(x)\mathrm{d}x \\
=&C(\lambda) + \frac{\alpha}{\mu_3 U'(0)} \left\{ \int_0^\infty \xi(c) \left[\int_{-\infty}^0 \exp\left((\lambda+1)\frac{c-\mu_3}{c\mu_3}x\right) \right.\right. \\
&\left.\left. \exp\left(\frac{\lambda}{c}x\right) K(x)\mathrm{d}x \right] \mathrm{d}c \right\} \psi(\lambda,0) + \frac{\beta}{\mu_3 U'(Z_3)} \exp\left(\frac{\lambda+1}{\mu_3}Z_3\right) \\
&\cdot \left\{ \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu_3\tau - Z_3} \exp\left(\frac{\lambda+1}{\mu_3}x\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau \right\} \psi(\lambda,Z_3) \\
&+ \frac{1}{\mu_3} \int_{-\infty}^0 \exp\left(\frac{\lambda+1}{\mu_3}x\right) \kappa(x)\mathrm{d}x,
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi(\lambda,Z_3) = C(\lambda) \exp\left(-\frac{\lambda+1}{\mu_3}Z_3\right) + \frac{\alpha}{\mu_3 U'(0)} \left\{ \int_0^\infty \xi(c) \left[\int_{-\infty}^{Z_3} \frac{c}{c+s(x)\mu_3} \right.\right. \\
&\left.\left. \cdot \exp\left(\frac{\lambda+1}{\mu_3}(x-Z_3)\right) \exp\left(-\frac{\lambda|x|}{c+s(x)\mu_3}\right) K\left(\frac{cx}{c+s(x)\mu_3}\right) \mathrm{d}x \right] \mathrm{d}c \right\} \psi(\lambda,0) \\
&+ \frac{\beta}{\mu_3 U'(Z_3)} \left\{ \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{Z_3} \exp\left(\frac{\lambda+1}{\mu_3}(x-Z_3)\right) \exp(-\lambda\tau) \right.\right. \\
&\left.\left. \cdot W(x-\mu_3\tau-Z_3)\mathrm{d}x \right] \mathrm{d}\tau \right\} \psi(\lambda,Z_3) + \frac{1}{\mu_3} \int_{-\infty}^{Z_3} \exp\left[\frac{\lambda+1}{\mu_3}(x-Z_3)\right] \kappa(x)\mathrm{d}x \\
=&C(\lambda) \exp\left(-\frac{\lambda+1}{\mu_3}Z_3\right) + \frac{\alpha}{\mu_3 U'(0)} \left\{ \int_0^\infty \xi(c) \left[\int_{-\infty}^{Z_3} \frac{c}{c+s(x)\mu_3} \right.\right. \\
&\left.\left. \cdot \exp\left(\frac{\lambda+1}{\mu_3}(x-Z_3)\right) \exp\left(-\frac{\lambda|x|}{c+s(x)\mu_3}\right) K\left(\frac{cx}{c+s(x)\mu_3}\right) \mathrm{d}x \right] \mathrm{d}c \right\} \psi(\lambda,0) \\
&+ \frac{\beta}{\mu_3 U'(Z_3)} \left\{ \int_0^\infty \eta(\tau) e^\tau \left[\int_{-\infty}^{-\mu_3\tau} \exp\left(\frac{\lambda+1}{\mu_3}x\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau \right\} \psi(\lambda,Z_3) \\
&+ \frac{1}{\mu_3} \int_{-\infty}^0 \exp\left(\frac{\lambda+1}{\mu_3}x\right) \kappa(x+Z_3)\mathrm{d}x.
\end{aligned}
$$

Finally, by using these equations and by canceling out $\psi(\lambda, Z_3)$, we obtain the valuable equation

$$\begin{aligned}
&\left\{1-\frac{\alpha}{\mu_3 U'(0)}\int_0^{\infty}\xi(c)\left[\int_{-\infty}^{0}\exp\left((\lambda+1)\frac{c-\mu_3}{c\mu_3}x\right)\exp\left(\frac{\lambda}{c}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\right\}\\
&\cdot\left\{1-\frac{\beta}{\mu_3 U'(Z_3)}\int_0^{\infty}\eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu_3\tau}\exp\left(\frac{\lambda+1}{\mu_3}x\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}\psi(\lambda,0)\\
&-\left\{\frac{\alpha}{\mu_3 U'(0)}\int_0^{\infty}\xi(c)\left[\int_{-\infty}^{Z_3}\frac{c}{c+s(x)\mu_3}\exp\left(\frac{\lambda+1}{\mu_3}(x-Z_3)\right)\right.\right.\\
&\left.\left.\cdot\ \exp\left(-\frac{\lambda|x|}{c+s(x)\mu_3}\right)K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\right\}\\
&\cdot\left\{\frac{\beta}{\mu_3 U'(Z_3)}\exp\left(\frac{\lambda+1}{\mu_3}Z_3\right)\int_0^{\infty}\eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu_3\tau-Z_3}\exp\left(\frac{\lambda+1}{\mu_3}x\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}\\
&\cdot\psi(\lambda,0)\\
=&\left\{C(\lambda)+\frac{1}{\mu_3}\int_{-\infty}^{0}\exp\left(\frac{\lambda+1}{\mu_3}x\right)\kappa(x)\mathrm{d}x\right\}\\
&\cdot\left\{1-\frac{\beta}{\mu_3 U'(Z_3)}\int_0^{\infty}\eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu_3\tau}\exp\left(\frac{\lambda+1}{\mu_3}x\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}\\
&+\left\{C(\lambda)\exp\left(-\frac{\lambda+1}{\mu_3}Z_3\right)+\frac{1}{\mu_3}\int_{-\infty}^{0}\exp\left(\frac{\lambda+1}{\mu_3}x\right)\kappa(x+Z_3)\mathrm{d}x\right\}\\
&\cdot\left\{\frac{\beta}{\mu_3 U'(Z_3)}\exp\left(\frac{\lambda+1}{\mu_3}Z_3\right)\int_0^{\infty}\eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu_3\tau-Z_3}\exp\left(\frac{\lambda+1}{\mu_3}x\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}.
\end{aligned}$$

3.4. The Stability Index Functions

The main purpose of this subsection is to define the stability index functions (that is, the Evans functions) in the right half complex plane $\Omega = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > -1\}$ by using the above valuable equation.

Definition 3.1. Define the stability index functions for the first two small traveling wave fronts of (1.1) by

$$\begin{aligned}
\mathcal{E}_1(\lambda) &= 1-\frac{\alpha}{\mu_1 U'(0)}\int_0^{\infty}\xi(c)\left[\int_{-\infty}^{0}\exp\left((\lambda+1)\frac{c-\mu_1}{c\mu_1}x\right)\exp\left(\frac{\lambda}{c}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c,\\
\mathcal{E}_2(\lambda) &= 1-\frac{\beta}{\mu_2 U'(0)}\int_0^{\infty}\eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu_2\tau}\exp\left(\frac{\lambda+1}{\mu_2}x\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau,
\end{aligned}$$

in the open simply connected domain $\Omega = \{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda > -1\}$.

Definition 3.2. (I) Define the stability index function for the large traveling wave front of (1.1) for the case $\theta = \Theta$ by

$$\begin{aligned}
\mathcal{E}_3(\lambda) =& 1-\frac{\alpha}{\mu_3 U'(0)}\left\{\int_0^{\infty}\xi(c)\left[\int_{-\infty}^{0}\exp\left((\lambda+1)\frac{c-\mu_3}{c\mu_3}x\right)\exp\left(\frac{\lambda}{c}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\right\}\\
&-\frac{\beta}{\mu_3 U'(0)}\left\{\int_0^{\infty}\eta(\tau)e^{\tau}\left[\int_{-\infty}^{-\mu_3\tau}\exp\left(\frac{\lambda+1}{\mu_3}x\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\},
\end{aligned}$$

in the open simply connected domain $\Omega = \{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda > -1\}$.
(II) Define the stability index function for the large traveling wave front of (1.1) for the case $\theta < \Theta$ by

$$\begin{aligned}
\mathcal{E}_3(\lambda) = &\left\{1 - \frac{\alpha}{\mu_3 U'(0)} \int_0^{\infty} \xi(c) \Big[\int_{-\infty}^{0} \exp\Big((\lambda+1)\frac{c-\mu_3}{c\mu_3}x\Big) \exp\big(\frac{\lambda}{c}x\big) K(x)\mathrm{d}x \Big] \mathrm{d}c \right\} \\
&\cdot \left\{1 - \frac{\beta}{\mu_3 U'(Z_3)} \int_0^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{-\mu_3\tau} \exp\left(\frac{\lambda+1}{\mu_3}x\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau \right\} \\
&- \left\{ \frac{\alpha}{\mu_3 U'(0)} \int_0^{\infty} \xi(c) \left[\int_{-\infty}^{Z_3} \frac{c}{c+s(x)\mu_3} \exp\left(\frac{\lambda+1}{\mu_3}(x-Z_3)\right) \right.\right. \\
&\cdot \left.\left. \exp\left(-\frac{\lambda|x|}{c+s(x)\mu_3}\right) K\left(\frac{cx}{c+s(x)\mu_3}\right) \mathrm{d}x \right] \mathrm{d}c \right\} \\
&\cdot \left\{ \frac{\beta}{\mu_3 U'(Z_3)} \exp\big(\frac{\lambda+1}{\mu_3}Z_3\big) \int_0^{\infty} \eta(\tau) e^{\tau} \Big[\int_{-\infty}^{-\mu_3\tau - Z_3} \exp\big(\frac{\lambda+1}{\mu_3}x\big) W(x)\mathrm{d}x \Big] \mathrm{d}\tau \right\},
\end{aligned}$$

in the open simply connected domain $\Omega = \{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda > -1\}$.

3.5. The Compatible Solutions of the General Eigenvalue Problem

The main purpose of this subsection is to find the compatible solutions of the general eigenvalue problem.

Definition 3.3. Define the following complex auxiliary functions in the open simply connected domain $\Omega = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > -1\}$:

$$\begin{aligned}
\mathcal{F}(\lambda) = &\left\{ \frac{1}{\mu_3} \int_{-\infty}^{0} \exp\left(\frac{\lambda+1}{\mu_3}x\right) \kappa(x)\mathrm{d}x \right\} \\
&\cdot \left\{1 - \frac{\beta}{\mu_3 U'(Z_3)} \int_0^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{-\mu_3\tau} \exp\left(\frac{\lambda+1}{\mu_3}x\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau \right\} \\
&+ \left\{ \frac{1}{\mu_3} \int_{-\infty}^{0} \exp\left(\frac{\lambda+1}{\mu_3}x\right) \kappa(x+Z_3)\mathrm{d}x \right\} \\
&\cdot \left\{ \frac{\beta}{\mu_3 U'(Z_3)} \exp\big(\frac{\lambda+1}{\mu_3}Z_3\big) \int_0^{\infty} \eta(\tau) e^{\tau} \Big[\int_{-\infty}^{-\mu_3\tau - Z_3} \exp\big(\frac{\lambda+1}{\mu_3}x\big) W(x)\mathrm{d}x \Big] \mathrm{d}\tau \right\}, \\
\mathcal{G}(\lambda) = &\left\{1 - \frac{\beta}{\mu_3 U'(Z_3)} \int_0^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{-\mu_3\tau} \exp\left(\frac{\lambda+1}{\mu_3}x\right) W(x)\mathrm{d}x \right] \mathrm{d}\tau \right\} \\
&+ \left\{ \frac{\beta}{\mu_3 U'(Z_3)} \exp\big(\frac{\lambda+1}{\mu_3}Z_3\big) \int_0^{\infty} \eta(\tau) e^{\tau} \Big[\int_{-\infty}^{-\mu_3\tau - Z_3} \exp\big(\frac{\lambda+1}{\mu_3}x\big) W(x)\mathrm{d}x \Big] \mathrm{d}\tau \right\} \\
&\cdot \exp\left(-\frac{\lambda+1}{\mu_3}Z_3\right).
\end{aligned}$$

The above valuable equation becomes

$$\mathcal{E}_3(\lambda)\psi(\lambda,0) = C(\lambda)\mathcal{G}(\lambda) + \mathcal{F}(\lambda).$$

Therefore, we have appropriately determined the complex constant $C(\lambda)$:

$$C(\lambda) = \frac{\mathcal{E}_3(\lambda)\psi(\lambda,0) - \mathcal{F}(\lambda)}{\mathcal{G}(\lambda)}.$$

Now, we find the compatible solutions of the general eigenvalue problem $\mathcal{L}(\lambda)\psi+\kappa = \lambda\psi$:

$$\psi(\lambda, z) = \frac{\mathcal{E}_3(\lambda)\psi(\lambda,0) - \mathcal{F}(\lambda)}{\mathcal{G}(\lambda)} \exp\left(-\frac{\lambda+1}{\mu_3}z\right)$$
$$+ \frac{\alpha}{\mu_3 U'(0)} \left\{\int_0^\infty \xi(c) \left[\int_{-\infty}^z \frac{c}{c+s(x)\mu_3} \exp\left(\frac{\lambda+1}{\mu_3}(x-z)\right)\right.\right.$$
$$\left.\left.\cdot \exp\left(-\frac{\lambda|x|}{c+s(x)\mu_3}\right) K\left(\frac{cx}{c+s(x)\mu_3}\right) \mathrm{d}x\right] \mathrm{d}c\right\} \psi(\lambda, 0)$$
$$+ \frac{\beta}{\mu_3 U'(Z_3)} \left\{\int_0^\infty \eta(\tau)\right.$$
$$\left.\cdot \left[\int_{-\infty}^z \exp\left(\frac{\lambda+1}{\mu_3}(x-z)\right) \exp(-\lambda\tau) W(x-\mu_3\tau - Z_3)\mathrm{d}x\right] \mathrm{d}\tau\right\} \psi(\lambda, Z_3)$$
$$+ \frac{1}{\mu_3}\int_{-\infty}^z \exp\left[\frac{\lambda+1}{\mu_3}(x-z)\right] \kappa(x)\mathrm{d}x.$$

3.6. The Properties of the Stability Index Functions

The main purpose of this subsection is to study the properties of the stability index functions.

Theorem 3.3. *(The relationships between the speed index functions and the stability index functions)*

(I) *The speed index function $\phi_1 = \phi_1(\mu)$ and the stability index function $\mathcal{E}_1 = \mathcal{E}_1(\lambda)$ for the first small traveling wave front $u(x,t) = U_{\mathrm{front}-1}(x+\mu_1 t)$ of (1.1) are related through the mathematical equation*

$$\mathcal{E}_1(\lambda) = 1 - \frac{1}{\phi_1(\mu_1)}\phi_1\left(\frac{\mu_1}{\lambda+1}\right).$$

(II) *The speed index function $\phi_2 = \phi_2(\mu)$ and the stability index function $\mathcal{E}_2 = \mathcal{E}_2(\lambda)$ for the second small traveling wave front $u(x,t) = U_{\mathrm{front}-2}(x+\mu_2 t)$ of (1.1) are related through the mathematical equation*

$$\mathcal{E}_2(\lambda) = 1 - \frac{1}{\phi_2(\mu_2)}\phi_2\left(\frac{\mu_2}{\lambda+1}\right).$$

(III) *The speed index functions $\phi_1 = \phi_1(\mu)$ and $\phi_2 = \phi_2(\mu)$ and the stability index function $\mathcal{E}_3 = \mathcal{E}_3(\lambda)$ for the large traveling wave front $u(x,t) = U_{\mathrm{front}-3}(x+\mu_3 t)$ of (1.1) for the case $\theta = \Theta$ are related through the mathematical equation*

$$\mathcal{E}_3(\lambda) = 1 - \frac{1}{\phi_1(\mu_3)}\phi_1\left(\frac{\mu_3}{\lambda+1}\right) - \frac{1}{\phi_2(\mu_3)}\phi_2\left(\frac{\mu_3}{\lambda+1}\right).$$

(IV) *The speed index functions $\phi_1 = \phi_1(\mu)$ and $\phi_2 = \phi_2(\mu)$ and the stability index function $\mathcal{E}_3 = \mathcal{E}_3(\lambda)$ for the large traveling wave front $u(x,t) = U_{\mathrm{front}-3}(x+\mu_3 t)$ of (1.1) for the case $\theta < \Theta$ are related through the mathematical equation*

$$\mathcal{E}_3(\lambda) = 1 - \frac{1}{\phi_1(\mu_3)}\phi_1\left(\frac{\mu_3}{\lambda+1}\right) - \frac{1}{\phi_2(\mu_3)}\phi_2\left(\frac{\mu_3}{\lambda+1}\right).$$

Proof. This is easy to prove by using Definitions 2.1, 3.1, 3.2. □

Theorem 3.4. *(The properties of the stability index functions) The following statements are correct, for all positive integers $k = 1, 2, 3$.*

(I) *The stability index functions $\mathcal{E}_1 = \mathcal{E}_1(\lambda)$, $\mathcal{E}_2 = \mathcal{E}_2(\lambda)$ and $\mathcal{E}_3 = \mathcal{E}_3(\lambda)$ are complex analytic functions of $\lambda \in \Omega$ and they are real-valued if λ is real.*

(II) *The complex number λ_0 is an eigenvalue of the eigenvalue problem $\mathcal{L}_k(\lambda)\psi = \lambda\psi$ if and only if λ_0 is a zero of the stability index function $\mathcal{E}_k = \mathcal{E}_k(\lambda)$, that is, $\mathcal{E}_k(\lambda_0) = 0$. In particular, $\mathcal{E}_k(0) = 0$.*

(III) *The algebraic multiplicity of any eigenvalue λ_0 of the eigenvalue problem $\mathcal{L}_k(\lambda)\psi = \lambda\psi$ is equal to the order of λ_0 as a zero of the stability index function $\mathcal{E}_k(\lambda)$.*

(IV) *The stability index functions $\mathcal{E}_1 = \mathcal{E}_1(\lambda)$, $\mathcal{E}_2 = \mathcal{E}_2(\lambda)$ and $\mathcal{E}_3 = \mathcal{E}_3(\lambda)$ enjoy the following limit*

$$\lim_{|\lambda| \to \infty} \mathcal{E}_k(\lambda) = 1,$$

in the right half plane $\{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda \geq 0\}$.

(V) *There hold the following results on the imaginary axis:*

$$\sup_{\lambda \in \mathbf{i}\mathbb{R}} |\mathcal{E}_k(\lambda)| = 1, \qquad \sup_{\lambda \in \mathbf{i}\mathbb{R}} |1 - \mathcal{E}_k(\lambda)| = 1.$$

(VI) *The real parts of the stability index functions enjoy the estimate*

$$Re\mathcal{E}_k(\lambda) > Re\mathcal{E}_k(0) = 0,$$

for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$, $Re\lambda \geq 0$ and $|\lambda| \leq c_k$, for some positive constant $c_k > 0$.

(VII) *There exists no nonzero eigenvalue of $\mathcal{L}_k(\lambda)\psi = \lambda\psi$ in the right half complex plane $\{\lambda \in \mathbb{C}\colon Re\lambda \geq 0\}$ for the traveling wave front $u(x,t) = U(x + \mu_k t)$ of (1.1). (VIII) The derivative of the stability index function $\mathcal{E}_k = \mathcal{E}_k(\lambda)$ at $\lambda_0 = 0$ is positive, that is, ${\mathcal{E}_k}'(0) > 0$.*

Proof. The proofs of (I), (II), (III), (IV) are standard and omitted. Please see [77]. The proof of (V) follows from Van Der Corput Lemma in [63]. The proofs of (VI), (VII) and (VIII) follow from Lemma 5.4 and Lemma 5.5 (in Section 4). In the general eigenvalue problem $\mathcal{L}(\lambda)\psi + \kappa = \lambda\psi$, letting $\lambda = 0$ and $\kappa(z) = U'(z)$, we find that all solutions of the eigenvalue problem $\mathcal{L}(0)\psi + U'(z) = 0$ are unbounded on $\mathbb{R}$. Therefore, $\lambda = 0$ is a simple eigenvalue. □

3.7. The Completion of the Stability Analysis

The proof of Theorem 1.3. It is finished by coupling together the results of Theorem 3.2, Theorem 3.3 and Theorem 3.4. □

The proof of Theorem 1.4. The main ideas in the rigorous mathematical analysis of the stability of the large traveling wave back of (1.2) are the same as those in the proof of Theorem 1.3. All technical details are very similar. The main difference is to replace $U'_{\mathrm{front}}(0)$ and $U'_{\mathrm{front}}(Z_3)$ by $|U'_{\mathrm{back}}(0)|$ and $|U'_{\mathrm{back}}(-Z_3)|$, respectively. □

4. Conclusion

4.1. Summary

We study the following nonlinear scalar integral differential equations

$$\begin{aligned}\frac{\partial u}{\partial t}+u=&\alpha\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c\\ &+\beta\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial t}+u=&\alpha\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c\\ &+\beta\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau-w_k,\end{aligned}$$

arising from synaptically coupled neuronal networks. We have accomplished the existence and stability of the traveling wave solutions of these model equations. Additionally, we have found sufficient conditions the neurobiological mechanisms (represented by the synaptic couplings, by the probability density functions, by the synaptic rate constants and by the thresholds) must satisfy, so that the traveling wave fronts of (1.1) and the traveling wave backs of (1.2) exist. The rigorous mathematical analysis of the traveling wave solutions of these nonlinear scalar integral differential equations involves many new methods and techniques.

For equation (1.1), we construct speed index functions and stability index functions. We built valuable relationships between the speed index functions and the stability index functions. These stability index functions are very important to establish the exponential stability of the traveling wave fronts of equation (1.1) and the traveling wave backs of equation (1.2).

Nonlinear scalar reaction diffusion equations may support stable traveling wave fronts, see [1, 3, 4, 13]. Nonlinear singularly perturbed systems of reaction diffusion equations support fast stable traveling pulse solutions and slow unstable traveling pulse solutions, see [41, 57, 58]. These results are similar to our main results - Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4, Theorem 4.2 (without proof, please see the next subsection).

Neurobiologists have found spiral waves in neuronal networks, see [32, 71, 72]. However, except for some numerical results [44], there has been no rigorous mathematical theory to explain or support the existence and stability of spiral waves. Mathematicians have discovered lurching waves in neuronal networks [68]. Up to today, there has been no corresponding mathematical theory to study the existence and stability of the lurching waves. These are important open problems in mathematical neuroscience worth of rigorous mathematical analysis.

4.2. Some Related Results

We present some interesting results related to the nonlinear scalar integral differential equation (1.1) and to the nonlinear singularly perturbed system of integral differential equations (1.3)-(1.4).

Theorem 4.1. *Suppose that the assumptions (1.5)-(1.16) hold. If* $\xi > 0$ *almost everywhere on some open interval* $(0, c_0)$*, for a positive constant* $c_0 > 0$*, then there exists only standing wave fronts to the nonlinear scalar integral differential equation (1.1). In another word, if there exists a traveling wave front with a positive wave speed* μ_0 *to the nonlinear scalar integral differential equation (1.1), then* $\xi = 0$ *almost everywhere on* $(0, c_0)$*, for some positive constant* $c_0 > 0$*.*

Proof. Suppose that there exists a traveling wave front $u(x,t) = U(x + \mu_0 t)$ to the nonlinear scalar integral differential equation

$$\begin{aligned}\frac{\partial u}{\partial t} + u =&\alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \Theta) \mathrm{d}y \right] \mathrm{d}\tau,\end{aligned}$$

where $\mu_0 > 0$ represents the wave speed and $z = x + \mu_0 t$ represents a moving coordinate. Then the traveling wave front and the wave speed μ_0 satisfy the equation

$$\begin{aligned}\mu_0 U' + U =&\alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ &+ \beta \int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(z-y) H(U(y - \mu_0 \tau) - \Theta) \mathrm{d}y \right] \mathrm{d}\tau.\end{aligned}$$

Suppose that the traveling wave front meets the boundary conditions

$$\lim_{z \to -\infty} U(z) = 0, \qquad \lim_{z \to \infty} U(z) = \alpha + \beta, \qquad \lim_{z \to \pm\infty} U'(z) = 0.$$

Suppose that $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$, $U'(0) > 0$ and $U > \theta$ on $(0, \infty)$. Suppose also that $U < \Theta$ on $(-\infty, Z_0)$, $U(Z_0) = \Theta$, $U'(Z_0) > 0$ and $U > \Theta$ on (Z_0, ∞), for some nonnegative constant $Z_0 \geq 0$, to be determined later. As before, $Z_0 = 0$ if $\theta = \Theta$ and $Z_0 > 0$ if $\theta < \Theta$. Now, for the first integral, we have

$$\begin{aligned}&\alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ =&\alpha \int_{\mu_0}^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ &+ \alpha \int_0^{\mu_0} \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ =&\alpha \int_{\mu_0}^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ &+ \alpha \int_0^{\mu_0} \xi(c) \left[\int_{-\infty}^z K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\ &+ \alpha \int_0^{\mu_0} \xi(c) \left[\int_z^\infty K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c.\end{aligned}$$

We will simplify the last three integrals by using three cases. (I) Let $0 < \mu_0 < c$. Let

$$\omega = y - \frac{\mu_0}{c}|z-y|.$$

Then

$$\begin{aligned} z-\omega &= z-y+\frac{\mu_0}{c}|z-y| \\ &= (z-y)\left[1+\frac{\mu_0}{c}s(z-y)\right] \\ &= (z-y)\left[1+\frac{\mu_0}{c}s(z-\omega)\right]. \end{aligned}$$

Hence

$$z-y=\frac{c}{c+s(z-\omega)\mu_0}(z-\omega),$$

and

$$\mathrm{d}y=\frac{c}{c+s(z-\omega)\mu_0}\mathrm{d}\omega-\frac{c\mu_0}{[c+s(z-\omega)\mu_0]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.$$

Therefore, we have

$$\begin{aligned} &\int_{\mathbb{R}} K(z-y)H\left(U\left(y-\frac{\mu_0}{c}|z-y|\right)-\theta\right)\mathrm{d}y \\ &= \int_{\mathbb{R}} \frac{c}{c+s(z-\omega)\mu_0}K\left(\frac{c(z-\omega)}{c+s(z-\omega)\mu_0}\right)H(U(\omega)-\theta)\mathrm{d}\omega \\ &= \int_0^{\infty} \frac{c}{c+s(z-\omega)\mu_0}K\left(\frac{c(z-\omega)}{c+s(z-\omega)\mu_0}\right)\mathrm{d}\omega \\ &= \int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x)\mathrm{d}x, \end{aligned}$$

where

$$x=\frac{c}{c+s(z-\omega)\mu_0}(z-\omega),$$

and

$$\mathrm{d}x=-\frac{c}{c+s(z-\omega)\mu_0}\mathrm{d}\omega+\frac{c\mu_0}{[c+s(z-\omega)\mu_0]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.$$

(II) Let $\mu_0>c$ and $z>y$. Let

$$\omega=y-\frac{\mu_0}{c}|z-y|.$$

Then

$$\begin{aligned} z-\omega &= z-y+\frac{\mu_0}{c}|z-y| \\ &= (z-y)\left[1+\frac{\mu_0}{c}s(z-y)\right] \\ &= (z-y)\left[1+\frac{\mu_0}{c}s(z-\omega)\right]. \end{aligned}$$

Hence

$$z-y=\frac{c}{c+s(z-\omega)\mu_0}(z-\omega),$$

and

$$\mathrm{d}y = \frac{c}{c+s(z-\omega)\mu_0}\mathrm{d}\omega - \frac{c\mu_0}{[c+s(z-\omega)\mu_0]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.$$

Moreover

$$\begin{aligned}
&\int_{-\infty}^{z} K(z-y)H\left(U\left(y-\frac{\mu_0}{c}|z-y|\right)-\theta\right)\mathrm{d}y \\
=&\int_{-\infty}^{z} \frac{c}{c+s(z-\omega)\mu_0}K\left(\frac{c(z-\omega)}{c+s(z-\omega)\mu_0}\right)H(U(\omega)-\theta)\mathrm{d}\omega \\
=&H(z)\int_{0}^{z} \frac{c}{c+s(z-\omega)\mu_0}K\left(\frac{c(z-\omega)}{c+s(z-\omega)\mu_0}\right)\mathrm{d}\omega \\
=&H(z)\int_{0}^{cz/(c+s(z)\mu_0)} K(x)\mathrm{d}x,
\end{aligned}$$

where

$$x = \frac{c}{c+s(z-\omega)\mu_0}(z-\omega),$$

and

$$\mathrm{d}x = -\frac{c}{c+s(z-\omega)\mu_0}\mathrm{d}\omega + \frac{c\mu_0}{[c+s(z-\omega)\mu_0)]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.$$

(III) Let $\mu_0 > c$ and $z < y$. Let

$$\omega = y - \frac{\mu_0}{c}|z-y|.$$

Then

$$\begin{aligned}
z-\omega &= z-y+\frac{\mu_0}{c}|z-y| \\
&= (z-y)\left[1+\frac{\mu_0}{c}s(z-y)\right] \\
&= (z-y)\left[1-\frac{\mu_0}{c}s(z-\omega)\right].
\end{aligned}$$

Hence

$$z-y \quad = \quad \frac{c}{c-s(z-\omega)\mu_0}(z-\omega),$$

and

$$\mathrm{d}y = \frac{c}{c-s(z-\omega)\mu_0}\mathrm{d}\omega + \frac{c\mu_0}{[c-s(z-\omega)\mu_0]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.$$

Moreover

$$\begin{aligned}
&\int_{z}^{\infty} K(z-y)H\left(U\left(y-\frac{\mu_0}{c}|z-y|\right)-\theta\right)\mathrm{d}y \\
=&\int_{-\infty}^{z} \frac{c}{c-s(z-\omega)\mu_0}K\left(\frac{c(z-\omega)}{c-s(z-\omega)\mu_0}\right)H(U(\omega)-\theta)\mathrm{d}\omega \\
=&H(z)\int_{0}^{z} \frac{c}{c-s(z-\omega)\mu_0}K\left(\frac{c(z-\omega)}{c-s(z-\omega)\mu_0}\right)\mathrm{d}\omega \\
=&H(z)\int_{cz/(c-s(z)\mu_0)}^{0} K(x)\mathrm{d}x,
\end{aligned}$$

where

$$x = \frac{c}{c - s(z-\omega)\mu_0}(z-\omega),$$

and

$$\mathrm{d}x = -\frac{c}{c - s(z-\omega)\mu_0}\mathrm{d}\omega - \frac{c\mu_0}{[c - s(z-\omega)\mu_0]^2}(z-\omega)s'(z-\omega)\mathrm{d}\omega.$$

Now the first integral in the traveling wave equation becomes

$$\begin{aligned}
&\alpha \int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(z-y) H\left(U\left(y - \frac{\mu_0}{c}|z-y| \right) - \theta \right) \mathrm{d}y \right] \mathrm{d}c \\
=&\alpha \int_{\mu_0}^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x)\mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_0^{\mu_0} \xi(c) \left[H(z) \int_0^{cz/(c+s(z)\mu_0)} K(x)\mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_0^{\mu_0} \xi(c) \left[H(z) \int_{cz/(c-s(z)\mu_0)}^0 K(x)\mathrm{d}x \right] \mathrm{d}c.
\end{aligned}$$

Now the above traveling wave equation becomes

$$\begin{aligned}
\mu_0 U' + U =&\alpha \int_{\mu_0}^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x)\mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_0^{\mu_0} \xi(c) \left[H(z) \int_0^{cz/(c+s(z)\mu_0)} K(x)\mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_0^{\mu_0} \xi(c) \left[H(z) \int_{cz/(c-s(z)\mu_0)}^0 K(x)\mathrm{d}x \right] \mathrm{d}c \\
&+ \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu_0\tau - Z_0} W(x)\mathrm{d}x \right] \mathrm{d}\tau.
\end{aligned}$$

This is a first order nonhomogeneous linear differential equation. Solving it by using the method of integrating factor, we obtain the solution

$$\begin{aligned}
U(z) =&\alpha \int_{\mu_0}^\infty \xi(c) \left[\int_{-\infty}^{cz/(c+s(z)\mu_0)} K(x)\mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_0^{\mu_0} \xi(c) \left[H(z) \int_0^{cz/(c+s(z)\mu_0)} K(x)\mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_0^{\mu_0} \xi(c) \left[H(z) \int_{cz/(c-s(z)\mu_0)}^0 K(x)\mathrm{d}x \right] \mathrm{d}c \\
&+ \beta \int_0^\infty \eta(\tau) \left[\int_{-\infty}^{z-\mu_0\tau - Z_0} W(x)\mathrm{d}x \right] \mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&- \alpha \int_{\mu_0}^{\infty} \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu_0}\right) \frac{c}{c+s(x)\mu_0} K\left(\frac{cx}{c+s(x)\mu_0}\right) \mathrm{d}x \right] \mathrm{d}c \\
&- \alpha \int_{0}^{\mu_0} \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu_0}\right) H(x) \frac{c}{c+s(x)\mu_0} K\left(\frac{cx}{c+s(x)\mu_0}\right) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_{0}^{\mu_0} \xi(c) \left[\int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu_0}\right) H(x) \frac{c}{c-s(x)\mu_0} K\left(\frac{cx}{c-s(x)\mu_0}\right) \mathrm{d}x \right] \mathrm{d}c \\
&- \beta \exp\left(\frac{Z_0}{\mu_0}\right) \int_{0}^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{z-\mu_0\tau-Z_0} \exp\left(\frac{x-z}{\mu_0}\right) W(x) \mathrm{d}x \right] \mathrm{d}\tau.
\end{aligned}
$$

Setting $z = 0$ and $z = Z_0$, $U(0) = \theta$ and $U(Z_0) = \Theta$, respectively, we have the system of compatible equations

$$
\begin{aligned}
&\alpha \int_{\mu_0}^{\infty} \xi(c) \left[\int_{-\infty}^{0} K(x) \mathrm{d}x \right] \mathrm{d}c + \beta \int_{0}^{\infty} \eta(\tau) \left[\int_{-\infty}^{-\mu_0\tau-Z_0} W(x) \mathrm{d}x \right] \mathrm{d}\tau \\
&- \alpha \int_{\mu_0}^{\infty} \xi(c) \left[\int_{-\infty}^{0} \exp\left(\frac{c-\mu_0}{c\mu_0} x\right) K(x) \mathrm{d}x \right] \mathrm{d}c \\
&- \beta \exp\left(\frac{Z_0}{\mu_0}\right) \int_{0}^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{-\mu_0\tau-Z_0} \exp\left(\frac{x}{\mu_0}\right) W(x) \mathrm{d}x \right] \mathrm{d}\tau = \theta,
\end{aligned}
$$

and

$$
\begin{aligned}
&\alpha \int_{\mu_0}^{\infty} \xi(c) \left[\int_{-\infty}^{cZ_0/(c+s(Z_0)\mu_0)} K(x) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_{0}^{\mu_0} \xi(c) \left[H(Z_0) \int_{0}^{cZ_0/(c+s(Z_0)\mu_0)} K(x) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_{0}^{\mu_0} \xi(c) \left[H(Z_0) \int_{cZ_0/(c-s(Z_0)\mu_0)}^{0} K(x) \mathrm{d}x \right] \mathrm{d}c \\
&+ \beta \int_{0}^{\infty} \eta(\tau) \left[\int_{-\infty}^{-\mu_0\tau} W(x) \mathrm{d}x \right] \mathrm{d}\tau \\
&- \alpha \int_{\mu_0}^{\infty} \xi(c) \left[\int_{-\infty}^{Z_0} \exp\left(\frac{x-Z_0}{\mu_0}\right) \frac{c}{c+s(x)\mu_0} K\left(\frac{cx}{c+s(x)\mu_0}\right) \mathrm{d}x \right] \mathrm{d}c \\
&- \alpha \int_{0}^{\mu_0} \xi(c) \left[\int_{-\infty}^{Z_0} \exp\left(\frac{x-Z_0}{\mu_0}\right) H(x) \frac{c}{c+s(x)\mu_0} K\left(\frac{cx}{c+s(x)\mu_0}\right) \mathrm{d}x \right] \mathrm{d}c \\
&+ \alpha \int_{0}^{\mu_0} \xi(c) \left[\int_{-\infty}^{Z_0} \exp\left(\frac{x-Z_0}{\mu_0}\right) H(x) \frac{c}{c-s(x)\mu_0} K\left(\frac{cx}{c-s(x)\mu_0}\right) \mathrm{d}x \right] \mathrm{d}c \\
&- \beta \exp\left(\frac{Z_0}{\mu_0}\right) \int_{0}^{\infty} \eta(\tau) e^{\tau} \left[\int_{-\infty}^{-\mu_0\tau} \exp\left(\frac{x-Z_0}{\mu_0}\right) W(x) \mathrm{d}x \right] \mathrm{d}\tau = \Theta.
\end{aligned}
$$

Of course, there exists no solution to this system of compatible equations, unless $\xi = 0$ on $(0, c_0)$. This is a contradiction to the initial assumption that there exists a traveling wave front. Therefore, there exists no such traveling wave solution $u(x,t) = U(x + \mu_0 t)$ with a positive wave speed $\mu_0 > 0$. The proof of Theorem 4.1 is finished. □

Theorem 4.2. *Suppose that the assumptions (1.5)-(1.16) hold. There exist three fast exponentially stable fast traveling pulse solutions*

$$(u(x,t), w(x,t)) = (U_{\text{fast-pulse}}(\varepsilon,\cdot), W_{\text{fast-pulse}}(\varepsilon,\cdot))$$

to the nonlinear singularly perturbed system of integral differential equations (1.3)-(1.4).

(I) *The first traveling pulse solution*

$$(u(x,t), w(x,t)) = (U_{\text{pulse}-1}(\varepsilon,\cdot), W_{\text{pulse}-1}(\varepsilon,\cdot))$$

satisfies the following traveling wave equations and boundary conditions

$$\begin{aligned}
&\mu_1(\varepsilon){U_1}' + U_1 + W_1 \\
=&\alpha\int_0^\infty \xi(c)\Big[\int_{\mathbb{R}} K(z-y)H\big(U_1\big(y-\frac{\mu_1(\varepsilon)}{c}|z-y|\big)-\theta\big)\mathrm{d}y\Big]\mathrm{d}c, \\
&\mu_1(\varepsilon){W_1}' = \varepsilon(U_1 - \gamma W_1), \\
&\lim_{z\to\pm\infty}(U_1(z), W_1(z)) = (0,0), \qquad \lim_{z\to\pm\infty}({U_1}'(z), {W_1}'(z)) = (0,0),
\end{aligned}$$

where $z = x + \mu_1(\varepsilon)t$. Additionally, there exists a unique positive constant $\mathcal{Z}_{\text{pulse}-1}(\varepsilon) > 0$, such that $U_1(\varepsilon, 0) = \theta$ and $U_1(\varepsilon, \mathcal{Z}_{\text{pulse}-1}(\varepsilon)) = \theta$, $U_1 > \theta$ on $(0, \mathcal{Z}_{\text{pulse}-1}(\varepsilon))$ and $U_1 < \theta$ on $(-\infty, 0) \cup (\mathcal{Z}_{\text{pulse}-1}(\varepsilon), \infty)$.

(II) *The second traveling pulse solution*

$$(u(x,t), w(x,t)) = (U_{\text{pulse}-2}(\varepsilon,\cdot), W_{\text{pulse}-2}(\varepsilon,\cdot))$$

satisfies the following traveling wave equations and boundary conditions

$$\begin{aligned}
&\mu_2(\varepsilon){U_2}' + U_2 + W_2 = \beta\int_0^\infty \eta(\tau)\Big[\int_{\mathbb{R}} W(z-y)H(U_2(y-\mu_2(\varepsilon)\tau)-\Theta)\mathrm{d}y\Big]\mathrm{d}\tau, \\
&\mu_2(\varepsilon){W_2}' = \varepsilon(U_2 - \gamma W_2), \\
&\lim_{z\to\pm\infty}(U_2(z), W_2(z)) = (0,0), \qquad \lim_{z\to\pm\infty}({U_2}'(z), {W_2}'(z)) = (0,0),
\end{aligned}$$

where $z = x + \mu_2(\varepsilon)t$. Additionally, there exists a unique positive constant $\mathcal{Z}_{\text{pulse}-2}(\varepsilon) > 0$, such that $U_2(\varepsilon, 0) = \Theta$ and $U_2(\varepsilon, \mathcal{Z}_{\text{pulse}-2}(\varepsilon)) = \Theta$, $U_2 > \Theta$ on $(0, \mathcal{Z}_{\text{pulse}-2}(\varepsilon))$ and $U_2 < \Theta$ on $(-\infty, 0) \cup (\mathcal{Z}_{\text{pulse}-2}(\varepsilon), \infty)$.

(III) *The large traveling pulse solution*

$$(u(x,t), w(x,t)) = (U_{\text{pulse}-3}(\varepsilon,\cdot), W_{\text{pulse}-3}(\varepsilon,\cdot))$$

satisfies the following traveling wave equations and boundary conditions

$$\begin{aligned}
&\mu_3(\varepsilon){U_3}' + U_3 + W_3 \\
=&\alpha\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(z-y)H\left(U_3\left(y-\frac{\mu_3(\varepsilon)}{c}|z-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c \\
&+\beta\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(z-y)H(U_3(y-\mu_3(\varepsilon)\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau, \\
&\mu_3(\varepsilon){W_3}' = \varepsilon(U_3 - \gamma W_3), \\
&\lim_{z\to\pm\infty}(U_3(z), W_3(z)) = (0,0), \qquad \lim_{z\to\pm\infty}({U_3}'(z), {W_3}'(z)) = (0,0),
\end{aligned}$$

where $z = x + \mu_3(\varepsilon)t$. Additionally, there exist three positive constants $\Gamma(\varepsilon) > 0$, $\Lambda(\varepsilon) > 0$ and $\mathcal{Z}_{\text{pulse}-3}(\varepsilon) > 0$, such that $U_3(\varepsilon, 0) = \theta$ and $U_3(\varepsilon, \mathcal{Z}_{\text{pulse}-3}(\varepsilon)) = \theta$, $U_3 > \theta$ on $(0, \mathcal{Z}_{\text{pulse}-3}(\varepsilon))$ and $U_3 < \theta$ on $(-\infty, 0) \cup (\mathcal{Z}_{\text{pulse}-3}(\varepsilon), \infty)$; similarly, $U_3(\Gamma(\varepsilon)) = \Theta$ and $U_3(\Lambda(\varepsilon)) = \Theta$, $U_3 > \Theta$ on $(\Gamma(\varepsilon), \Lambda(\varepsilon))$ and $U_3 < \Theta$ on $(-\infty, \Gamma(\varepsilon)) \cup (\Lambda(\varepsilon), \infty)$.

Proof. The results in Theorem 4.2 will be accomplished in another paper. □

5. Appendix - Some Technical Lemmas

The following technical lemmas have been used in this paper.

Lemma 5.1. *Let $\alpha > 0$, $\beta > 0$ and $\mu > 0$ be positive constants, let λ be a complex constant, such that $Re\lambda > -1$.*

(I) *Suppose that $N_1 = N_1(z)$ and $N_2 = N_2(z)$ are at least piecewise smooth functions defined on $\mathbb{R}$, such that*

$$\lim_{z\to-\infty} N_1(z) = 0, \qquad \lim_{z\to\infty} N_1(z) = 1, \qquad \lim_{z\to\pm\infty} {N_1}'(z) = 0,$$
$$\lim_{z\to-\infty} N_2(z) = 0, \qquad \lim_{z\to\infty} N_2(z) = 1, \qquad \lim_{z\to\pm\infty} {N_2}'(z) = 0.$$

Then the following boundary value problem

$$\mu U' + U = \alpha N_1(z) + \beta N_2(z),$$
$$\lim_{z\to-\infty} U(z) = 0, \qquad \lim_{z\to\infty} U(z) = \alpha + \beta,$$

has the bounded solution

$$\begin{aligned} U(z) =& \alpha N_1(z) + \beta N_2(z) - \alpha \int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu}\right) {N_1}'(x)\mathrm{d}x \\ &- \beta \int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu}\right) {N_2}'(x)\mathrm{d}x, \\ U'(z) =& \frac{\alpha}{\mu} \int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu}\right) {N_1}'(x)\mathrm{d}x + \frac{\beta}{\mu} \int_{-\infty}^{z} \exp\left(\frac{x-z}{\mu}\right) {N_2}'(x)\mathrm{d}x. \end{aligned}$$

(II) *Suppose that $\kappa = \kappa(z)$, $N_1 = N_1(\lambda, z)$ and $N_2 = N_2(\lambda, z)$ are complex at least piecewise smooth functions of z defined on $\mathbb{R}$, such that*

$$\lim_{z\to-\infty} N_1(\lambda, z) = 0, \qquad \lim_{z\to-\infty} N_2(\lambda, z) = 0,$$
$$\lim_{z\to\pm\infty} \frac{\partial N_1}{\partial z}(\lambda, z) = 0, \qquad \lim_{z\to\pm\infty} \frac{\partial N_2}{\partial z}(\lambda, z) = 0.$$

Then the following boundary value problem

$$\mu\psi' + (\lambda + 1)\psi = \alpha N_1(\lambda, z)\psi(\lambda, 0) + \beta N_2(\lambda, z)\psi(\lambda, Z) + \kappa(z),$$
$$\lim_{z\to-\infty} \psi(\lambda, z) = 0,$$

has the formal solution

$$\begin{aligned}\psi(\lambda,z)=&C(\lambda)\exp\left(-\frac{\lambda+1}{\mu}z\right)\\&+\frac{\alpha}{\mu}\left\{\int_{-\infty}^{z}\exp\left[\frac{\lambda+1}{\mu}(x-z)\right]N_1(\lambda,x)\mathrm{d}x\right\}\psi(\lambda,0)\\&+\frac{\beta}{\mu}\left\{\int_{-\infty}^{z}\exp\left[\frac{\lambda+1}{\mu}(x-z)\right]N_2(\lambda,x)\mathrm{d}x\right\}\psi(\lambda,Z)\\&+\frac{1}{\mu}\left\{\int_{-\infty}^{z}\exp\left[\frac{\lambda+1}{\mu}(x-z)\right]\kappa(x)\mathrm{d}x\right\}.\end{aligned}$$

Proof. The proof of (I) is simple and is omitted. Let us prove the result in (II).

Multiplying the differential equation by the integrating factor $\exp\left(\frac{\lambda+1}{\mu}z\right)$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\left\{\mu\exp\left(\frac{\lambda+1}{\mu}z\right)\psi(\lambda,z)\right\}\\=&\alpha\exp\left(\frac{\lambda+1}{\mu}z\right)N_1(\lambda,z)\psi(\lambda,0)\\&+\beta\exp\left(\frac{\lambda+1}{\mu}z\right)N_2(\lambda,z)\psi(\lambda,Z)+\exp\left(\frac{\lambda+1}{\mu}z\right)\kappa(z).\end{aligned}$$

Integrating this equation with respect to z from $-\infty$, we get

$$\begin{aligned}&\mu\exp\left(\frac{\lambda+1}{\mu}z\right)\psi(\lambda,z)\\=&\mu C(\lambda)+\alpha\left\{\int_{-\infty}^{z}\exp\left(\frac{\lambda+1}{\mu}x\right)N_1(\lambda,x)\mathrm{d}x\right\}\psi(\lambda,0)\\&+\beta\left\{\int_{-\infty}^{z}\exp\left(\frac{\lambda+1}{\mu}x\right)N_2(\lambda,x)\mathrm{d}x\right\}\psi(\lambda,Z)+\int_{-\infty}^{z}\exp\left(\frac{\lambda+1}{\mu}x\right)\kappa(x)\mathrm{d}x,\end{aligned}$$

where $C(\lambda)$ is a complex constant. Finally, we obtain the formal solution

$$\begin{aligned}&\psi(\lambda,z)\\=&C(\lambda)\exp\left(-\frac{\lambda+1}{\mu}z\right)+\frac{\alpha}{\mu}\left\{\int_{-\infty}^{z}\exp\left[\frac{\lambda+1}{\mu}(x-z)\right]N_1(\lambda,x)\mathrm{d}x\right\}\psi(\lambda,0)\\&+\frac{\beta}{\mu}\left\{\int_{-\infty}^{z}\exp\left[\frac{\lambda+1}{\mu}(x-z)\right]N_2(\lambda,x)\mathrm{d}x\right\}\psi(\lambda,Z)\\&+\frac{1}{\mu}\left\{\int_{-\infty}^{z}\exp\left[\frac{\lambda+1}{\mu}(x-z)\right]\kappa(x)\mathrm{d}x\right\}.\end{aligned}$$

The proof of Lemma 5.1 is finished. □

Lemma 5.2. *Let* $(\mu,Z)=(\mu_3,Z_3)$.

(I) *If there exists a real number* z_1*, such that the large traveling wave front of equation (1.1) satisfies* $U(z_1)=\theta$*, then* $U'(z_1)>0$*. If there exists a real number* z_2*, such that the large traveling wave front of equation (1.1) satisfies* $U(z_2)=\Theta$*, then* $U'(z_2)>0$.

(II) *If there exists a real number z_3, such that the large traveling wave back of equation (1.2) satisfies $U(z_3) = \theta$, then $U'(z_3) < 0$. If there exists a real number z_4, such that the large traveling wave back of equation (1.2) satisfies $U(z_4) = \Theta$, then $U'(z_4) < 0$.*

Proof. Suppose that there exist real numbers z_1 and z_2, such that the large traveling wave front crosses the thresholds at these points, namely, $U(z_1) = \theta$ and $U(z_2) = \Theta$, respectively. Then we find the derivatives

$$\begin{aligned}
U'(z_1) =& \frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{-\infty}^{z_1} \exp\left(\frac{x-z_1}{\mu_3}\right)\frac{c}{c+s(x)\mu_3}K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu_3}\exp\left(\frac{Z_3}{\mu_3}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{z_1-\mu_3\tau-Z_3}\exp\left(\frac{x-z_1}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz_1/(c+s(z_1)\mu_3)} K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu_3}\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z_1-\mu_3\tau-Z_3} W(x)\mathrm{d}x\right]\mathrm{d}\tau-\frac{\theta}{\mu_3}>0,\\
U'(z_2) =& \frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{-\infty}^{z_2} \exp\left(\frac{x-z_2}{\mu_3}\right)\frac{c}{c+s(x)\mu_3}K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu_3}\exp\left(\frac{Z_3}{\mu_3}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{z_2-\mu_3\tau-Z_3}\exp\left(\frac{x-z_2}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{-\infty}^{cz_2/(c+s(z_2)\mu_3)} K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu_3}\int_0^\infty \eta(\tau)\left[\int_{-\infty}^{z_2-\mu_3\tau-Z_3} W(x)\mathrm{d}x\right]\mathrm{d}\tau-\frac{\Theta}{\mu_3}>0.
\end{aligned}$$

Suppose that there exist real numbers z_3 and z_4, such that the large traveling wave back crosses the thresholds at these points, namely, $U(z_3) = \theta$ and $U(z_4) = \Theta$, respectively. Then we find the derivatives

$$\begin{aligned}
U'(z_3) =& -\frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{-\infty}^{z_3} \exp\left(\frac{x-z_3}{\mu_3}\right)\frac{c}{c+s(x)\mu_3}K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\\
&-\frac{\beta}{\mu_3}\exp\left(\frac{Z_3}{\mu_3}\right)\int_{-\infty}^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{z_3-\mu_3\tau-Z_3}\exp\left(\frac{x-z_3}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{cz_3/(c+s(z_3)\mu_3)}^{\infty} K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu_3}\int_0^\infty \eta(\tau)\left[\int_{z_3-\mu_3\tau-Z_3}^{\infty} W(x)\mathrm{d}x\right]\mathrm{d}\tau-\frac{\theta+w_3}{\mu_3}<0,\\
U'(z_4) =& -\frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{-\infty}^{z_4} \exp\left(\frac{x-z_4}{\mu_3}\right)\frac{c}{c+s(x)\mu_3}K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\\
&-\frac{\beta}{\mu_3}\exp\left(\frac{Z_3}{\mu_3}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{z_4-\mu_3\tau-Z_3}\exp\left(\frac{x-z_4}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{cz_4/(c+s(z_4)\mu_3)}^\infty K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu_3}\int_0^\infty \eta(\tau)\left[\int_{z_4-\mu_3\tau-Z_3}^\infty W(x)\mathrm{d}x\right]\mathrm{d}\tau-\frac{\Theta+w_3}{\mu_3}<0.
\end{aligned}
$$

In particular, for the large traveling wave front, we have

$$
\begin{aligned}
U'(0)=&\frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{-\infty}^0 \exp\left(\frac{c-\mu_3}{c\mu_3}x\right)K(x)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu_3}\exp\left(\frac{Z_3}{\mu_3}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu_3\tau-Z_3}\exp\left(\frac{x}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{1}{\mu_3}\left\{\frac{\alpha+\beta}{2}-\theta-\beta\int_0^\infty \eta(\tau)\left[\int_{-\mu_3\tau-Z_3}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau\right\}>0,
\end{aligned}
$$

and

$$
\begin{aligned}
U'(Z_3)=&\frac{\alpha}{\mu_3}\int_0^\infty \xi(c)\left[\int_{-\infty}^{Z_3}\exp\left(\frac{x-Z_3}{\mu_3}\right)\frac{c}{c+s(x)\mu_3}K\left(\frac{cx}{c+s(x)\mu_3}\right)\mathrm{d}x\right]\mathrm{d}c\\
&+\frac{\beta}{\mu_3}\exp\left(\frac{Z_3}{\mu_3}\right)\int_0^\infty \eta(\tau)e^\tau\left[\int_{-\infty}^{-\mu_3\tau}\exp\left(\frac{x-Z_3}{\mu_3}\right)W(x)\mathrm{d}x\right]\mathrm{d}\tau\\
=&\frac{1}{\mu_3}\left\{\frac{\alpha+\beta}{2}-\Theta+\alpha\int_0^\infty \xi(c)\left[\int_0^{cZ_3/(c+s(Z_3)\mu_3)}K(x)\mathrm{d}x\right]\mathrm{d}c\right\}\\
&-\frac{\beta}{\mu_3}\int_0^\infty \eta(\tau)\left[\int_{-\mu_3\tau-Z_3}^0 W(x)\mathrm{d}x\right]\mathrm{d}\tau>0.
\end{aligned}
$$

The proof of Lemma 5.2 is finished. □

Lemma 5.3. *Suppose that the function $U=U(z)$ satisfies the conditions $U(a)=\theta$ and $U'(a)>0$, $U(z)\neq\theta$, for all $z\neq a$, where a is a real constant. Then for all real functions K, κ and P defined in $\mathbb{R}$, if neglecting higher order terms and only keeping linear term, then we have*

$$
\begin{aligned}
&\int_{\mathbb{R}} K(y)H(P(y)-\theta)\mathrm{d}y-\int_{\mathbb{R}} K(y)H(U(y)-\theta)\mathrm{d}y\\
\approx&\frac{K(a)}{U'(a)}\big[P(a)-U(a)\big],\\
&\int_{\mathbb{R}} K(y)H(P(y,\kappa(y))-\theta)\mathrm{d}y-\int_{\mathbb{R}} K(y)H(U(y)-\theta)\mathrm{d}y\\
\approx&\frac{K(a)}{U'(a)}\big[P(a,\kappa(a))-U(a)\big].
\end{aligned}
$$

Proof. Without loss of generality, let $U'>0$ on $\mathbb{R}$. Let $r=U(z)-\theta$. Then the inverse function $z=V(r)$ exists, such that $r=U(V(r))-\theta$ for all r and $z=V(U(z)-\theta)$ for all z. Moreover $V(0)=a$ because $U(a)=\theta$. Furthermore,

$\mathrm{d}r = U'(z)\mathrm{d}z$, hence $\mathrm{d}z = \frac{1}{U'(z)}\mathrm{d}r = \frac{1}{U'(V(r))}\mathrm{d}r$. Therefore, we have

$$\begin{aligned}
&\int_{\mathbb{R}} K(y)H(P(y)-\theta)\mathrm{d}y - \int_{\mathbb{R}} K(y)H(U(y)-\theta)\mathrm{d}y \\
\approx &\int_{\mathbb{R}} K(y)H'(U(y)-\theta)\big[P(y)-U(y)\big]\mathrm{d}y \\
= &\int_{\mathbb{R}} \frac{K(V(r))}{U'(V(r))}\delta(r)\big[P(V(r))-U(V(r))\big]\mathrm{d}r \\
= &\frac{K(V(0))}{U'(V(0))}\big[P(V(0))-U(V(0))\big] = \frac{K(a)}{U'(a)}\big[P(a)-U(a)\big].
\end{aligned}$$

The second approximation may be proved similarly. The proof of Lemma 5.3 is finished now. □

Lemma 5.4. *Suppose that $\mathcal{E} = \mathcal{E}(\lambda)$ is a complex analytic function defined in the open simply connected domain $\Omega = \{\lambda \in \mathbb{C}\text{: } Re\lambda > -1\}$ in the complex plane $\mathbb{C}$. If $|\mathcal{E}(\lambda)|$ attains a global maximum at some interior point $\lambda_0 \in \Omega$, then $\mathcal{E}(\lambda)$ is a constant function in Ω.*

Proof. We write the complex analytic function as

$$\mathcal{E}(\lambda) = E_{\mathrm{real}}(\lambda) + \mathbf{i}E_{\mathrm{imag}}(\lambda), \qquad \lambda = x + \mathbf{i}y,$$

where $E_{\mathrm{real}}(\lambda)$ and $E_{\mathrm{imag}}(\lambda)$ are the real part and the imaginary part of the complex analytic function $\mathcal{E}(\lambda)$, respectively. As real harmonic functions of the real variables x and y, both parts satisfy the mean value formula:

$$\begin{aligned}
E_{\mathrm{real}}(\lambda_0) &= \left[\int_{B(\lambda_0,R)} E_{\mathrm{real}}(\lambda)\mathrm{d}\lambda\right] \Big/ \left[\int_{B(\lambda_0,R)} 1\mathrm{d}\lambda\right], \\
E_{\mathrm{imag}}(\lambda_0) &= \left[\int_{B(\lambda_0,R)} E_{\mathrm{imag}}(\lambda)\mathrm{d}\lambda\right] \Big/ \left[\int_{B(\lambda_0,R)} 1\mathrm{d}\lambda\right],
\end{aligned}$$

for all $B(\lambda_0, R) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < R\} \subset \Omega$. See Evans [26]. Therefore, the complex analytic function $\mathcal{E} = \mathcal{E}(\lambda)$ also satisfies the mean value formula:

$$\mathcal{E}(\lambda_0) = \left[\int_{B(\lambda_0,R)} \mathcal{E}(\lambda)\mathrm{d}\lambda\right] \Big/ \left[\int_{B(\lambda_0,R)} 1\mathrm{d}\lambda\right],$$

for all $B(\lambda_0, R) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < R\} \subset \Omega$, where the complex integral is defined by

$$\int_{B(\lambda_0,R)} \mathcal{E}(\lambda)\mathrm{d}\lambda \equiv \int_{B(\lambda_0,R)} E_{\mathrm{real}}(\lambda)\mathrm{d}x\mathrm{d}y + \mathbf{i}\int_{B(\lambda_0,R)} E_{\mathrm{imag}}(\lambda)\mathrm{d}x\mathrm{d}y.$$

If $|\mathcal{E}(\lambda_0)| = \sup_{\lambda \in B(\lambda_0,R)} |\mathcal{E}(\lambda)|$, that is, $|\mathcal{E}(\lambda)|$ attains a local maximum at an interior point $\lambda_0 \in \Omega$, then it is easy to see that

$$|\mathcal{E}(\lambda_0)| \leq \left[\int_{B(\lambda_0,R)} |\mathcal{E}(\lambda)|\mathrm{d}\lambda\right] \Big/ \left[\int_{B(\lambda_0,R)} 1\mathrm{d}\lambda\right] \leq |\mathcal{E}(\lambda_0)|.$$

Hence, $|\mathcal{E}(\lambda)| = |\mathcal{E}(\lambda_0)|$, for all $\lambda \in B(\lambda_0, R)$. Recall that the domain Ω is connected. Therefore, $|\mathcal{E}(\lambda)| = |\mathcal{E}(\lambda_0)|$ and $\mathcal{E}(\lambda) = \mathcal{E}(\lambda_0)$, for all $\lambda \in \Omega$. The proof of Lemma 5.4 is finished. □

Lemma 5.5. *Suppose that $\mathcal{E} = \mathcal{E}(\lambda)$ is a complex analytic function defined in the right half complex plane $\Omega = \{\lambda \in \mathbb{C} : Re\lambda > -1\}$.*

(I) *Suppose that $\mathcal{E}(0) = 0$, and*

$$\lim_{|\lambda|\to\infty} |\mathcal{E}(\lambda)| = 1, \qquad \lim_{|\lambda|\to\infty} \big|1 - \mathcal{E}(\lambda)\big| = 0,$$
$$\sup_{\mathbf{i}\mathbb{R}} |\mathcal{E}(\lambda)| = 1, \qquad \sup_{\mathbf{i}\mathbb{R}} \big|1 - \mathcal{E}(\lambda)\big| = 1.$$

Then $0 < |\mathcal{E}(\lambda)| < 1$ in $\{\lambda \in \mathbb{C} : Re\lambda > 0\}$.

(II) *Suppose that $\mathcal{E}(\lambda) = E_{\mathrm{real}}(\lambda) + \mathbf{i}E_{\mathrm{imag}}(\lambda)$. Suppose also that $E_{\mathrm{real}}(0) < E_{\mathrm{real}}(\lambda)$, for all $\lambda \neq 0$ with $Re\lambda \geq 0$ and $|\lambda| \leq c_0$, for some positive constant $c_0 > 0$, and that ${E_{\mathrm{imag}}}'(0) = 0$. Then*

$$\mathcal{E}'(0) > 0.$$

Proof. (I) Clearly, $\mathcal{E}(\lambda)$ is not a constant function. Thus, both $|\mathcal{E}(\lambda)|$ and $|1-\mathcal{E}(\lambda)|$ cannot attain a global maximum in Ω. In another word, the global maximum of $|\mathcal{E}(\lambda)|$ and $|1-\mathcal{E}(\lambda)|$ can only be attained on the boundary of Ω. Since $\sup |\mathcal{E}(\lambda)| = 1$ and $\sup |1 - \mathcal{E}(\lambda)| = 1$ over the boundary of the open simply connected domain Ω, we see $|\mathcal{E}(\lambda)| < 1$ and $|1 - \mathcal{E}(\lambda)| < 1$ inside Ω. Additionally, we have the estimates

$$1 - |\mathcal{E}(\lambda)| < |1 - \mathcal{E}(\lambda)| < 1,$$

for all λ with $\mathrm{Re}\lambda > 0$. Therefore, we find that

$$0 < |\mathcal{E}(\lambda)| < 1.$$

(II) We have

$$\mathcal{E}'(0) = {E_{\mathrm{real}}}'(0) + \mathbf{i}{E_{\mathrm{imag}}}'(0) = {E_{\mathrm{real}}}'(0) > 0.$$

The proof of the last inequality ${E_{\mathrm{real}}}'(0) > 0$ follows from the assumption and the Hopf Lemma [26]. The proof of Lemma 5.5 is finished. ☐

Lemma 5.6. *Suppose that the kernel function K satisfies the following conditions*

$$|K(x)| \leq C\exp(-\rho|x|), \qquad \text{on } \mathbb{R},$$
$$\int_{\mathbb{R}} K(x)\mathrm{d}x = 1, \qquad \int_{-\infty}^{0} K(x)\mathrm{d}x = \frac{1}{2},$$

for two positive constants $C > 0$ and $\rho > 0$. Then there holds the following estimate

$$\left|\int_{-\infty}^{0} \exp\left(\frac{\lambda + 1}{\mu}x\right) K(x)\mathrm{d}x\right| \leq \int_{-\infty}^{0} K(x)\mathrm{d}x,$$

for all positive constants $\mu > 0$ and for all complex constants $\lambda \neq 0$ with $Re\lambda \geq 0$.

Proof. First of all, the estimate is correct for nonnegative kernel functions $K \geq 0$ satisfying the above conditions. Second, the estimate is correct for all rapidly decreasing functions $K \in \Phi$, where

$$\Phi = \Big\{K \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \big[\exp(\rho|x|^2)|K(x)|\big] < \infty, \text{ for some positive constant } \rho > 0\Big\}.$$

Actually, let $\rho > 0$ be a positive parameter and let z be a real number. By using integration by parts for M times, we have

$$
\begin{aligned}
&\int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) K(x) \mathrm{d}x \\
=&\int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) \frac{\mathrm{d}}{\mathrm{d}x} \left[- \int_{x}^{0} K(y_1) \mathrm{d}y_1 \right] \mathrm{d}x \\
=&\exp(\rho x + \mathbf{i} xz) \Big[- \int_{x}^{0} K(y_1) \mathrm{d}y_1 \Big] \Big|_{-\infty}^{0} \\
&+ (\rho + \mathbf{i} z) \int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) \Big[\int_{x}^{0} K(y_1) \mathrm{d}y_1 \Big] \mathrm{d}x \\
=&(\rho + \mathbf{i} z) \int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) \left[\int_{x}^{0} K(y_1) \mathrm{d}y_1 \right] \mathrm{d}x \\
=&(\rho + \mathbf{i} z) \int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) \frac{\mathrm{d}}{\mathrm{d}x} \left[- \int_{x}^{0} \int_{y_2}^{0} K(y_1) \mathrm{d}y_1 \mathrm{d}y_2 \right] \mathrm{d}x \\
=& (\rho + \mathbf{i} z) \exp(\rho x + \mathbf{i} xz) \left\{ - \int_{x}^{0} \left[\int_{y_2}^{0} K(y_1) \mathrm{d}y_1 \right] \mathrm{d}y_2 \right\} \Bigg|_{-\infty}^{0} \\
&+ (\rho + \mathbf{i} z)^2 \int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) \left\{ \int_{x}^{0} \left[\int_{y_2}^{0} K(y_1) \mathrm{d}y_1 \right] \mathrm{d}y_2 \right\} \mathrm{d}x \\
=&(\rho + \mathbf{i} z)^2 \int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) \left\{ \int_{x}^{0} \left[\int_{y_2}^{0} K(y_1) \mathrm{d}y_1 \right] \mathrm{d}y_2 \right\} \mathrm{d}x = \cdots \\
=&(\rho + \mathbf{i} z)^M \int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) \\
&\cdot \left\{ \int_{x}^{0} \int_{y_M}^{0} \left[\cdots \int_{y_3}^{0} \int_{y_2}^{0} K(y_1) \mathrm{d}y_1 \mathrm{d}y_2 \cdots \right] \mathrm{d}y_{M-1} \mathrm{d}y_M \right\} \mathrm{d}x.
\end{aligned}
$$

Note that for all real numbers z, there holds the following estimate

$$
\begin{aligned}
&\left| \int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) \left\{ \int_{x}^{0} \int_{y_M}^{0} \left[\cdots \int_{y_3}^{0} \int_{y_2}^{0} K(y_1) \mathrm{d}y_1 \mathrm{d}y_2 \cdots \right] \mathrm{d}y_{M-1} \mathrm{d}y_M \right\} \mathrm{d}x \right| \\
\leq & \left| \int_{-\infty}^{0} \exp(\rho x) \left\{ \int_{x}^{0} \int_{y_M}^{0} \left[\cdots \int_{y_3}^{0} \int_{y_2}^{0} K(y_1) \mathrm{d}y_1 \mathrm{d}y_2 \cdots \right] \mathrm{d}y_{M-1} \mathrm{d}y_M \right\} \mathrm{d}x \right|.
\end{aligned}
$$

Therefore, we have the estimate

$$
\begin{aligned}
&\left| \int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) K(x) \mathrm{d}x \right| \\
=&\Bigg| (\rho + \mathbf{i} z)^M \int_{-\infty}^{0} \exp(\rho x + \mathbf{i} xz) \\
&\cdot \left\{ \int_{x}^{0} \int_{y_M}^{0} \left[\cdots \int_{y_3}^{0} \int_{y_2}^{0} K(y_1) \mathrm{d}y_1 \mathrm{d}y_2 \cdots \right] \mathrm{d}y_{M-1} \mathrm{d}y_M \right\} \mathrm{d}x \Bigg|
\end{aligned}
$$

$$\leq \left| (\rho + \mathbf{i}z)^M \int_{-\infty}^{0} \exp(\rho x) \right.$$
$$\left. \cdot \left\{ \int_x^0 \int_{y_M}^0 \left[\cdots \int_{y_3}^0 \int_{y_2}^0 K(y_1)\mathrm{d}y_1\mathrm{d}y_2 \cdots \right] \mathrm{d}y_{M-1}\mathrm{d}y_M \right\} \mathrm{d}x \right|$$
$$= \left| \left(\frac{\rho + \mathbf{i}z}{\rho} \right)^M \int_{-\infty}^{0} K(x)\mathrm{d}x \right|.$$

Letting $\rho \to \infty$, we get the estimate

$$\left| \int_{-\infty}^{0} \exp(\mathbf{i}xz)K(x)\mathrm{d}x \right| \leq \left| \int_{-\infty}^{0} K(x)\mathrm{d}x \right|.$$

Note that Φ is dense in $L^1(\mathbb{R})$. For any kernel function satisfying the assumptions in Lemma 5.6, there exists a sequence of rapidly decreasing functions $\{K_l : l = 1, 2, 3, \cdots\} \subset \Phi$, such that

$$\left| \int_{-\infty}^{0} \exp\left(\frac{\lambda + 1}{\mu} x \right) K_l(x)\mathrm{d}x \right| \leq \int_{-\infty}^{0} K_l(x)\mathrm{d}x,$$
$$\lim_{l \to \infty} \left| \int_{\mathbb{R}} |K_l(x) - K(x)|\mathrm{d}x \right| = 0.$$

Let $l \to \infty$, we finish the proof of the estimate in Lemma 5.6. □

5.1. The Nonlinear Scalar Integral Differential Equations and the Nonlinear Singularly Perturbed Systems of Integral Differential Equations Generalized by (1.1) and (1.3) - (1.4)

The nonlinear scalar integral differential equations (1.1) and the nonlinear singularly perturbed system of integral differential equations (1.3) -(1.4) arise from synaptically coupled neuronal networks. They generalize the following model equations

$$\frac{\partial u}{\partial t} + u = \alpha \int_{\mathbb{R}} K(x-y)H(u(y,t) - \theta)\mathrm{d}y.$$
$$\frac{\partial u}{\partial t} + u = \alpha \int_{\mathbb{R}} K(x-y)H(u(y,t) - \theta)\mathrm{d}y + \beta \int_{\mathbb{R}} W(x-y)H(u(y,t) - \Theta)\mathrm{d}y.$$
$$\begin{cases} \dfrac{\partial u}{\partial t} + u + w = \alpha \displaystyle\int_{\mathbb{R}} K(x-y)H(u(y,t) - \theta)\mathrm{d}y, \\ \dfrac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \end{cases}$$
$$\begin{cases} \dfrac{\partial u}{\partial t} + u + w = \alpha \displaystyle\int_{\mathbb{R}} K(x-y)H(u(y,t) - \theta)\mathrm{d}y \\ \qquad\qquad + \beta \displaystyle\int_{\mathbb{R}} W(x-y)H(u(y,t) - \Theta)\mathrm{d}y, \\ \dfrac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u + w = \alpha \displaystyle\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) \mathrm{d}y, \\ \dfrac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u + w = \alpha \displaystyle\int_0^\infty \xi(c) \left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) \mathrm{d}y\right] \mathrm{d}c, \\ \dfrac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \end{cases}$$

$$\frac{\partial u}{\partial t} + u + w = \beta \int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \Theta) \mathrm{d}y, \qquad \frac{\partial w}{\partial t} = \varepsilon(u - \gamma w).$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u + w = \beta \displaystyle\int_0^\infty \eta(\tau) \left[\int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \Theta) \mathrm{d}y\right] \mathrm{d}\tau, \\ \dfrac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u + w = \alpha \displaystyle\int_{\mathbb{R}} K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) \mathrm{d}y \\ \qquad\qquad\qquad + \beta \displaystyle\int_{\mathbb{R}} W(x-y) H(u(y, t-\tau) - \Theta) \mathrm{d}y, \\ \dfrac{\partial w}{\partial t} = \varepsilon(u - \gamma w). \end{cases}$$

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