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NUMERICAL APPROXIMATION OF THE PHASE-FIELD TRANSITION SYSTEM WITH NON-HOMOGENEOUS CAUCHY-NEUMANN BOUNDARY CONDITIONS IN BOTH UNKNOWN FUNCTIONS VIA FRACTIONAL STEPS METHOD*

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Abstract The paper concerns with the proof of the convergence for an iterative scheme of fractional steps type associated to the phase-field transition system endowed with non-homogeneous Cauchy-Neumann boundary conditions, in both unknown functions. The advantage of such method consists in simplifying the numerical computation necessary to be done in order to approximate the solution of nonlinear parabolic system. On the basis of this approach, a numerical algorithm in 2D case is introduced and an industrial implementation is made.

Keywords Boundary value problems for nonlinear parabolic PDE, stability and convergence of numerical method, finite element method, thermodynamics and heat transfer.

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1. Introduction

Consider the following nonlinear parabolic system in $Q = (0, T] \times \Omega$, where $T > 0$ and Ω is a bounded domain in $\mathbb{R}^n$ $(n \leq 3)$ whose boundary is smooth enough:

$$\begin{cases} C_p \dfrac{\partial}{\partial t} u + \dfrac{\ell}{2} \dfrac{\partial}{\partial t} \varphi = k \Delta u + f(t, x) & \text{on} \quad Q = (0, T] \times \Omega \\ \alpha \xi \dfrac{\partial}{\partial t} \varphi = \xi \Delta \varphi + \dfrac{1}{2\xi} (\varphi - \varphi^3) + s_\xi u + g(t, x) & \text{on} \quad Q, \end{cases} \tag{1.1}$$

subject to the non-homogeneous Cauchy-Neumann boundary conditions, in both unknown functions u and φ:

$$\begin{cases} k \dfrac{\partial}{\partial \nu} u + hu = w_1(t, x) & \text{on} \quad \Sigma = (0, T] \times \partial\Omega \\ \xi \dfrac{\partial}{\partial \nu} \varphi + c_0 \varphi = w_2(t, x) & \text{on} \quad \Sigma, \end{cases} \tag{1.2}$$

and the initial conditions:

$$u(0, x) = u_0(x), \quad \varphi(0, x) = \varphi_0(x) \quad \text{on} \quad \Omega, \tag{1.3}$$

where

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- $u(t,x)$ - represents the *reduced temperature distribution* on Q, i.e. $u(t,x) = \theta(t,x) - \theta_M$, with $\theta(t,x)$ representing the temperature of the material at $(t,x) \in Q$ and θ_M *the melting temperature* (the temperature at which solid and liquid may coexist in equilibrium, separated by a planar interface);
- $\varphi(t,x)$ - is the *phase function* (the order parameter) used to distinguish between the states (phases) of material which occupies the region Ω at every moment of time $t \in [0,T]$;
- $C_p = \rho c$; ρ - the *density*, c - the *specific heat capacity*;
- $\ell, \kappa, \alpha, \xi, h, c_0$ are physical parameters representing: the *latent heat*,the *thermal conductivity*, the *relaxation time*, the *measure of the interface thickness*, the *heat transfer coefficient*, a positive constant, respectively;
- $s_\xi = \frac{m[S]_E}{2\sigma} T_E$ is a bounded and positive quantity, expressed by positive and bounded physical values: $m = \int\limits_{-1}^{1} \big(2F(s)\big)^{\frac{1}{2}} ds$, $F(s) = \dfrac{1}{4}(s^2-1)^2$, $[S]_E$ - the *entropy difference between phases per volume*, σ - the *interfacial tension*, T_E - the *equilibrium melting temperature* (see [11]);
- p, q are given numbers assumed to satisfy

$$q \geq p \geq 2; \tag{1.4}$$

- $f \in L^p(Q)$, $g \in L^q(Q)$ are given functions (also, can be interpreted as *distributed control*);
- $w_1(t,x), w_2(t,x) \in W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)$ - are given functions depending on two variables which also can be interpreted as *boundary control*;
- $u_0(x) \in W_p^{2-\frac{2}{p}}(\Omega)$, $\varphi_0(x) \in W_\infty^{2-\frac{2}{q}}(\Omega)$, provided $k\frac{\partial}{\partial \nu} u_0(x) + h u_0(x) = w_1(0,x)$, $\xi \frac{\partial}{\partial \nu}\varphi_0(x) + c_0\varphi_0(x) = w_2(0,x)$ on $\partial\Omega$.

In the formulation of problem (1.1) we have started from the phase field equations describing the phenomenon of solidification (see [11]) to which we have added some new physics parameters, as well as appropriate boundary conditions, in order to cover a wide variety of industrial applications (see [18–20]).

The non-homogeneous Cauchy-Neumann boundary conditions in both unknown functions u and φ (see relation (1.2)), untreated until now in mathematical literature, include a broad class of complex phenomenas at $\partial\Omega$ and will thus allow the formulation of new boundary optimal control problems.

At the moment t the material is considered to be *liquid* if the phase function φ is close to $+1+\delta_1$ and $u(t,x) \geq 1+\delta_2$, while it is considered to be *solid* if the phase function φ is close to $-1-\delta_1$ and $u(t,x) \leq -1-\delta_2$, with δ_1, δ_2 prescribed positive numbers.

We define the *separating region* (the interface at the moment t) as being the set:

$$\Omega_t = \{x \in \Omega;\ |u(t,x)| \leq 1+\delta_2,\ |\varphi(t,x)| \leq 1+\delta_1\}.$$

Regarding the existence and regularity of solutions in (1.1)-(1.3), we have

Theorem 1.1. *Problem* (1.1)-(1.3) *has a unique solution* (u, φ) *with* $u \in W_p^{1,2}(Q)$ *and* $\varphi \in W_\nu^{1,2}(Q)$, *where* $\nu = min\{p, \mu\}$. *In addition* (u, φ) *satisfies*

$$\begin{aligned} & \|u\|_{W_p^{1,2}(Q)} + \|\varphi\|_{W_\nu^{2,1}(Q)} \\ \leq \quad & C \Big\{ 1 + \|u_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\varphi_0\|^{3-\frac{2}{q}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|w_1\|_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)} \\ & + \|w_2\|_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)} + \|f\|_{L^p(Q)} + \|g\|_{L^q(Q)} \Big\}, \end{aligned} \tag{1.5}$$

where the constant C *depends on* $|\Omega|$ *(the measure of* Ω*),* T, n, p, q *and physical parameters.*

Moreover, given any number $M > 0$, *if* (u_1, φ_1) *and* (u_2, φ_2) *are solutions of (1.1)-(1.3) for the same initial conditions, corresponding to the dates*

$$(f_1, g_1, w_1^1, w_2^1), (f_2, g_2, w_1^2, w_2^2) \in L^p(Q) \times L^q(Q) \times \left(W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma) \right)^2,$$

such that $\|\varphi_1\|_{L^\nu(Q)}$, $\|\varphi_2\|_{L^\nu(Q)} \leq M$, *then the estimate below holds*

$$\begin{aligned} & \|u_1 - u_2\|_{W_p^{1,2}(Q)} + \|\varphi_1 - \varphi_2\|_{W_\nu^{1,2}(Q)} \\ \leq \quad & C \Big\{ \|f_1 - f_2\|_{L^p(Q)} + \|g_1 - g_2\|_{L^q(Q)} + \|w_1^1 - w_1^2\|_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)} \\ & + \|w_2^1 - w_2^2\|_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)} \Big\}, \end{aligned} \tag{1.6}$$

where the constant C *depends on* $|\Omega|$, T, M, n, p, q *and physical parameters.*

Remark 1.1. The result established by Theorem 1.1 is also valid for the linear system (1.7)-(1.9).

The sketch proof of Theorem 1.1 can be found in [9].

The phase-field transition system (1.1)-(1.2) with constant physical parameters, subject to the non-homogeneous Cauchy-Neumann boundary conditions for unknown u, namely: $\frac{\partial}{\partial \nu} u + hu = w(t, x)$ on Σ, has been analyzed in [8] and [9]. Endowed with dynamic boundary conditions and *singular potentials*, the system (1.1)-(1.2) was treated in [12].

Numerical investigations of the phase-field subject to different boundary conditions, can be found in [2, 13, 15, 21], where finite difference scheme is used, and [8, 16, 18–20] which authors use finite element method (**fem**).

For other detailed discussions on the phase-field transition system we refer to [2, 8–13, 15–24] and references there in.

In order to approximate the nonlinear problem let's associate to (1.1)–(1.3) for every $\varepsilon > 0$ the following approximating scheme (see also [8, 15, 16, 22]):

$$\begin{cases} C_p \dfrac{\partial}{\partial t} u^\varepsilon + \dfrac{\ell}{2} \dfrac{\partial}{\partial t} \varphi^\varepsilon = k \Delta u^\varepsilon + f(t, x) & \text{in} \quad Q_i^\varepsilon = (i\varepsilon, (i+1)\varepsilon) \times \Omega \\ \alpha \xi \dfrac{\partial}{\partial t} \varphi^\varepsilon = \xi \Delta \varphi^\varepsilon + \dfrac{1}{2\xi} \varphi^\varepsilon + s_\xi u^\varepsilon + g(t, x) & \text{in} \quad Q_i^\varepsilon, \end{cases} \tag{1.7}$$

$$\begin{cases} k \dfrac{\partial}{\partial \nu} u^\varepsilon + h u^\varepsilon = w_1(t, x) & \text{on} \quad \Sigma_i^\varepsilon = (i\varepsilon, (i+1)\varepsilon) \times \partial\Omega \\ \xi \dfrac{\partial}{\partial \nu} \varphi^\varepsilon + c_0 \varphi^\varepsilon = w_2(t, x) & \text{on} \quad \Sigma_i^\varepsilon, \end{cases} \tag{1.8}$$

$$\begin{cases} u^\varepsilon_+(i\varepsilon,x)=u^\varepsilon_-(i\varepsilon,x) \quad u^\varepsilon_-(0,x)=u_0(x) & \text{on} \quad \Omega \\ \varphi^\varepsilon_+(i\varepsilon,x)=z((i+1)\varepsilon,\varphi^\varepsilon_-(i\varepsilon,x)) & \text{on} \quad \Omega, \end{cases} \tag{1.9}$$

below where $z(\cdot,\varphi^\varepsilon_-(i\varepsilon,x))$ is the solution of Cauchy problem:

$$\begin{cases} z'(s)+\dfrac{1}{2\xi}z^3(s)=0, \; s\in(i\varepsilon,(i+1)\varepsilon), \\ z(i\varepsilon)=\varphi^\varepsilon_-(i\varepsilon,x), \; \varphi^\varepsilon_-(0,x)=\varphi_0(x), \end{cases} \tag{1.10}$$

for $i=0,1,\cdots,M_\varepsilon-1$, with $M_\varepsilon=\left[\frac{T}{\varepsilon}\right]$, $Q^\varepsilon_{M_\varepsilon-1}=[(M_\varepsilon-1)\varepsilon,T]\times\Omega$, $\varphi^\varepsilon_+(i\varepsilon,x)=\lim\limits_{t\downarrow i\varepsilon}\varphi^\varepsilon(t,x)$ and $\varphi^\varepsilon_-(i\varepsilon,x)=\lim\limits_{t\uparrow i\varepsilon}\varphi^\varepsilon(t,x)$.

Definition 1.1. By weak solution of the nonlinear system (1.1)-(1.3) we mean a pair of functions $u,\varphi\in L^2([0,T];H^1(\Omega))\cap L^\infty(Q)$ which satisfies (1.1)-(1.3) in the following sense:

$$\begin{aligned} &\int\limits_Q\left(C_p\frac{\partial}{\partial t}u+\frac{\ell}{2}\frac{\partial}{\partial t}\varphi\right)\phi\,dx\,dt+k\int\limits_Q\nabla u\nabla\phi\,dx\,dt+h\int\limits_\Sigma u\phi d\gamma\,dt \\ &=\int\limits_\Sigma w_1\phi d\gamma\,dt+\int\limits_Q f\phi dx\,dt, \end{aligned} \tag{1.11}$$

$$\begin{aligned} &\alpha\xi\int\limits_Q\left(\frac{\partial}{\partial t}\varphi\right)\psi\,dx\,dt+\xi\int\limits_Q\nabla\varphi\nabla\psi dx\,dt+c_0\int\limits_\Sigma\varphi\psi d\gamma\,dt \\ &=\int\limits_\Sigma w_2\psi d\gamma\,dt+\frac{1}{2\xi}\int\limits_Q(\varphi-\varphi^3)\psi\,dx\,dt+s_\xi\int\limits_Q u\psi\,dx\,dt+\int\limits_Q g\psi\,dx\,dt, \end{aligned} \tag{1.12}$$

$\forall\phi,\ \psi\in L^2([0,T];H^1(\Omega))$ and $u(0,x)=u_0(x)$, $\varphi(0,x)=\varphi_0(x)$ in Ω.

Definition 1.2. By weak solution of the linear system (1.7)-(1.9) we mean a pair of functions $u,\varphi\in L^2([0,T];H^1(\Omega))\cap L^\infty(Q)$ which satisfies (1.7)-(1.9) in the following sense:

$$\begin{aligned} &\int\limits_Q\left(C_p\frac{\partial}{\partial t}u^\varepsilon+\frac{\ell}{2}\frac{\partial}{\partial t}\varphi^\varepsilon\right)\phi\,dx\,dt+k\int\limits_Q\nabla u^\varepsilon\nabla\phi\,dx\,dt+h\int\limits_\Sigma u^\varepsilon\phi d\gamma\,dt \\ &=\int\limits_\Sigma w_1\phi d\gamma\,dt+\int\limits_Q f\phi dx\,dt, \end{aligned} \tag{1.13}$$

$$\begin{aligned} &\alpha\xi\int\limits_Q\left(\frac{\partial}{\partial t}\varphi^\varepsilon\right)\psi\,dx\,dt+\xi\int\limits_Q\nabla\varphi^\varepsilon\nabla\psi dx\,dt+c_0\int\limits_\Sigma\varphi^\varepsilon\psi d\gamma\,dt \\ &=\int\limits_\Sigma w_2\psi d\gamma\,dt+\frac{1}{2\xi}\int\limits_Q\varphi^\varepsilon\psi\,dx\,dt+s_\xi\int\limits_Q u^\varepsilon\psi\,dx\,dt+\int\limits_Q g\psi\,dx\,dt. \end{aligned} \tag{1.14}$$

$\forall\phi,\ \psi\in L^2([0,T];H^1(\Omega))$ and $u^\varepsilon_-(0,x)=u_0(x)$, $\varphi^\varepsilon_-(0,x)=\varphi_0(x)$ in Ω.

The symbol $\int\limits_Q$ above denotes the duality between $L^2([0,T];H^1(\Omega))$ and $L^2([0,T];H^1(\Omega)')$.

Remark 1.2. We choose $u, \varphi \in L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)$ in Definitions 1.1 and 1.2 making use of the continuous embedding $W_p^{1,2}(Q) \subset L^\infty(Q)$ when $n = 2,\ p > 2$ (see Theorem 1.1) which is relevant for industrial applications. Such a choice can be made also in the case $n = 3,\ p > \frac{5}{2}$ (see also [9]).

Throughout this paper when is not clearly precised, we will denote by C a constant which may change from line to line.

This paper is divided as follows: we start by giving the convergence of the linear approximating scheme (1.7)-(1.9) associated to the nonlinear transition system (1.1)-(1.3) and finish by a numerical algorithm in the 2D case and industrial implementation.

2. Convergence and weak stability of the approximating scheme

In this section, we will prove the convergence of the iterative scheme (1.7)-(1.10) of fractional steps type for the phase-field transition system (1.1)-(1.3). We have

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n \quad (n \leq 3)$ be a bounded domain with a smooth boundary. Assume that $u_0(x) \in W_p^{2-\frac{2}{p}}(\Omega)$, $\varphi_0(x) \in W_\infty^{2-\frac{2}{q}}(\Omega)$, satisfying $k\frac{\partial}{\partial\nu}u_0(x) + hu_0(x) = w_1(0,x)$, $\xi\frac{\partial}{\partial\nu}\varphi_0(x) + c_0\varphi_0(x) = w_2(0,x)$ on $\partial\Omega$ and $w_1(t,x), w_2(t,x) \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$. Let $(u^\varepsilon,\ \varphi^\varepsilon)$ be the solution of the approximating scheme* (1.7)-(1.9)*. Then for $\varepsilon \to 0$, one has*

$$(u^\varepsilon,\ \varphi^\varepsilon) \to (u^\star,\ \varphi^\star) \quad \text{strongly in} \quad L^2(\Omega) \quad \text{for any} \quad t \in (0,T], \tag{2.1}$$

where $u^\star,\ \varphi^\star \in L^2([0,T]; H^1(\Omega))$ is the weak solution to the nonlinear phase transition system (1.1)-(1.3)*.*

The following lemmas, which targets the Cauchy problem (1.10) and which are very useful in the proof of the main result of this Section (Theorem 2.1) were established for the first time in the work [16]. For reader convenience we fully reproduce their proofs.

Lemma 2.1. *If $\varphi_-^\varepsilon(i\varepsilon,x) \in L^\infty(\Omega)$, $i = 0, 1, ..., M_\varepsilon - 1$, then $z((i+1)\varepsilon,x) \in L^\infty(\Omega)$.*

Proof. From the Cauchy problem (1.10), using the method of separation of variables and integrating on $(i\varepsilon,(i+1)\varepsilon)$, we get

$$z^2((i+1)\varepsilon,x) = \frac{\xi\, z^2(i\varepsilon,x)}{\varepsilon\, z^2(i\varepsilon,x) + \xi},$$

i.e, by $(1.10)_2$

$$z^2((i+1)\varepsilon,x) = \frac{\xi\, \varphi_-^\varepsilon(i\varepsilon,x)^2}{\varepsilon\, \varphi_-^\varepsilon(i\varepsilon,x)^2 + \xi}. \tag{2.2}$$

This gives us

$$z^2((i+1)\varepsilon,x) \leq \varphi_-^\varepsilon(i\varepsilon,x)^2, \quad a.e\ x \in \Omega. \tag{2.3}$$

Since $\varphi_-^\varepsilon(i\varepsilon,x) \in L^\infty(\Omega)$, we conclude that $z((i+1)\varepsilon,x) \in L^\infty(\Omega)$ for all $i \in \{0, 1, ..., M_\varepsilon - 1\}$. $\square$

Lemma 2.2. *For $i = 0, 1, ..., M_\varepsilon - 1$, the estimate below holds*

$$\|\varphi_+^\varepsilon(i\varepsilon, x)\|^2_{L^2(\Omega)} \leq \|\varphi_-^\varepsilon(i\varepsilon, x)\|^2_{L^2(\Omega)}. \tag{2.4}$$

Proof. The proof follows directly to Lemma 2.1. In fact, using (2.3) and relation $(1.9)_2$, we deduce

$$\|\varphi_+^\varepsilon(i\varepsilon, x)\|^2_{L^2(\Omega)} \leq \|\varphi_-^\varepsilon(i\varepsilon, x)\|^2_{L^2(\Omega)}$$

as claimed. □

Lemma 2.3. *For $i = 0, 1, ..., M_\varepsilon - 1$, the estimate below holds*

$$\|\nabla\varphi_+^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \leq \|\nabla\varphi_-^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)}. \tag{2.5}$$

Proof. Let us set $\theta(t, x) = \nabla z(t, x)$. Thus (1.10) becomes

$$\begin{aligned} &\theta'(s, x) + \frac{3}{2\xi}\theta z^2(s, x) = 0, \ \ s \in (i\varepsilon, (i+1)\varepsilon), \\ &\theta(i\varepsilon, x) = \nabla\varphi_-^\varepsilon(i\varepsilon, x). \end{aligned} \tag{2.6}$$

The solution of (2.6) is then

$$\theta((i+1)\varepsilon, x) = e^{\int_{i\varepsilon}^{(i+1)\varepsilon} -\frac{3}{2\xi} z^2(t,.)dt}\theta(i\varepsilon, x). \tag{2.7}$$

After a small majoration, the proof is well done. □

Lemma 2.4. *The following estimate holds*

$$\|z(\varepsilon, x) - \varphi_-^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \leq \varepsilon L, \tag{2.8}$$

where $L > 0$ is a constant depending on Ω, $\|\varphi_-^\varepsilon\|_{L^\infty(\Omega)}$ and on the parameter ξ.

Proof. From (1.10), using the inequality $(a^3 - b^3)(a - b) \geq 0 \quad \forall a, b \in \mathbb{R}$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|z(t, x) - z(i\varepsilon, x)|^2 &\leq -\frac{1}{2\xi}z^3(t, x)\big(z(t, x) - z(i\varepsilon, x)\big) \\ &\leq -\frac{1}{2\xi}z^3(i\varepsilon, x)\big(z(t, x) - z(i\varepsilon, x)\big). \end{aligned} \tag{2.9}$$

Integrating (2.9) on $(i\varepsilon, (i+1)\varepsilon)$, we get

$$|z((i+1)\varepsilon, x) - z(i\varepsilon, x)| \leq \frac{\varepsilon}{2\xi}|z^3(i\varepsilon, x)| = \frac{\varepsilon}{2\xi}|\varphi_-^\varepsilon(i\varepsilon, x)^3|. \tag{2.10}$$

Hence

$$\|z(\varepsilon, x) - \varphi_-^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \leq \varepsilon L,$$

where $L > 0$ is a constant depending on Ω, $\|\varphi_-^\varepsilon\|_{L^\infty(\Omega)}$ and on the parameter ξ. □

Proof of Theorem 2.1. Consider $i = 0$. In this case, from lemma 2.1 we derive that the the solution of the Cauchy problem (1.10) $z((i+1)\varepsilon, x)$ belongs to $L^\infty(\Omega)$. Since $W_\infty^{2-\frac{2}{p}}(\Omega) \subset W_\infty^1(\Omega)$ then $z((i+1)\varepsilon, x) \in W_\infty^1(\Omega)$. Using Remark 1.1 to the problem (1.7)-(1.9) we ensure the existence of a solution u^ε, $\varphi^\varepsilon \in W_p^{1,2}(Q_0^\varepsilon) \cap L^\infty(Q_0^\varepsilon) \times W_\nu^{1,2}(Q_0^\varepsilon) \cap L^\infty(Q_0^\varepsilon)$. Thus, by induction $\varphi_-^\varepsilon(i\varepsilon, x) \in L^\infty(\Omega)$,

$i = 1, 2, ..., M_\varepsilon - 1$. Finally, we deduce that problem (1.7)-(1.9) has the solution $u^\varepsilon,\ \varphi^\varepsilon \in W_p^{1,2}(Q_i^\varepsilon) \cap L^\infty(Q_i^\varepsilon) \times W_\nu^{1,2}(Q_i^\varepsilon) \cap L^\infty(Q_i^\varepsilon)$, for all $i \in \{0, 1, ..., M_\varepsilon - 1\}$.

Let us now complete our reasoning by giving some a priori estimates in $Q_i^\varepsilon,\ i = 0, 1, ..., M_\varepsilon - 1$.

Multiplying $(1.7)_1$ by $\frac{2}{\ell} s_\xi u^\varepsilon$ and $(1.7)_2$ by φ_t^ε, using Green's formula and integrating by parts, we obtain

$$\begin{aligned}
&\frac{2\,C_p}{\ell} s_\xi \frac{d}{dt} \int\limits_\Omega |u^\varepsilon|^2 dx + s_\xi \int\limits_\Omega \varphi_t^\varepsilon u^\varepsilon\, dx + \frac{2k}{\ell} s_\xi \int\limits_\Omega |\nabla u^\varepsilon|^2 dx + \frac{2h}{\ell} s_\xi \int\limits_{\partial\Omega} |u^\varepsilon|^2 d\gamma \\
= \ &\frac{2}{\ell} s_\xi \int\limits_{\partial\Omega} w_1\, u^\varepsilon\, d\gamma + \frac{2}{\ell} s_\xi \int\limits_\Omega f\, u^\varepsilon\, dx,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&\alpha\xi \int\limits_\Omega |\varphi_t^\varepsilon|^2 dx + \frac{\xi}{2}\frac{d}{dt} \int\limits_\Omega |\nabla\varphi^\varepsilon|^2 dx + \frac{c_0}{2}\frac{d}{dt} \int\limits_{\partial\Omega} |\varphi^\varepsilon|^2 d\gamma \\
= \ &\frac{1}{4\xi}\frac{d}{dt} \int\limits_\Omega |\varphi^\varepsilon|^2 dx + s_\xi \int\limits_\Omega \varphi_t^\varepsilon u^\varepsilon\, dx + \int\limits_{\partial\Omega} w_2\, \varphi_t^\varepsilon\, d\gamma + \int\limits_\Omega g\, \varphi_t^\varepsilon\, dx.
\end{aligned} \tag{2.12}$$

Using Hölder's inequality

$$\frac{2}{\ell} s_\xi \int\limits_{\partial\Omega} w_1\, u^\varepsilon\, d\gamma \le \frac{2h}{\ell} s_\xi \int\limits_{\partial\Omega} |u^\varepsilon|^2\, d\gamma + \frac{s_\xi}{2\ell h} \int\limits_{\partial\Omega} |w_1|^2\, d\gamma, \tag{2.13}$$

$$\int\limits_\Omega g\, \varphi_t^\varepsilon\, dx \le \frac{\alpha\xi}{2} \int\limits_\Omega |\varphi_t^\varepsilon|^2\, dx + \frac{1}{2\alpha\xi} \int\limits_\Omega |g|^2\, dx. \tag{2.14}$$

Hence, adding (2.11)-(2.12) and making use (2.13)-(2.14), we obtain

$$\begin{aligned}
&\frac{2\,C_p}{\ell} s_\xi \frac{d}{dt} \int\limits_\Omega |u^\varepsilon|^2 dx + \frac{2k}{\ell} s_\xi \int\limits_\Omega |\nabla u^\varepsilon|^2 dx + \frac{\alpha\xi}{2} \int\limits_\Omega |\varphi_t^\varepsilon|^2 dx \\
&+ \frac{\xi}{2}\frac{d}{dt} \int\limits_\Omega |\nabla\varphi^\varepsilon|^2 dx + \frac{c_0}{2}\frac{d}{dt} \int\limits_{\partial\Omega} |\varphi^\varepsilon|^2 d\gamma \\
\le \ &\frac{s_\xi}{2\ell h} \int\limits_{\partial\Omega} |w_1|^2\, d\gamma + \frac{2}{\ell} s_\xi \int\limits_\Omega f\, u^\varepsilon\, dx + \frac{1}{4\xi}\frac{d}{dt} \int\limits_\Omega |\varphi^\varepsilon|^2 dx + \int\limits_{\partial\Omega} w_2\, \varphi_t^\varepsilon\, d\gamma \\
&+ \frac{1}{2\alpha\xi} \int\limits_\Omega |g|^2\, dx.
\end{aligned} \tag{2.15}$$

Multiplying $(1.7)_2$ by $\frac{1}{\alpha\xi^2}\varphi^\varepsilon$, integrating over Ω and using Green's formula we get

$$\begin{aligned}
&\frac{1}{2\xi}\frac{d}{dt} \int\limits_\Omega |\varphi^\varepsilon|^2 dx + \frac{1}{\alpha\xi} \int\limits_\Omega |\nabla\varphi^\varepsilon|^2 dx + \frac{c_0}{\alpha\xi^2} \int\limits_{\partial\Omega} |\varphi^\varepsilon|^2 d\gamma \\
= &\frac{1}{\alpha\xi^2} \int\limits_{\partial\Omega} w_2 \varphi^\varepsilon\, d\gamma + \frac{1}{2\alpha\xi^3} \int\limits_\Omega |\varphi^\varepsilon|^2 dx + \frac{s_\xi}{\alpha\xi^2} \int\limits_\Omega u^\varepsilon \varphi^\varepsilon\, dx + \frac{1}{\alpha\xi^2} \int\limits_\Omega g\varphi^\varepsilon\, dx.
\end{aligned} \tag{2.16}$$

Using again Hölder's inequality

$$\frac{1}{\alpha} \int\limits_{\partial\Omega} w_2\, \varphi^\varepsilon\, d\gamma \le \frac{c_0}{\alpha\ell} \int\limits_{\partial\Omega} |\varphi^\varepsilon|^2\, d\gamma + \frac{1}{4c_0\alpha} \int\limits_{\partial\Omega} |w_2|^2\, d\gamma, \tag{2.17}$$

from (2.16) and (2.17) we obtain,

$$\begin{aligned}&\frac{1}{2\xi}\frac{d}{dt}\int\limits_{\Omega}|\varphi^\varepsilon|^2dx+\frac{1}{\alpha\xi}\int\limits_{\Omega}|\nabla\varphi^\varepsilon|^2dx\\&\leq\frac{1}{4c_0\alpha\xi^2}\int\limits_{\partial\Omega}|w_2|^2\,d\gamma+\frac{1}{2\alpha\xi^3}\int\limits_{\Omega}|\varphi^\varepsilon|^2dx+\frac{s_\xi}{\alpha\xi^2}\int\limits_{\Omega}u^\varepsilon\varphi^\varepsilon\,dx+\frac{1}{\alpha\xi^2}\int\limits_{\Omega}g\varphi^\varepsilon\,dx.\end{aligned}\tag{2.18}$$

Adding (2.15) and (2.18) and performing with Cauchy's inequality, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\Big[\frac{2\,C_p}{\ell}s_\xi\int\limits_{\Omega}|u^\varepsilon|^2dx+\frac{\xi}{2}\int\limits_{\Omega}|\nabla\varphi^\varepsilon|^2dx+\frac{1}{4\xi}\int\limits_{\Omega}|\varphi^\varepsilon|^2dx+\frac{c_0}{2}\int\limits_{\partial\Omega}|\varphi^\varepsilon|^2d\gamma\Big]\\&+\frac{2k}{\ell}s_\xi\int\limits_{\Omega}|\nabla u^\varepsilon|^2dx+\frac{\alpha\xi}{2}\int\limits_{\Omega}|\varphi_t^\varepsilon|^2dx+\frac{1}{\alpha\xi}\int\limits_{\Omega}|\nabla\varphi^\varepsilon|^2dx\\&\leq\int\limits_{\partial\Omega}w_2\,\varphi_t^\varepsilon\,d\gamma+C\Big\{\int\limits_{\Omega}|\varphi^\varepsilon|^2dx+\int\limits_{\Omega}|u^\varepsilon|^2dx+\int\limits_{\partial\Omega}|w_1|^2d\gamma\\&+\int\limits_{\partial\Omega}|w_2|^2d\gamma+\int\limits_{\Omega}|f|^2dx+\int\limits_{\Omega}|g|^2dx\Big\}.\end{aligned}\tag{2.19}$$

Integrating over $(0,\varepsilon)$ gives

$$\begin{aligned}&\frac{2\,C_p}{\ell}s_\xi\|u^\varepsilon(\varepsilon)\|^2_{L^2(\Omega)}+\frac{1}{4\xi}\|\varphi^\varepsilon_-(\varepsilon)\|^2_{L^2(\Omega)}+\frac{\xi}{2}\|\nabla\varphi^\varepsilon_-(\varepsilon)\|^2_{L^2(\Omega)}+\frac{c_0}{2}\|\varphi^\varepsilon_-(\varepsilon)\|^2_{L^2(\partial\Omega)}\\&+\frac{2k}{\ell}s_\xi\int\limits_0^\varepsilon\|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds+\frac{1}{\alpha\xi}\int\limits_0^\varepsilon\|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds+\frac{\alpha\xi}{2}\int\limits_0^\varepsilon\|\varphi_t^\varepsilon\|^2_{L^2(\Omega)}ds\\&\leq\frac{2\,C_p}{\ell}s_\xi\|u_0^\varepsilon\|^2_{L^2(\Omega)}+\frac{1}{4\xi}\|\varphi_0^\varepsilon\|^2_{L^2(\Omega)}+\frac{\xi}{2}\|\nabla\varphi_0^\varepsilon\|^2_{L^2(\Omega)}+\frac{c_0}{2}\|\varphi_0^\varepsilon\|^2_{L^2(\partial\Omega)}\\&+\int\limits_{\Sigma_0^\varepsilon}w_2\,\varphi_t^\varepsilon\,d\gamma\,ds+C\Big\{\int\limits_0^\varepsilon\|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)}ds+\int\limits_0^\varepsilon\|u^\varepsilon(s)\|^2_{L^2(\Omega)}ds\\&+\|w_1\|^2_{L^2(\Sigma_0^\varepsilon)}+\|w_2\|^2_{L^2(\Sigma_0^\varepsilon)}+\|f\|^2_{L^2(Q_0^\varepsilon)}+\|g\|^2_{L^2(Q_0^\varepsilon)}\Big\}.\end{aligned}\tag{2.20}$$

We now focus on the right term $\int_{\Sigma_0^\varepsilon}w_2\,\varphi_t^\varepsilon\,d\gamma\,ds$ in the previous inequality. We have

$$\int\limits_{\Sigma_0^\varepsilon}w_2\varphi_t^\varepsilon d\gamma\,ds=\int\limits_{\Sigma_0^\varepsilon}\frac{\partial}{\partial t}\big(w_2\varphi^\varepsilon\big)d\gamma\,ds-\int\limits_{\Sigma_0^\varepsilon}w_2'\varphi^\varepsilon\,d\gamma\,ds\tag{2.21}$$

and

$$\begin{aligned}&\int\limits_{\partial\Omega}w_2\varphi^\varepsilon d\gamma\leq\frac{1}{c_0}\int\limits_{\partial\Omega}|w_2|^2d\gamma+\frac{c_0}{4}\int\limits_{\partial\Omega}|\varphi^\varepsilon|^2d\gamma,\\&\int\limits_{\Sigma_0^\varepsilon}w_2'\varphi^\varepsilon\,d\gamma\,ds\leq\frac{1}{2}\int\limits_{\Sigma_0^\varepsilon}|\varphi^\varepsilon|^2d\gamma\,ds+\frac{1}{2}\int\limits_{\Sigma_0^\varepsilon}|w_2'|^2d\gamma\,ds.\end{aligned}\tag{2.22}$$

Combining (2.20), (2.21) and (2.22), we obtain

$$\begin{aligned}
&\frac{2\,C_p}{\ell}s_\xi\|u^\varepsilon(\varepsilon)\|^2_{L^2(\Omega)}+\frac{1}{4\xi}\|\varphi^\varepsilon_-(\varepsilon)\|^2_{L^2(\Omega)}+\frac{\xi}{2}\|\nabla\varphi^\varepsilon_-(\varepsilon)\|^2_{L^2(\Omega)}+\frac{c_0}{4}\|\varphi^\varepsilon_-(\varepsilon)\|^2_{L^2(\partial\Omega)}\\
&+\frac{2k}{\ell}s_\xi\int_0^\varepsilon\|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds+\frac{1}{\alpha\xi}\int_0^\varepsilon\|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds+\frac{\alpha\xi}{2}\int_0^\varepsilon\|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds\\
\le&\frac{2\,C_p}{\ell}s_\xi\|u^\varepsilon_0\|^2_{L^2(\Omega)}+\frac{1}{4\xi}\|\varphi^\varepsilon_0\|^2_{L^2(\Omega)}+\frac{\xi}{2}\|\nabla\varphi^\varepsilon_0\|^2_{L^2(\Omega)}+\frac{c_0}{4}\|\varphi^\varepsilon_0\|^2_{L^2(\partial\Omega)}\\
&+C\Big\{\int_0^\varepsilon\|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)}ds+\int_0^\varepsilon\|\varphi^\varepsilon(s)\|^2_{L^2(\partial\Omega)}ds+\int_0^\varepsilon\|u^\varepsilon(s)\|^2_{L^2(\Omega)}ds\\
&+\|w_1\|^2_{L^2(\Sigma^\varepsilon_0)}+\|w_2'\|^2_{L^2(\Sigma^\varepsilon_0)}+\|w_2\|^2_{L^2(\Sigma^\varepsilon_0)}+\|f\|^2_{L^2(Q^\varepsilon_0)}+\|g\|^2_{L^2(Q^\varepsilon_0)}\Big\}.
\end{aligned} \tag{2.23}$$

Similarly for $Q^\varepsilon_i, \quad i=1,2,...,M_\varepsilon-2$,

$$\begin{aligned}
&\frac{2\,C_p}{\ell}s_\xi\|u^\varepsilon((i+1)\varepsilon)\|^2_{L^2(\Omega)}+\frac{1}{4\xi}\|\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\Omega)}+\frac{\xi}{2}\|\nabla\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\Omega)}\\
&+\frac{c_0}{4}\|\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\partial\Omega)}+\frac{2k}{\ell}s_\xi\int_{i\varepsilon}^{(i+1)\varepsilon}\|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds\\
&+\frac{1}{\alpha\xi}\int_{i\varepsilon}^{(i+1)\varepsilon}\|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds+\frac{\alpha\xi}{2}\int_{i\varepsilon}^{(i+1)\varepsilon}\|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds\\
\le&\frac{2\,C_p}{\ell}s_\xi\|u^\varepsilon(i\varepsilon)\|^2_{L^2(\Omega)}+\frac{1}{4\xi}\|\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\Omega)}+\frac{\xi}{2}\|\nabla\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\Omega)}\\
&+\frac{c_0}{4}\|\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\partial\Omega)}+C\Big\{\int_{i\varepsilon}^{(i+1)\varepsilon}\|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)}ds+\int_{i\varepsilon}^{(i+1)\varepsilon}\|\varphi^\varepsilon(s)\|^2_{L^2(\partial\Omega)}ds\\
&+\int_{i\varepsilon}^{(i+1)\varepsilon}\|u^\varepsilon(s)\|^2_{L^2(\Omega)}ds+\|w_1\|^2_{L^2(\Sigma^\varepsilon_i)}+\|w_2'\|^2_{L^2(\Sigma^\varepsilon_i)}+\|w_2\|^2_{L^2(\Sigma^\varepsilon_i)}\\
&+\|f\|^2_{L^2(Q^\varepsilon_i)}+\|g\|^2_{L^2(Q^\varepsilon_i)}\Big\}.
\end{aligned} \tag{2.24}$$

Again to $Q^T_{M_\varepsilon-1}$,

$$\begin{aligned}
&\frac{2\,C_p}{\ell}s_\xi\|u^\varepsilon(T)\|^2_{L^2(\Omega)}+\frac{1}{4\xi}\|\varphi^\varepsilon_-(T)\|^2_{L^2(\Omega)}+\frac{\xi}{2}\|\nabla\varphi^\varepsilon_-(T)\|^2_{L^2(\Omega)}\\
&+\frac{c_0}{4}\|\varphi^\varepsilon_-(T)\|^2_{L^2(\partial\Omega)}+\frac{2k}{\ell}s_\xi\int_{M_\varepsilon-1}^{T}\|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds+\frac{1}{\alpha\xi}\int_{M_\varepsilon-1}^{T}\|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds\\
&+\frac{\alpha\xi}{2}\int_{M_\varepsilon-1}^{T}\|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{2\,C_p}{l} s_\xi \|u^\varepsilon((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{1}{4\xi}\|\varphi^\varepsilon_+((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\Omega)}\\
&\quad + \frac{\xi}{2}\|\nabla\varphi^\varepsilon_+((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{c_0}{4}\|\varphi^\varepsilon_+((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\partial\Omega)}\\
&\quad + C\Big\{ \int_{M_\varepsilon-1}^{T} \|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)}ds + \int_{M_\varepsilon-1}^{T} \|\varphi^\varepsilon(s)\|^2_{L^2(\partial\Omega)}ds + \int_{M_\varepsilon-1}^{T} \|u^\varepsilon(s)\|^2_{L^2(\Omega)}ds\\
&\quad + \|w_1\|^2_{L^2(\Sigma^T_{M_\varepsilon-1})} + \|w_2'\|^2_{L^2(\Sigma^T_{M_\varepsilon-1})} + \|w_2\|^2_{L^2(\Sigma^T_{M_\varepsilon-1})} + \|f\|^2_{L^2(Q^T_{M_\varepsilon-1})}\\
&\quad + \|g\|^2_{L^2(Q^T_{M_\varepsilon-1})}\Big\}.
\end{aligned}
\tag{2.25}
$$

Considering inequalities (2.4) and (2.5) given respectively by Lemmas 2.2 and 2.3, we remark

$$
\begin{aligned}
&E^1_{ie} + \frac{1}{4\xi}\|\varphi^\varepsilon_+((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{c_0}{4}\|\varphi^\varepsilon_+((i+1)\varepsilon)\|^2_{L^2(\partial\Omega)} + \frac{\xi}{2}\|\nabla\varphi^\varepsilon_+((i+1)\varepsilon)\|^2_{L^2(\Omega)}\\
\le& E^1_{ie} + \frac{1}{4\xi}\|\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{c_0}{4}\|\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\partial\Omega)} + \frac{\xi}{2}\|\nabla\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\Omega)}\\
\le& E^2_{ie} + \|\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\Omega)} + \|\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\partial\Omega)} + \|\nabla\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\Omega)},
\end{aligned}
\tag{2.26}
$$

where

$$
\begin{aligned}
E^1_{ie} =& \frac{2\,C_p}{\ell} s_\xi \|u^\varepsilon((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{2k}{\ell} s_\xi \int_{i\varepsilon}^{(i+1)\varepsilon} \|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds\\
&+ \frac{1}{\alpha\xi}\int_{i\varepsilon}^{(i+1)\varepsilon} \|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds + \frac{\alpha\xi}{2}\int_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds
\end{aligned}
\tag{2.27}
$$

and

$$
\begin{aligned}
E^2_{ie} =& \frac{2\,C_p}{\ell} s_\xi \|u^\varepsilon(i\varepsilon)\|^2_{L^2(\Omega)} + C\Big\{ \int_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)}ds\\
&+ \int_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi^\varepsilon(s)\|^2_{L^2(\partial\Omega)}ds + \int_{i\varepsilon}^{(i+1)\varepsilon} \|u^\varepsilon(s)\|^2_{L^2(\Omega)}ds + \|w_1\|^2_{L^2(\Sigma^\varepsilon_i)}\\
&+ \|w_2'\|^2_{L^2(\Sigma^\varepsilon_i)} + \|w_2\|^2_{L^2(\Sigma^\varepsilon_i)} + \|f\|^2_{L^2(Q^\varepsilon_i)} + \|g\|^2_{L^2(Q^\varepsilon_i)}\Big\}.
\end{aligned}
\tag{2.28}
$$

Adding (2.23), (2.25), (2.26) and doing some calculations

$$
\begin{aligned}
&\frac{2\,C_p}{\ell} s_\xi \|u^\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{1}{4\xi}\|\varphi^\varepsilon_-(T)\|^2_{L^2(\Omega)} + \frac{\xi}{2}\|\nabla\varphi^\varepsilon_-(T)\|^2_{L^2(\Omega)} + \frac{c_0}{4}\|\varphi^\varepsilon_-(T)\|^2_{L^2(\partial\Omega)}\\
&+ \frac{2k}{\ell} s_\xi \int_0^T \|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds + \frac{1}{\alpha\xi}\int_0^T \|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds + \frac{\alpha\xi}{2}\sum_{i=0}^{M_\varepsilon-1}\int_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds
\end{aligned}
$$

$$\leq \frac{2\,C_p}{\ell} s_\xi \|u_0\|^2_{L^2(\Omega)} + \frac{1}{4\xi}\|\varphi_0\|^2_{L^2(\Omega)} + \frac{\xi}{2}\|\nabla\varphi_0\|^2_{L^2(\Omega)} + \frac{c_0}{4}\|\varphi_0\|^2_{L^2(\partial\Omega)}$$
$$+ C\Big\{ \int_0^T \|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)}ds + \int_0^T \|\varphi^\varepsilon(s)\|^2_{L^2(\partial\Omega)}ds + \int_0^T \|u^\varepsilon(s)\|^2_{L^2(\Omega)}ds + \|w_1\|^2_{L^2(\Sigma)}$$
$$+ \|w_2'\|^2_{L^2(\Sigma)} + \|w_2\|^2_{L^2(\Sigma)} + \|f\|^2_{L^2(Q)} + \|g\|^2_{L^2(Q)}\Big\}. \tag{2.29}$$

Continuing by applying Gronwall inequality, we finally deduce

$$\int_0^T \|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds + \int_0^T \|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds + \int_0^T \|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds + \int_0^T \|\varphi^\varepsilon(s)\|^2_{L^2(\partial\Omega)}ds \leq C, \tag{2.30}$$

where C does not depend to ε and M_ε.

Furthermore multiplying $(1.7)_1$ by u^ε_t, using Green's formula and integrating over $[i\varepsilon,(i+1)\varepsilon]$, $i = 0\cdots M_\varepsilon - 1$, we get

$$\begin{aligned} &C_p \int_{Q_i^\varepsilon} |u^\varepsilon_t|^2 dxds + \frac{k}{2}\int_\Omega |\nabla u^\varepsilon|^2 dxds + \frac{h}{2}\int_{\partial\Omega} |u^\varepsilon|^2 d\gamma + \frac{\ell}{2}\int_{Q_i^\varepsilon} \varphi^\varepsilon_t u^\varepsilon_t dxds \\ &= \int_{\Sigma_i^\varepsilon} w_1 u^\varepsilon_t\, dxds + \int_{Q_i^\varepsilon} f u^\varepsilon_t\, dxds. \end{aligned} \tag{2.31}$$

Using Hölder's inequality

$$\begin{aligned} &\frac{\ell}{2}\int_{Q_i^\varepsilon} \varphi^\varepsilon_t u^\varepsilon_t dxds \leq \frac{C_p}{4}\int_{Q_i^\varepsilon} |u^\varepsilon_t|^2 dxds + \frac{\ell^2}{4C_p}\int_{Q_i^\varepsilon} |\varphi^\varepsilon_t|^2 dxds, \\ &\int_{Q_i^\varepsilon} f u^\varepsilon_t dxds \leq \frac{C_p}{4}\int_{Q_i^\varepsilon} |u^\varepsilon_t|^2 dxds + \frac{1}{C_p}\int_{Q_i^\varepsilon} |f|^2 dxds. \end{aligned} \tag{2.32}$$

From (2.30),(2.31),(2.32) and after summing

$$\frac{C_p}{2}\int_Q |u^\varepsilon_t|^2 dxds + \frac{k}{2}\int_\Omega |\nabla u^\varepsilon|^2 dxds + \frac{h}{2}\int_{\partial\Omega} |u^\varepsilon|^2 d\gamma \leq C + \int_\Sigma w_1 u^\varepsilon_t\, dxds. \tag{2.33}$$

Since

$$\int_\Sigma w_1 u^\varepsilon_t d\gamma\, ds = \int_\Sigma \frac{\partial}{\partial t}\big(w_1 u^\varepsilon\big) d\gamma\, ds - \int_\Sigma w_1' u^\varepsilon\, d\gamma\, ds \tag{2.34}$$

and

$$\begin{aligned} &\int_{\partial\Omega} w_1 u^\varepsilon d\gamma \leq \frac{1}{h}\int_{\partial\Omega} |w_1|^2 d\gamma + \frac{h}{4}\int_{\partial\Omega} |u^\varepsilon|^2 d\gamma \\ &\int_\Sigma w_1' u^\varepsilon\, d\gamma\, ds \leq \frac{1}{2}\int_\Sigma |u^\varepsilon|^2 d\gamma\, ds + \frac{1}{2}\int_\Sigma |w_1'|^2 d\gamma\, ds. \end{aligned} \tag{2.35}$$

Performing (2.33) by using (2.32)-(2.34) and Gronwall's inequality, we obtain

$$\int_Q |u_t^\varepsilon|^2 dxds + \int_\Omega |\nabla u^\varepsilon|^2 dxds + \int_{\partial\Omega} |u^\varepsilon|^2 d\gamma \le C, \tag{2.36}$$

where C does not depend to ε and M_ε. Due to estimate (2.8) we have

$$\begin{aligned} &\sum_{i=0}^{M_\varepsilon - 1} \|\varphi_+^\varepsilon(i\varepsilon, x) - \varphi_-^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \le TL = C_1 \\ &\sum_{i=0}^{M_\varepsilon - 1} \|\varphi_+^\varepsilon(i\varepsilon, x) - \varphi_-^\varepsilon(i\varepsilon, x)\|_{L^2(\partial\Omega)} \le C_2, \end{aligned} \tag{2.37}$$

where C_1 and C_2 **do not depend** on M_ε and ε.

Adding (2.30), (2.36) and (2.37), we deduce

$$\begin{aligned} &\overset{T}{\underset{0}{V1}}\,\varphi^\varepsilon + \overset{T}{\underset{0}{V2}}\,\varphi^\varepsilon + \int_0^T \|u_t^\varepsilon(t)\|^2_{L^2(\Omega)} dt + \int_0^T \|\varphi_t^\varepsilon(t)\|^2_{L^2(\Omega)} dt \\ &+ \int_0^T \|\nabla u^\varepsilon\|^2_{L^2(\Omega)} dt + \int_0^T \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega)} dt \le C, \qquad \varepsilon > 0, \end{aligned} \tag{2.38}$$

where $V1_0^T\,\varphi^\varepsilon$ and $V2_0^T\,\varphi^\varepsilon$ stand respectively for the variation of $\varphi^\varepsilon : [0,T] \to L_2(\Omega)$ and $\varphi^\varepsilon : [0,T] \to L_2(\partial\Omega)$. Since the injection of $L_2(\Omega)$ into $H^{-1}(\Omega)$ is compact and the set $\{\varphi_t^\varepsilon(t)\}$ is bounded in $L_2(\Omega)$ for every $t \in [0,T]$, we deduce that there exists (Helly-Foiaş theorem) a bounded variation $\varphi^*(t) \in BV([0,T]; H^{-1}(\Omega))$ and a subsequence $\varphi^\varepsilon(t)$ such that

$$\varphi^\varepsilon(t) \to \varphi^*(t) \quad \text{strongly in} \quad H^{-1}(\Omega) \quad \text{for every} \quad t \in [0,T]. \tag{2.39}$$

In addition we deduce from (2.38)

$$\varphi^\varepsilon \to \varphi^* \quad \textit{weakly in} \quad L^2(0,T; H^1(\Omega)). \tag{2.40}$$

Remember that due to the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ then (see [5]) for each $\kappa > 0$, there exists some constant c_κ depending on κ (and on the spaces $H^1(\Omega), H^{-1}(\Omega), L^2(\Omega)$) such that

$$\|\varphi^\varepsilon(t) - \varphi^*(t)\|_{L^2(\Omega)} \le \kappa \|\varphi^\varepsilon(t) - \varphi^*(t)\|_{H^1(\Omega)} + c_\kappa \|\varphi^\varepsilon(t) - \varphi^*(t)\|_{H^{-1}(\Omega)}, \tag{2.41}$$

$\forall \varepsilon > 0$ and $\forall t \in [0,T]$, where $c_\kappa \to 0$ as $\kappa \to 0$.

Using (2.39)-(2.41), we get

$$\varphi^\varepsilon \to \varphi^* \quad \text{strongly in} \quad L^2(\Omega) \quad \text{for any} \quad t \in [0,T]. \tag{2.42}$$

Therefore, by (1.7) and (2.38) we also have

$$\begin{aligned} &\int_0^T \|\Delta \varphi^\varepsilon\|_{L^2(\Omega)}\, dt \le C, \quad \forall t \in [0,T], \\ &\int_0^T \|\Delta u^\varepsilon\|_{L^2(\Omega)}\, dt \le C, \quad \forall t \in [0,T], \end{aligned} \tag{2.43}$$

and also

$$\begin{aligned} \|\varphi^{\varepsilon}\|_{L^2(0,T;H^2(\Omega))} &\leq C \\ \|u^{\varepsilon}\|_{L^2(0,T;H^2(\Omega))} &\leq C. \end{aligned} \tag{2.44}$$

Hence, due to the inclusion $H^2(\Omega) \subset H^1(\Omega)$ is compact, we get that the sequence $\{u^{\varepsilon}\}$ is compact in $L^2(0, T; H^1(\Omega))$. Thus, up to a subsequence denoted u^{ε}, we get

$$\begin{aligned} u^{\varepsilon} &\to u^* \quad \text{strongly in} \quad L^2([0,T]; H^1(\Omega)), \\ u^{\varepsilon} &\to u^* \quad \text{weakly in} \quad L^2([0,T]; H^2(\Omega)), \\ u_t^{\varepsilon} &\to u_t^* \quad \text{weakly in} \quad L^2([0,T]; L^2(\Omega)) \end{aligned} \tag{2.45}$$

and, by the Ascoli-Arzelà theorem

$$u^{\varepsilon} \to u^* \quad \text{strongly in} \quad C([0,T]; L^2(\Omega)). \tag{2.46}$$

Our assertion (2.1) holds true from (2.42) and (2.46). This achieves the proof of Theorem 2.1. □

As regards the unknwown functions $u^*(t), \varphi^*(t)$, one can prove as in [6] that it is absolutely continuous in t on $[0, T]$ and satisfies a.e the phase-field system (1.1)-(1.3), which means that the pair $(u^*(t), \varphi^*(t))$ is a strong solution to our nonlinear problem. So, Theorem 2.1 can be regarding as a constructive way to prove the existence and regularity of solutions for the nonlinear parabolic system (1.1)-(1.3).

Corollary 2.1. *Let* $u_0, \varphi_0 \in W_{\infty}^{2-\frac{2}{p}}(\Omega)$ *satisfying* $k\frac{\partial}{\partial\nu}u_0(x) + hu_0(x) = w_1(0,x)$, $\xi\frac{\partial}{\partial\nu}\varphi_0(x) + c_0\varphi_0(x) = w_2(0,x)$ *and* $w_1(t,x), w_2(t,x) \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$. *Then* $(u^\star, \varphi^\star) \in \left(L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)\right)^2$ *is a weak solution of nonlinear system* (1.1)-(1.3).

3. Approximation of phase-field transition system in 2D by finite element method Algorithm Armel-fracfem2D

In this Section we are concerned with the numerical approximation of the weak solution corresponding with (1.7)-(1.9) (see Definition 1.2) by finite element method (**fem**) i.e. with the numerical approximation of the weak solution of the following equations:

$$\begin{aligned} &\left(C_p\, u_t^{\varepsilon} + \frac{\ell}{2}\varphi_t^{\varepsilon}, \phi\right) + k(\nabla u^{\varepsilon}, \nabla\phi) + h\int\limits_{\partial\Omega} u^{\varepsilon}\phi\, dxdy \\ = &\int\limits_{\partial\Omega} w_1(\cdot,x,y)\phi\, dxdy + \int\limits_{\Omega} f\phi\, dxdy \quad \forall\, \phi \in H^1(\Omega), \quad \text{a.e.} \quad \text{in } [0,T], \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\alpha\xi(\varphi_t^{\varepsilon}, \psi) + \xi(\nabla\varphi^{\varepsilon}, \nabla\psi) + c_0\int\limits_{\partial\Omega} u^{\varepsilon}\psi\, dxdy \\ = &\int\limits_{\partial\Omega} w_2(\cdot,x,y)\psi\, dxdy + \frac{1}{2\xi}(\varphi^{\varepsilon}, \psi) + s_{\xi}(u^{\varepsilon}, \psi) + \int\limits_{\Omega} g\psi\, dxdy \\ &\forall\, \psi \in H^1(\Omega), \quad \text{a.e.} \quad \text{in } [0,T], \end{aligned} \tag{3.2}$$

together with the initial conditions

$$u(0,x) = u_0(x), \quad \varphi(0,x) = \varphi_0(x), \quad x \in \Omega. \tag{3.3}$$

In (3.1)-(3.2), $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.

Considering $M = M_\varepsilon$ as the number of equidistant nodes in which is divided the time-interval $[0,T]$, we set

$$\varepsilon = dt = T/M, \quad t_i = i\,dt, \quad i = 0,1,2,\ldots,M.$$

We assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain. Let T_ρ be the triangulation (mesh) over Ω and $\bar{\Omega} = \cup\{K,\ K \in T_\rho\}$ and let $N_j = (x_k, y_l),\ j = \overline{1,nn}$, be the nodes associated to T_ρ. If we denote by V_{nn} the corresponding finite element space to T_ρ then, the basis functions $\{b_j\}_{j=1}^{nn}$ of V_{nn} are defined by

$$b_j(N_i) = \delta_{ji}, \quad i,j = \overline{1,nn},$$

and

$$V_{nn} = \text{SPAN } \{b_1, b_2, ..., b_{nn}\}.$$

We say that the function $v(x,y)$ belongs to V_{nn} only if it can be expressed as

$$v(x,y) = \sum_{l=1}^{nn} c_l b_l(x,y), \quad (x,y) \in \bar{\Omega}.$$

For $i = \overline{1,M}$, we denote by u^i and φ^i the V_{nn} interpolant of u^ε and φ^ε, respectively. Then $u^i,\ \varphi^i \in V_{nn}$ and

$$u^i(x,y) = \sum_{l=1}^{nn} u_l^i b_l(x,y), \quad i = \overline{1,M}, \tag{3.4}$$

$$\varphi^i(x,y) = \sum_{l=1}^{nn} \varphi_l^i b_l(x,y), \quad i = \overline{1,M}, \tag{3.5}$$

where the unknowns $u_l^i = u^\varepsilon(t_i, N_l)$, $\varphi_l^i = \varphi^\varepsilon(t_i, N_l)$, $i = \overline{1,M}$, $l = \overline{1,nn}$, represents the discrete solution of (3.10)-(3.11) below.

Let now $U,\ \Phi \in V_{nn}$ be two arbitrary functions, i.e.

$$U(x,y) = \sum_{l=1}^{nn} U_l b_l(x,y), \tag{3.6}$$

$$\Phi(x,y) = \sum_{l=1}^{nn} \Phi_l b_l(x,y). \tag{3.7}$$

Using an implicit (backward) finite difference scheme in time and taking into account the above notations, we introduce the discrete equations corresponding to (3.1)-(3.2) as follows ($i = \overline{1,M}$)

$$\begin{aligned} & C_p(u^i,U) + \frac{\ell}{2}(\varphi^i,U) + dt\,k(\nabla u^i, \nabla U) + dt\,h \int_{\partial\Omega} u^i U\,dxdy \\ = \;& dt \int_{\partial\Omega} w_1^i U\,dxdy + C_p(u^{i-1},U) + \frac{\ell}{2}(\varphi^{i-1},U) + dt \int_{\Omega} f\,U\,dxdy, \end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\alpha\xi(\varphi^i,\Phi)+dt\xi(\nabla\varphi^i,\nabla\Phi)+dt\,c_0\int\limits_{\partial\Omega}\varphi^i\Phi\,dxdy\\
&=\ dt\int\limits_{\partial\Omega}w_2^i\Phi\,dxdy+\frac{dt}{2\xi}(\varphi^i,\Phi)+dt\,s_\xi(u^i,\Phi)+\alpha\xi(\varphi^{i-1},\Phi)+dt\int\limits_{\Omega}g\Phi\,dxdy,
\end{aligned} \tag{3.9}$$

$\forall\ U,\Phi\in V_{nn}$, with $w_1^i=w_1(t_i,\cdot)$, $w_2^i=w_2(t_i,\cdot)$. Replaying (3.4)-(3.7) in (3.8)-(3.9), we get

$$\begin{aligned}
&C_p\sum_{k=1}^{nn}U_k\left(\sum_{l=1}^{nn}u_l^i\int\limits_{\Omega}b_kb_l\,dxdy\right)+\frac{\ell}{2}\sum_{k=1}^{nn}U_k\left(\sum_{l=1}^{nn}\varphi_l^i\int\limits_{\Omega}b_kb_l\,dxdy\right)\\
&+dt\,k\sum_{k=1}^{nn}U_k\left(\sum_{l=1}^{nn}u_l^i\int\limits_{\Omega}\nabla b_k\nabla b_l\,dxdy\right)+dt\,h\sum_{k=1}^{nn}U_k\left(\sum_{l=1}^{nn}u_l^i\int\limits_{\partial\Omega}b_kb_l\,dxdy\right)\\
&=dt\sum_{k=1}^{nn}U_k\left(\sum_{l=1}^{nn}w_{1l}^i\int\limits_{\partial\Omega}b_kb_l\,dxdy\right)+C_p\sum_{k=1}^{nn}U_k\left(\sum_{l=1}^{nn}u_l^{i-1}\int\limits_{\Omega}b_kb_l\,dxdy\right)\\
&+\frac{\ell}{2}\sum_{k=1}^{nn}U_k\left(\sum_{l=1}^{nn}\varphi_l^{i-1}\int\limits_{\Omega}b_kb_l\,dxdy\right)+dt\sum_{k=1}^{nn}U_k\left(\sum_{l=1}^{nn}f_l^i\int\limits_{\Omega}b_kb_l\,dxdy\right),
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\alpha\xi\sum_{k=1}^{nn}\Phi_k\left(\sum_{l=1}^{nn}\varphi_l^i\int\limits_{\Omega}b_kb_l\,dxdy\right)+dt\xi\sum_{k=1}^{nn}\Phi_k\left(\sum_{l=1}^{nn}\varphi_l^i\int\limits_{\Omega}\nabla b_k\nabla b_l\,dxdy\right)\\
&+dt\,c_0\sum_{k=1}^{nn}\Phi_k\left(\sum_{l=1}^{nn}\varphi_l^i\int\limits_{\partial\Omega}b_kb_l\,dxdy\right)\\
&=dt\sum_{k=1}^{nn}\Phi_k\left(\sum_{l=1}^{nn}w_{2l}^i\int\limits_{\partial\Omega}b_kb_l\,dxdy\right)+\frac{dt}{2\xi}\sum_{k=1}^{nn}\Phi_k\left(\sum_{l=1}^{nn}\varphi_l^i\int\limits_{\Omega}b_kb_l\,dxdy\right)\\
&+dt\,s_\xi\sum_{k=1}^{nn}\Phi_k\left(\sum_{l=1}^{nn}u_l^i\int\limits_{\Omega}b_kb_l\,dxdy\right)+\alpha\xi\sum_{k=1}^{nn}\Phi_k\left(\sum_{l=1}^{nn}\varphi_l^{i-1}\int\limits_{\Omega}b_kb_l\,dxdy\right)\\
&+dt\sum_{k=1}^{nn}\Phi_k\left(\sum_{l=1}^{nn}g_l^i\int\limits_{\Omega}b_kb_l\,dxdy\right),
\end{aligned} \tag{3.11}$$

for $i=\overline{1,M}$, where $w_{1l}^i=w_1(t_i,N_l)$, $w_{2l}^i=w_2(t_i,N_l)$, $f_l^i=f(t_i,N_l)$, $g_l^i=g(t_i,N_l)$, $l=\overline{1,nn}$.

Setting

$$\begin{aligned}
&b_{kl}=\int\limits_{\Omega}b_k(x,y)b_l(x,y)dxdy, && g_{kl}=\int\limits_{\Omega}\nabla b_k(x,y)\nabla b_l(x,y)dxdy\\
&B=(b_{kl})_{k,l=\overline{1,nn}}, &&\\
&R=(r_{kl})_{k,l=\overline{1,nn}}, && r_{kl}=C_p.b_{kl}+dt\,k.g_{kl},\\
&S=(s_{kl})_{k,l=\overline{1,nn}}, && s_{kl}=\alpha\xi.b_{kl}+dt\xi.g_{kl}-\frac{dt}{2\xi}b_{kl},\\
&FR=(fr_{kl})_{k,l=\overline{1,nn}}, && fr_{kl}=\int\limits_{\partial\Omega}b_k(x,y)b_l(x,y)dxdy,
\end{aligned}$$

the system (3.10)-(3.11) can be rewritten in the matrix form as

$$\begin{cases} R\,u_l^i + \dfrac{\ell}{2}B\,\varphi_l^i + hdt\,FRu_l^i = B\big(C_p\,u_l^{i-1} + \dfrac{\ell}{2}\varphi_l^{i-1} + dt\,f_l^i\big) + dt\,FRw_{1l}^i \\ S\,\varphi_l^i - s_\xi\,dtB\,u_l^i + c_0 dt\,FR\,\varphi_l^i = \alpha\xi B\big(\varphi_l^{i-1} + dtg_l^i\big) + dt\,FRw_{2l}^i, \end{cases} \tag{3.12}$$

where u_l^i, φ_l^i, and $l = \overline{1,nn}$ are unknown vectors corresponding to time the level i.

From the initial conditions (3.3), we have

$$u^0(x,y) \overset{\text{not}}{=} u_0(x,y) = \sum_{l=1}^{nn} u_0(N_l)b_l(x,y),$$
$$\varphi^0(x,y) \overset{\text{not}}{=} \varphi_0(x,y) = \sum_{l=1}^{nn} \varphi_0(N_l)b_l(x,y)$$

and then (see (3.4)-(3.5))

$$u_l^0 = u_0(N_l), \quad \varphi_l^0 = \varphi_0(N_l), \qquad l = \overline{1,nn}. \tag{3.13}$$

The numerical algorithm to calculate the approximate solution by fractional steps method can be obtained by the following sequence (i denotes the time level).

`Begin` **Armel-fracfem2D**
  `Choose` $T > 0$ and $\Omega \subset \mathbb{R}^2$;
  `Choose` $M > 0$, $nn > 0$ `and compute` $\varepsilon = dt, dx$;
  `Choose` $u_0, \varphi_0, w_1, w_2, f, g$;
  $i := 0 \quad \to \quad$ Compute u_l^0, φ_l^0, $l = \overline{1,nn}$ `from (3.13)`
  `For` $i := 1$ `to` M `do`
    `Compute` $z_l := z\,(\varepsilon, N_l)$, $l = \overline{1,nn}$ `from (2.2);`
    $\varphi^{i-1} := z_l, \quad l = \overline{1,nn}$;
    `Compute` u_l^i, φ_l^i, $l = \overline{1,nn}$`, solving the linear system (3.12);`
  `End-for;`
`End.`

Before to give some details regarding the numerical implementation of the algorithm `Armel_fracfem2D`, we recall that the convergence result established by Theorem 2.1 in previous Section guarantees that the approximate solution of the linear system (1.7)-(1.9) can be viewed as approximate solution of the nonlinear phase-field transition solution of the nonlinear phase-field transition System (1.1)-(1.3).

As it is well known, the finite element method (**fem**) is a general method for approximating the solution of boundary value problems for partial differential equations. This method is derived from the Ritz (or Gelerkin) method, characteristic for the finite element method being the choice of the finite dimensional space, namely, the *SPAN* of a set of finite element basis functions. The steps in solving a boundary value problem using **fem** are:

P0. (D) The direct formulation of the problem;

P1. (V) A variational (weak) formulation for problem (D);

P2. The construction of a finite element mesh (triangulation);

P3. The construction of the finite dimensional space of test function, called finite element basis functions;

P4. (V_{nn}) A discrete analogous of (V);

P5. Assemble the linear system of equations;

P6. Solve the system obtained in P5.

In our concrete case, on the position of (D) we have the problem given by relations (1.7)-(1.8) and, corresponding to (V), the relations (3.1)-(3.2). The next steps (**P2-P4**) ending with the discrete equations corresponding to (1.7)-(1.8), i.e. the linear system (3.12).

Corresponding to the physical parameters of the mathematical model (1.1)-(1.3), we have used industrial values indicated in the works [19,20]. To generate the triangulation T_ρ, we consider Ω a cross-section in a slab (thin) of $1300mm \times 220mm$. In Figure 1, the mesh can be seen in the directions of x_1 and x_2 - axis of a rectangular profile.

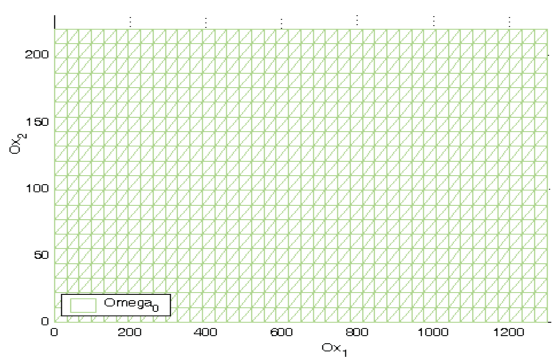

Figure 1. The triangulation over Ω =[0,1300]x[0,220]

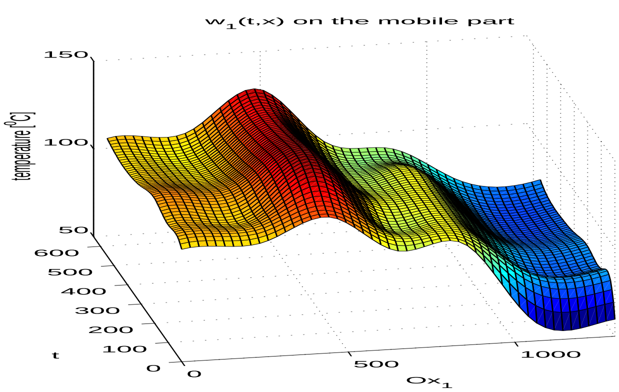

Figure 2. The values $w_1(t,x)$ on the mobile part

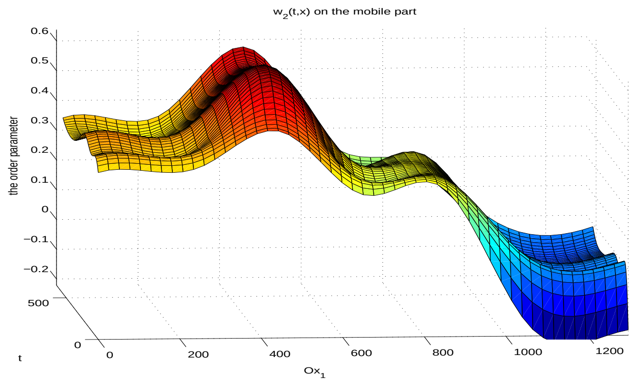

Figure 3. The values $w_2(t, x)$ on the mobile part

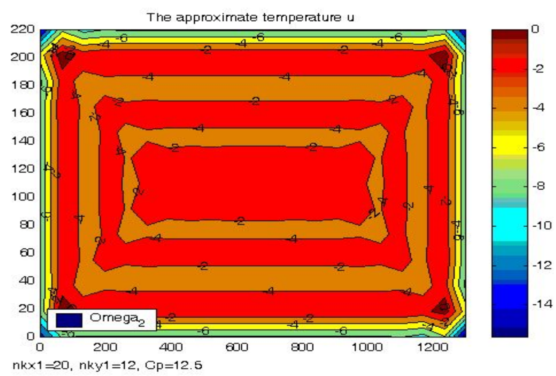

Figure 4. The approximate temperature u^2

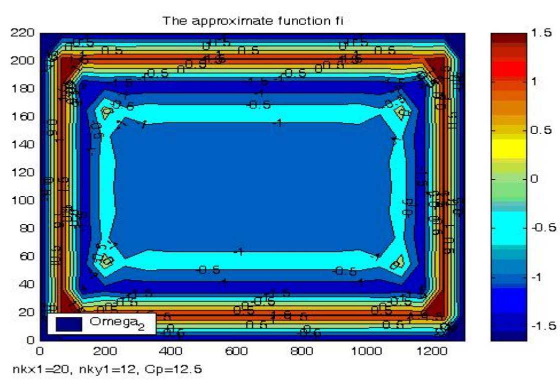

Figure 5. The approximate solution φ^2

The initial solution u_l^0, φ_l^0 in (3.13) was computed as solution of stationary equation $\varphi_t = \Delta\varphi = 0$ and as solution of Cauchy problem (1.10), respectively.

The values of $w_1(t, x)$ and $w_2(t, x) \in \Sigma$ are given as a spline interpolation (only the mobile part of the continuous casting machine is illustrated in Figures 2 and 3,

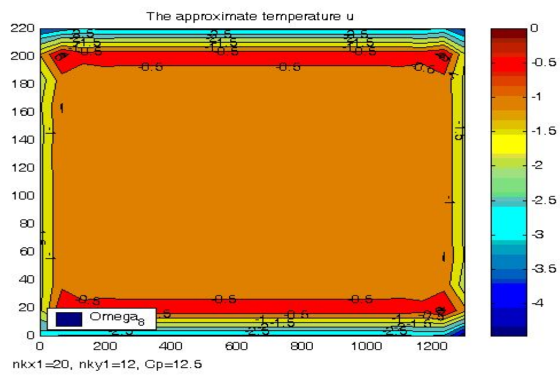

Figure 6. The approximate temperature u^8

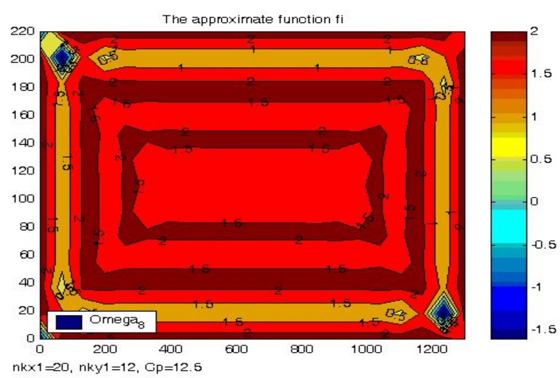

Figure 7. The approximate solution φ^8

respectively.

We shall present now the numerical experiments implementing the conceptual algorithm *Armel_fracfem2D*.

Figures 4 and 6 represent the approximate solutions u^i ($i = 2$ and $i = M = 8$, respectively), while Figures 5 and 7 represent the approximate solutions φ^i ($i = 2$ and $i = M = 8$, respectively).

The shape of the graphs shows the numerical stability and accuracy of the results obtained by implementing the *fractional steps method* (1.7) and (1.10). The most interesting aspect that we can observe while analyzing Figures 4-7 is the presence of *supercooling* and *superheating* phenomena (presence of solid fractions in the liquid, for example).

The numerical solution computed by this way can be considered as an admissible one for the corresponding boundary optimal control problem in order to improve the process optimization of continuous casting. Generally the fractional steps method considered here can used to approximate the solution of a nonlinear parabolic phase-field system containing a general nonlinear part.

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References

[1] R.A. Adams, *Sobolev spaces*, Academic Press, Orlando, San Diego, New-York, 1975.

[2] V. Arnăutu and C. Moroşanu, *Numerical approximation for the phase-field transition system*, Intern. J. Com. Math., 62(1996), 209-221.

[3] O. Axelson and V. Barker, *Finite element solution of boundary value problems*, Academic Press, 1984.

[4] V. Barbu, *A product formula approach to nonlinear optimal control problems*, SIAM J. Control and Optimiz., 26(1988), 496-520.

[5] V. Barbu, *Analysis and control of nonlinear infinite dimensional systems*, Academic Press, 190(1993).

[6] V. Barbu and M. Iannelli, *Approximating some non-linear equations by a Fractional step scheme,* Diff. and Integral Eqs., 1(1993), 15-26.

[7] V. Barbu and T. Precupanu, *Convexity and optimization in banach spaces*, 2^{rd} ed., Editura Academiei Bucureşti and D. Reidel Publ. Co., Dordrecht, Boston, Lancester, 1986.

[8] T. Benincasa and C. Moroşanu, *Fractional steps schemeto approximate the phase-field transition system with non-homogeneous Cauchy-Neumann boundary conditions*, Numer. Funct. Anal. & Optimiz., 30(2009), 199-213.

[9] T. Benincasa, A. Favini and C. Moroşanu, *A product formula approach to a non-homogeneous boundary optimal control problem governed by nonlinear phase-field transition system. PART I: A phase-field model*, J. Optim. Theory and Appl., 148(2011), 14-30.

[10] J.L. Boldrini, B.M.C. Caretta and E. Fernández-Cara, *Analysis of a two-phase field model for the solidification of an alloy,* J. Math. Anal. Appl., Vol., 357(2009), 25-44.

[11] G. Caginalp and X. Chen, *Convergence of the phase field model to its sharp interface limits*, Euro. Jnl of Applied Mathematics, 9(1998), 417-445.

[12] L. Cherfils, S. Gatti and A. Miranville, *Existence of global solutions to the Caginalp phase-field system with dynamic boundary conditions and singular potentials*, J.Math.Anal.Appl, 343(2002), 557-566.

[13] G. Fix, *Numerical simulation of free boundary problems using phase field models*, In: The Mathematics of Finite Elements and Applications IV, MAFELAP 1981, J.R. Whiteman, ed., Academic Press, London, New York, 265-279, 1982.

[14] O.A. Ladyzhenskaya, B.A. Solonnikov and N.N. Uraltzava, *Linear and quasilinear equations of parabolic type*, Prov. Amer. Math. Soc., 1968.

[15] C. Moroşanu, *Approximation and numerical results for phase field system by a fractional step scheme*, Revue d'Analyse Numérique et de Théorie de l'Approximation, 25(1996), 137-151.

[16] C. Moroşanu, *Approximation of the phase-field transition system via fractional steps method*, Numer. Funct. Anal. & Optimiz., 18(1997), 623-648.

[17] C. Moroşanu, *On the numerical stability of the cubic splines approximation to solution of phase-field transition system*, PanAmerican Math. J., 12(2002), 31-46.

[18] C. Moroşanu, *Approximation of the solid region in the continuous casting process of steel via phase-field transition system*, 6^{th} European Conference of Continuous Casting 2008, Riccione, Italy, 3-6 June, 2008.

[19] C. Moroşanu, Gh. Iorga and S. C. Cocindău, *Numerical simulation of the solid region via phase field transition system*, Metalurgia International, vol. XIII, 12(2008), 91-95.

[20] C. Moroşanu, Gh. Iorga and I. Tofan, *Numerical simulation of the thickness accretions in the secondary cooling zone of a continuous casting machine*, Metalurgia International, vol. XIV, 1(2009), 72-75.

[21] C. Moroşanu and Ana-Maria Moşneagu, *On the numerical approximation of the phase-field system with non-homogeneous Cauchy-Neumann boundary conditions. Case 1D*, ROMAI J., 9(2013), 91-110.

[22] C. Moroşanu and D. Motreanu, *An extension of the Lie-Trotter product formula*, Nonl. Funct. Anal. & Appl., 7(2002), 517-530.

[23] D. Motreanu and N. Pavel, *Tangency, flow invariance for differential equations, and optimization problems*, Marcel Dekker, Inc., New York, Basel, 1999.

[24] O. Penrose and P. C. Fife, *Thermodynamically consistent models of phase-field type for kinetics of phase transitions*, Phys. D., 43(1990), 44-62.

[25] C. Popa, *Trotter product formulae for Hamilton-Jacobi equations in infinite dimensions*, Diff. and Integral Eqs., 4(1991), 1251-1268.

[26] J.T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach eds., New York, 1969.