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DYNAMIC ANALYSIS OF A FRACTIONAL ORDER PHYTOPLANKTON MODEL

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Abstract The fractional order phytoplankton model (PM) can be written as $\frac{d^\alpha P_s}{dt^\alpha} = rP_s\big(1-\frac{P_s}{K}\big) - \frac{\upsilon P_s P_i}{P_s+1} + \gamma_1 P_i$, $\frac{d^\alpha P_{in}}{dt^\alpha} = \frac{\upsilon P_s P_i}{P_s+1} - \beta P_{in}$, $\frac{d^\alpha P_i}{dt^\alpha} = \beta_1 P_{in} - \delta P_i, P_s(\xi) = \varrho_0, \quad P_i(\xi) = \varrho_1, \quad P_{in}(\xi) = \varrho_2$, where P_s and P_i be the population densities of susceptible and infected phytoplankton respectively and P_{in} be the population density of population in incubated class. In this paper, stability analysis of the phytoplankton model is studied by using the fractional Routh-Hurwitz stability conditions. We have studied the local stability of the equilibrium points of PM. We applied an efficient numerical method based on converting the fractional derivative to integer derivative to solve the PM.

Keywords Phytoplankton model, Routh-Hurwitz stability conditions, numerical simulation.

MSC(2000) 26A33, 37B55.

1. Introduction

Phytoplankton are microscopic plants that live in the ocean. These small plants are very important to the ocean and to the whole planet! They are at the base of the food chain. Many small fish and whales eat them. Then bigger fish eat the little fish, etc. The food chain continues and at some point in time we (people) come into it when we eat the fish. So the energy of plankton becomes our energy too!. It has a major role in stabilizing the environment and survival of living population as it consumes half of the universal carbon-dioxide and releases oxygen. So far, there is a number of studies which show the presence of pathogenic viruses in the plankton community [7,13]. A good review of the nature of marine viruses and their ecological as well as their biological effects is given in [14]. Some researchers have shown using an electronic microscope that these viral diseases can affect bacteria and phytoplankton in coastal area and viruses are held responsible for the collapse of Emiliania huxleyi bloom in Mesocosms [10, 15]. Baghel et al. [3], proposed a three dimensional mathematical model of phytoplankton dynamics with the help of reaction-diffusion equations that studies the bifurcation and pattern formation mechanism. They provide an analytical explanation for understanding phytoplankton dynamics with three population classes: susceptible, incubated, and infected.

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In 2010, Dhar and Sharma [5] investigated the stability of the phytoplankton system

$$\begin{aligned} \frac{dP_s(t)}{dt} &= rP_s[1-\frac{P_s}{K}] - \upsilon P_s P_i + \gamma P_i, \\ \frac{dP_i(t)}{dt} &= \upsilon P_s P_i - \beta P_i, \end{aligned} \tag{1.1}$$

where P_s, P_i are the population densities of susceptible and infected phytoplankton at any instant of time t. r is the intrinsic growth rate of the population of susceptible phytoplankton, K is the carrying capacity of the population of susceptible phytoplankton, υ is the disease contact rate of the disease phytoplankton population, β is the removal rate of the disease phytoplankton population, out of which γ fraction of infected phytoplankton rejoin the susceptible phytoplankton population. Also the same system investigated by applying a frequency domain approach with time delay by Xu [18]. He use the delay as a bifurcation parameter; as it passes through a sequence of critical values, Hopf bifurcation occurs. A family of periodic solutions bifurcate from the equilibrium when the bifurcation parameter exceeds a critical value. Ghosh [9], proposed the interrelationship of latency period in viral infection and overall infection process in host community are of critical importance in context of pest control program. Both of them regulate the overall system stability as they are dynamically linked to predation by natural enemies in the system. In 2010, Dhar and Sharma [5], proposed the role of viral infection in phytoplankton dynamics without and with incubation population class is studied. It is observed that phytoplankton species in the absence of incubated class are unstable around an endemic equilibrium but the presence of delay in the form of incubated class has made it conditionally stable around an endemic equilibrium.

The authors of [16] proposed a prey-predator model for the phytoplankton-zooplankton system with the assumption that the viral disease is spreading only among the prey species, and, though the predator feeds on both the susceptible and infected prey, the infected prey is more vulnerable to predation as is seen in nature (see references quoted earlier). The dynamical behaviour of the system is investigated from the point of view of stability and persistence. The model shows that infection can be sustained only above a threshold of force of infection. Gakkhar and Negi [8] investigate the dynamical behaviour of toxin producing phytoplankton (TPP) and zooplankton. The phytoplanktons are divided into two groups, namely susceptible phytoplankton and infected phytoplankton. The conditions for coexistence for the populations are presented. Chattopadhyay et al. [4], deals with the problem of a nutrient-phytoplankton (N-P) populations where phytoplankton population is divided into two groups, namely susceptible phytoplankton and infected phytoplankton. Conditions for coexistence or extinction of populations are derived taking into account general nutrient uptake functions and Holling type-II functional response as an example.

2. Preliminaries

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The Riemann-Liouville and Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1} f(s) ds, $$

and

$$ {}_a^cD_{t_0}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, $$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$.

The initial value problem related to Definition 2.2 is

$$ \begin{cases} D^\alpha x(t) = f(t, x(t)), \\ x(t)|_{t=0^+} = x_0, \end{cases} \tag{2.1} $$

where $0 < \alpha < 1$ and $D^\alpha = D_0^\alpha$.

Now, some stability theorems on fractional-order systems are introduced.

Theorem 2.1 ([12]). *The following autonomous system:*

$$ \frac{d^\alpha x}{dt^\alpha} = Ax, \quad x(0) = x_0, \tag{2.2} $$

with $0 < \alpha \le 1$, $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, is asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ is satisfied for all eigenvalues of matrix A. Also, this system is stable if and only if $|\arg(\lambda)| \ge \frac{\alpha\pi}{2}$ is satisfied for all eigenvalues of matrix A with those critical eigenvalues satisfying $|\arg(\lambda)| = \frac{\alpha\pi}{2}$ having geometric multiplicity of one. The geometric multiplicity of an eigenvalue λ of the matrix A is the dimension of the subspace of vectors v for which $Av = \lambda v$.

Theorem 2.2 ([6]). *Consider the following commensurate fractional-order system:*

$$ \frac{d^\alpha x}{dt^\alpha} = f(x), \quad x(0) = x_0, \tag{2.3} $$

with $0 < \alpha \le 1$ and $x \in \mathbb{R}^n$. The equilibrium points of system (3.1) are calculated by solving the following equation: $f(x) = 0$. These points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ evaluated at the equilibrium points satisfy: $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

3. Mathematical Model

Let P_s and P_i be the population densities of susceptible and infected phytoplankton respectively. The population of susceptible phytoplankton is assumed to be growing logistically with intrinsic growth rate r and carrying capacity K. Now let P_{in} be the population density of population in incubated class. Here, we will use nonlinear Holling Type II functional responses for disease spreading because the disease conversion rates become saturated as victim densities increase. Let υ be the disease contact rate and it is volume-specific encounter rate between susceptible and infected phytoplankton, which is equivalent to the inverse of the average search time between successful spreading of disease. The coefficients δ and β are the total removal of phytoplankton from the infected and incubated class because of the death (including recovered) from disease and due to natural causes respectively. Again, γ_1 be the fraction of the population recovered from infected phytoplankton population and joined in the susceptible phytoplankton population and β_1 is the fraction of the incubated class population which will move to the infected class. Therefore, quantitatively $\delta > \gamma_1$ and $\beta > \beta_1$. Using these assumptions the dynamics of the system can be governed by the following set of differential equations:

$$\begin{aligned}
&\frac{dP_s}{dt} = rP_s\big(1 - \frac{P_s}{K}\big) - \frac{\upsilon P_s P_i}{P_s + 1} + \gamma_1 P_i,\\
&\frac{dP_{in}}{dt} = \frac{\upsilon P_s P_i}{P_s + 1} - \beta P_{in},\\
&\frac{dP_i}{dt} = \beta_1 P_{in} - \delta P_i,\\
&P_s(0) > 0, \quad P_i(0) > 0, \quad P_{in}(0) > 0.
\end{aligned} \tag{3.1}$$

The Holling type-II the functional response $\frac{\upsilon P_s P_i}{P_s+1}$ is used [11] and many other researchers. In this paper we investigate the following fractional order phytoplankton model with initial population; i.e., $P_s(\xi) > 0$, $P_{in}(\xi) > 0$, $P_i(\xi) > 0$ and the total population at any instant t is $N(t) = P_s(t) + P_{in}(t) + P_i(t)$:

$$\begin{aligned}
&\frac{d^\alpha P_s}{dt^\alpha} = rP_s\big(1 - \frac{P_s}{K}\big) - \frac{\upsilon P_s P_i}{P_s + 1} + \gamma_1 P_i,\\
&\frac{d^\alpha P_{in}}{dt^\alpha} = \frac{\upsilon P_s P_i}{P_s + 1} - \beta P_{in},\\
&\frac{d^\alpha P_i}{dt^\alpha} = \beta_1 P_{in} - \delta P_i,\\
&P_s(\xi) = \varrho_0, \quad P_i(\xi) = \varrho_1, \quad P_{in}(\xi) = \varrho_2.
\end{aligned} \tag{3.2}$$

where the parameters $0 < \alpha \le 1$, and $\frac{d^\alpha}{dt^\alpha}$ is in the sense of the Caputo fractional derivative defined in (2.2) with the initial time $t = \xi$.

4. Stability of equilibria

In this section we deal with the stability local dynamics of system (3.2). Let

$$\frac{d^\alpha P_s}{dt^\alpha} = 0, \quad \frac{d^\alpha P_{in}}{dt^\alpha} = 0, \quad \frac{d^\alpha P_i}{dt^\alpha} = 0. \tag{4.1}$$

Solving (4.1) for its roots, we can get that system (3.2) has three equilibria points as follows

$$O = (0,0,0), \quad E_1 = (K,0,0), \quad E_2 = (P^*, Q^*, R^*) \tag{4.2}$$

where

$$P^* = \frac{\delta\beta}{\upsilon\beta_1 - \delta\beta}, \quad Q^* = \frac{rP^*(P^*+1)(1 - \frac{P^*}{K})}{\upsilon P^* - \gamma_1(P^*+1)}, \quad R^* = \frac{r\delta P^*(P^*+1)(1 - \frac{P^*}{K})}{\beta_1(\upsilon P^* - \gamma_1(P^*+1))}.$$

If the Jacobian matrix of system (3.2) at the equilibrium point O is

$$J(O) = \begin{pmatrix} r & \gamma_1 & 0 \\ 0 & 0 & -\beta_1 \\ 0 & -\delta_1 & \beta_1 \end{pmatrix}, \tag{4.3}$$

with the characteristic equation

$$Q(\lambda) = det(\lambda - J(o)) = \lambda^3 + \big(-r - \beta_1\big)\lambda^2 + \big(-\delta\beta + r\beta_1\big)\lambda + r\delta\beta.$$

The eigenvalues corresponding to the equilibrium O are

$$\lambda_1 = r,$$
$$\lambda_{2,3} = \tfrac{\beta_1 \pm \sqrt{\beta_1^2 + 4\delta\beta}}{2}.$$

Then we have $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 > 0$. Whence it follows that the equilibrium O of system (3.2) is unstable. Thus the stable manifold of the origin $W^s(O)$ is one-dimensional and the unstable manifold of the origin $W^u(O)$ is two-dimensional.

The Jacobian matrix of (3.2) at equilibrium point $E_1 = (K, 0, 0)$ is

$$J(E_1) = \begin{pmatrix} -r & \gamma_1 - \frac{\upsilon K}{K+1} & 0 \\ 0 & \frac{\upsilon K}{K+1} & -\beta \\ 0 & -\delta & \beta_1 \end{pmatrix}, \tag{4.4}$$

with the characteristic equation

$$Q(\lambda) = det(\lambda - JE_1) = \lambda^3 + B_1\lambda^2 + B_2\lambda + B_3,$$

where

$$B_1 = r - \frac{\upsilon K}{K+1} - \beta_1,\ B_2 = \frac{\beta_1 \upsilon K}{K+1} - \delta\beta - r(\frac{\upsilon K}{K+1} + \beta_1),\ B_3 = r(\frac{\beta_1 \upsilon K}{K+1} - \delta\beta).$$

Let $D(Q)$ denote the discriminant of a polynomial $Q(\lambda)$. Then

$$D(Q) = 18B_1B_2B_3 + (B_1B_2)^2 - 4B3B_1^3 - 4B_2^3 - 27B_3^3.$$

Using the proposition given in [17], we have the following result by using Routh-Hurwitz conditions.

Theorem 4.1. *The equilibrium point E_1 of the system (3.2) is asymptotically stable if one of the following conditions holds for polynomial Q:*

(i) $D(Q) > 0, B_1 > 0, B_3 > 0$ *and* $B_1B_2 > B_3$.

(ii) $D(Q) < 0, B_1 \geq 0, B_2 \geq 0, B_3 > 0$ *and* $\alpha < \frac{2}{3}$.

(iii) $D(Q) < 0, B_1 < 0, B_2 < 0$ *and* $\alpha > \frac{2}{3}$.

The Jacobian matrix of (3.2) at equilibrium point $E_2 = (P^*, Q^*, R^*)$ is

$$J(E_2) = \begin{pmatrix} r - 2\frac{rP^*}{K} - \upsilon\frac{Q^*}{(P^*+1)^2} & \gamma_1 - \frac{\upsilon P^*}{P^*+1} & 0 \\ \upsilon\frac{Q^*}{(P^*+1)^2} & \frac{\upsilon P^*}{P^*+1} & -\beta \\ 0 & -\delta & \beta_1 \end{pmatrix}, \tag{4.5}$$

with the characteristic equation

$$Q(\lambda) = det(\lambda - JE_1) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3,$$

where

$$\begin{aligned}
A_1 &= -r + 2\frac{rP^*}{K} - \beta_1 - \upsilon\frac{Q^*}{(P^*+1)^2}) - \frac{\upsilon P^*}{P^*+1} - \beta_1,\\
A_2 &= -\frac{\upsilon P^*}{P^*+1}(-r + 2\frac{rP^*}{K} + \upsilon\frac{Q^*}{(P^*+1)^2}) + \upsilon^2\frac{(Q^*)^2}{(P^*+1)^4} - \delta\beta,\\
A_3 &= \frac{\beta_1\upsilon P^*}{P^*+1}(-r + 2\frac{rP^*}{K} + \upsilon\frac{Q^*}{(P^*+1)^2}) - \delta\beta(-r + 2\frac{rP^*}{K} + \upsilon\frac{Q^*}{(P^*+1)^2}).
\end{aligned}$$

Using the proposition given in [17], we have the following result by using Routh-Hurwitz conditions.

Theorem 4.2. *The equilibrium point E_2 of the system (3.2) is asymptotically stable if one of the following conditions holds for polynomial Q:*

(i) $D(Q) > 0, A_1 > 0, A_3 > 0$ *and* $A_1A_2 > A_3$.

(ii) $D(Q) < 0, A_1 \geq 0, A_2 \geq 0, A_3 > 0$ *and* $\alpha < \frac{2}{3}$.

(iii) $D(Q) < 0, A_1 < 0, A_2 < 0$ *and* $\alpha > \frac{2}{3}$.

5. The numerical decomposition method

In order to solve (3.2), we shall use a numerical method introduced by Atanackovic and Stankovic [1] to solve the single linear fractional differential equation (FDE). Also the same authors [2] developed the method to solve the nonlinear FDE. In [1] it was shown that for a function $f(t)$, the fractional derivative of order α with $0 < \alpha \leq 1$ may be expressed as

$$\begin{aligned}
D^\alpha f(t) = \quad & \frac{1}{\Gamma(2-\alpha)}\Big\{\frac{f^{(1)}(t)}{t^{\alpha-1}}[1 + \sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}]\\
& -[\frac{\alpha-1}{t^\alpha}f(t) + \sum_{p=2}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}(\frac{f(t)}{t^\alpha} + \frac{V_p(f)(t)}{t^{p-1+\alpha}})]\Big\},
\end{aligned} \tag{5.1}$$

where

$$V_p(f)(t) = -(p-1)\int_0^t \tau^{p-2}f(\tau)d\tau,\ p = 2, 3, \cdots, \tag{5.2}$$

with the following properties

$$\frac{d}{dt}V_p(f) = -(p-1)t^{p-2}f(t),\ p = 2, 3, \cdots. \tag{5.3}$$

We approximate $D^\alpha f(t)$ by using M terms in sums appearing in Eq. (5.1) as follows

$$\begin{aligned}
D^\alpha f(t) \simeq \quad & \frac{1}{\Gamma(2-\alpha)}\Big\{\frac{f^{(1)}(t)}{t^{\alpha-1}}[1 + \sum_{p=1}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}]\\
& -[\frac{\alpha-1}{t^\alpha}f(t) + \sum_{p=2}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}(\frac{f(t)}{t^\alpha} + \frac{V_p(f)(t)}{t^{p-1+\alpha}})]\Big\}.
\end{aligned} \tag{5.4}$$

We can rewrite Eq. (5.4) as follows

$$D^{\alpha}f(t) \simeq \Omega(\alpha,t,M)f^{(1)}(t) + \Phi(\alpha,t,M)f(t) + \sum_{p=2}^{M} A(\alpha,t,p)\frac{V_p(f)(t)}{t^{p-1+\alpha}}, \tag{5.5}$$

where

$$\begin{aligned}
\Omega(\alpha,t,M) &= \frac{1+\sum_{p=1}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}}{\Gamma(2-\alpha)t^{\alpha-1}},\\
R(\alpha,t) &= \frac{1-\alpha}{t^{\alpha}\Gamma(2-\alpha)},\ A(\alpha,t,p) = -\frac{\Gamma(p-1+\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)p!},\\
\Phi(\alpha,t,M) &= R(\alpha,t) + \sum_{p=2}^{M}\frac{A(\alpha,t,p)}{t^{\alpha}}.
\end{aligned}$$

We set

$$\begin{aligned}
&\Theta_1(t) = P_s(t), && \Theta_{M+1}(t) = P_i(t),\\
&\Theta_{2M+1}(t) = P_{in}(t), && \Theta_p(t) = V_p(P_s)(t),\\
&\Theta_{p+M}(t) = V_p(P_i)(t), && \Theta_{2M+p}(t) = V_p(P_{in})(t),
\end{aligned}$$

for $p = 2, 3, \cdots$.

We can rewrite system (3.1) as the following form

$$\begin{aligned}
&\Omega(\alpha,t,M)\Theta_1'(t) + \Phi(\alpha,t,M)\Theta_1(t) + \sum_{p=2}^{M} A(\alpha,t,p)\frac{\Theta_p(t)}{t^{p-1+\alpha}}\\
=\ & r\Theta_1(t)(1-\frac{\Theta_1(t)}{K}) - \upsilon\frac{\Theta_{M+1}(t)\Theta_1(t)}{\Theta_1(t)+1} + \gamma_1\Theta_{M+1}(t),\\
&\Omega(\alpha,t,M)\Theta_{M+1}'(t) + \Phi(\alpha,t,M)\Theta_{M+1}(t) + \sum_{p=2}^{M} A(\alpha,t,p)\frac{\Theta_{M+p}(t)}{t^{p-1+\alpha}}\\
=\ & \upsilon\frac{\Theta_{M+1}(t)\Theta_1(t)}{\Theta_1(t)+1} - \beta\Theta_{2M+1}(t),\\
&\Omega(\alpha,t,M)\Theta_{2M+1}'(t) + \Phi(\alpha,t,M)\Theta_{2M+1}(t) + \sum_{p=2}^{M} A(\alpha,t,p)\frac{\Theta_{2M+p}(t)}{t^{p-1+\alpha}}\\
=\ & \beta_1\Theta_{2M+1} - \delta\Theta_{M+1},
\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}
&\Theta_p(t) = -(p-1)\int_0^t \tau^{p-2}\Theta_1(t)(\tau)d\tau,\\
&\Theta_{M+p}(t) = -(p-1)\int_0^t \tau^{p-2}\Theta_{M+1}(t)(\tau)d\tau,\\
&\Theta_{2M+p}(t) = -(p-1)\int_0^t \tau^{p-2}\Theta_{2M+1}(t)(\tau)d\tau,\\
&p = 2, 3, \cdots, M.
\end{aligned} \tag{5.7}$$

Now we can rewrite (5.6) and (5.7) as the following form

$$
\begin{aligned}
\Theta_1'(t) &= \frac{1}{\Omega(\alpha,t,M)}(r\Theta_1(t)(1-\frac{\Theta_1(t)}{K}) - \upsilon\frac{\Theta_{M+1}(t)\Theta_1(t)}{\Theta_1(t)+1} + \gamma_1\Theta_{M+1}(t))\\
&\quad - \Phi(\alpha,t,M)\Theta_1(t) - \sum_{p=2}^{M} A(\alpha,t,p)\frac{\Theta_p(t)}{t^{p-1+\alpha}}),\\
\Theta_p'(t) &= -(p-1)t^{p-2}\Theta_1(t),\ p=2,3,\cdots,M,\\
\Theta_{M+1}'(t) &= \frac{1}{\Omega(\alpha,t,M)}(\upsilon\frac{\Theta_{M+1}(t)\Theta_1(t)}{\Theta_1(t)+1} - \beta\Theta_{2M+1}(t)\\
&\quad - \Phi(\alpha,t,M)\Theta_{M+1}(t) - \sum_{p=2}^{M} A(\alpha,t,p)\frac{\Theta_{M+p}(t)}{t^{p-1+\alpha}}),\\
\Theta_{M+p}'(t) &= -(p-1)t^{p-2}\Theta_{M+1}(t),\ p=2,3,\cdots,M,\\
\Theta_{2M+1}'(t) &= \frac{1}{\Omega(\alpha,t,M)}(\beta_1\Theta_{2M+1} - \delta\Theta_{M+1})\\
&\quad - \Phi(\alpha,t,M)\Theta_{2M+1}(t) - \sum_{p=2}^{M} A(\alpha,t,p)\frac{\Theta_{2M+p}(t)}{t^{p-1+\alpha}}),\\
\Theta_{2M+p}'(t) &= -(p-1)t^{p-2}\Theta_{2M+1}(t),\ p=2,3,\cdots,M,
\end{aligned}
\tag{5.8}
$$

with the following initial conditions

$$
\begin{aligned}
&\Theta_1(\xi) = \varrho_0, &\\
&\Theta_p\ (\xi) = 0, & p=2,3,\cdots,M,\\
&\Theta_{M+1}(\xi) = \varrho_1, &\\
&\Theta_{M+p}(\xi) = 0, & p=2,3,\cdots,M,\\
&\Theta_{2M+1}(\xi) = \varrho_2, &\\
&\Theta_{2M+p}(\xi) = 0, & p=2,3,\cdots,M.
\end{aligned}
\tag{5.9}
$$

Now we consider the numerical solution of system of ordinary differential equations (5.8) with the initial conditions (5.9) by using the well known Runge-Kutta method of order fourth.

6. Numerical Simulation

To verify the effectiveness of the obtained results, some numerical simulations for the fractional-order system (3.2) have been conducted. All the differential equations are solved using the method proposed in the previous section. In all numerical runs, the solution has been approximated at $\xi = \Delta t = 0.005$. Now we set $r = 0.1, \upsilon = \gamma_1 = \beta = \beta_1 = \delta = 0.1$ and $K = 1,$. In figure 1-3 we plot the numerical solution of system (3.2) at $\alpha = 0.80(0.02)0.96$.

Now we set $r = 5, \upsilon = 0.5, \gamma_1 = 0.1, \beta = 0.1, \beta_1 = 0.01, \delta = 0.4$ and $K = 100,$. In figure 4-6 we plot the numerical solution of system (3.2) at $\alpha = 0.80(0.02)0.96$.

In Figs. 7-9, we display the phase plane of PM respectively for $\alpha = 0.85, 0.9, 0.95$. The parameters have been set to $K = 50, \upsilon = 50, r = 8, \delta_1 = 0.002, \beta = 0.04, \beta_1 = 0.005, \delta = 0.02$ and $M = 5$.

7. Conclusion

In this work, we analyze the dynamic behavior of the fractional-order phytoplankton model. Firstly, we study the existence of extinction equilibrium and boundary equilibria, furthermore give stability criteria of the model from local point of view by using the fractional Routh-Hurwits criterion. The corresponding results are illustrated by the numerical simulation. Simulation results show the effectiveness of the method. Further, it has been shown that the supply rate α play an important role in shaping the dynamics of the system. A stable limit cycle is observed in Figs.7-9.

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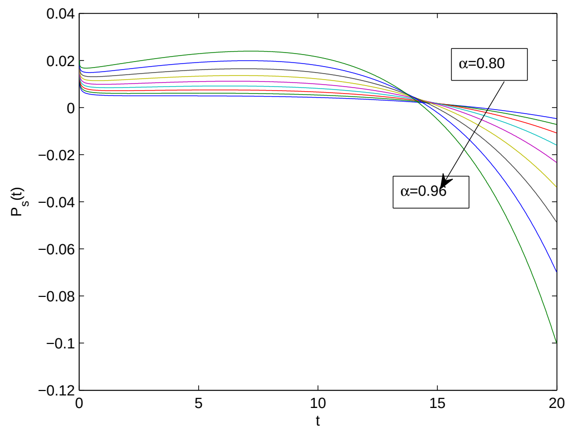

Figure 1. The numerical solution of system (3.2) at $\alpha = 0.80(0.02)0.96$

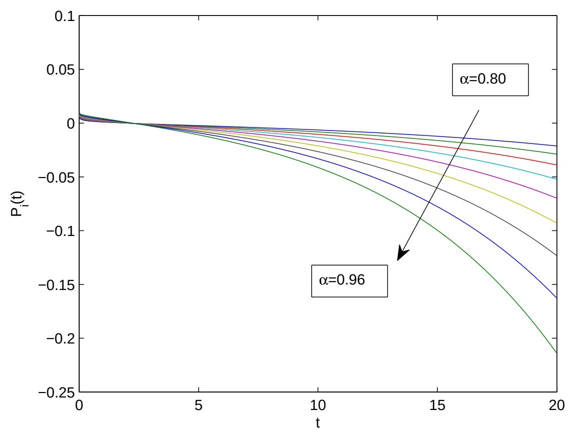

Figure 2. The numerical solution of system (3.2) at $\alpha = 0.80(0.02)0.96$

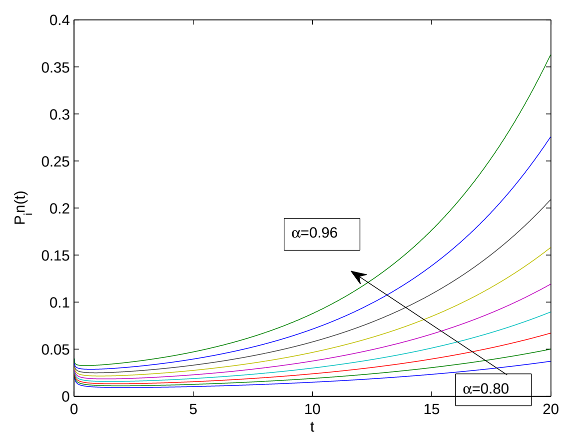

Figure 3. The numerical solution of system (3.2) at $\alpha = 0.80(0.02)0.96$

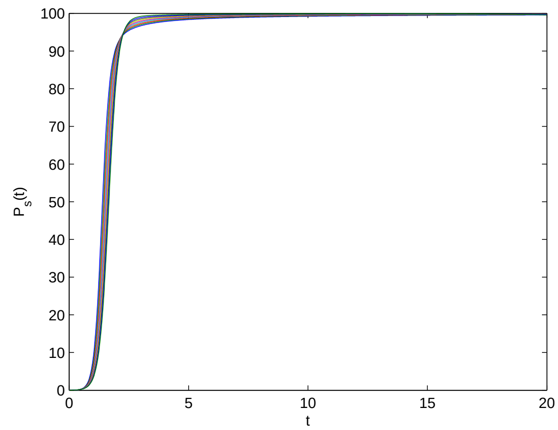

Figure 4. The numerical solution of system (3.2) at $\alpha = 0.80(0.02)0.96$

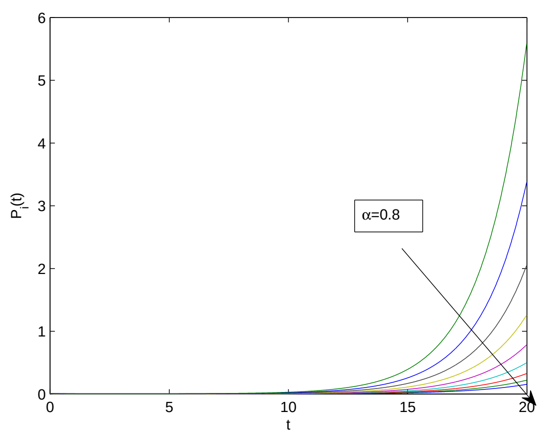

Figure 5. The numerical solution of system (3.2) at $\alpha = 0.80(0.02)0.96$

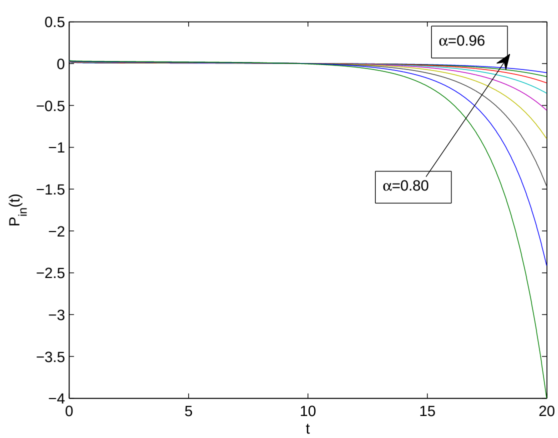

Figure 6. The numerical solution of system (3.2) at $\alpha = 0.80(0.02)0.96$

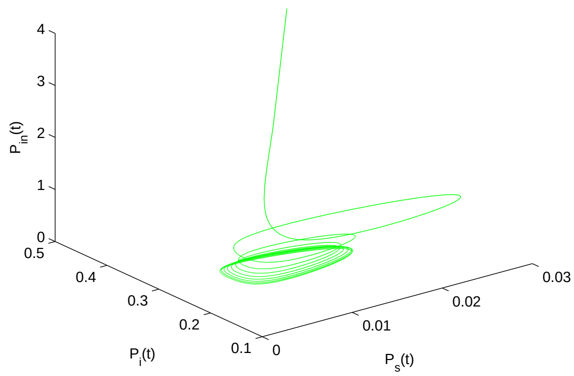

Figure 7. Phase plane of PM at $K = 50, \upsilon = 50, r = 8, \delta_1 = 0.002, \beta = 0.04, \beta_1 = 0.005, \delta = 0.02, \alpha = 0.85$ and $M = 5$.

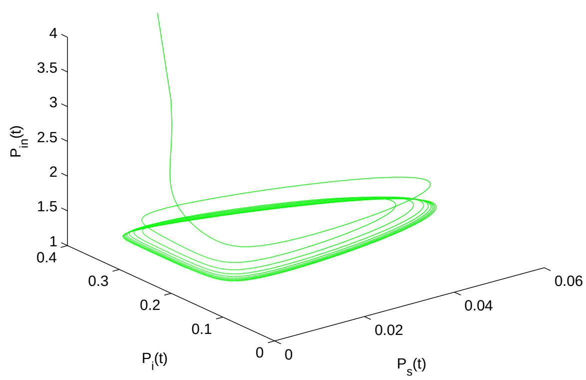

Figure 8. Phase plane of PM at $K = 50, \upsilon = 50, r = 8, \delta_1 = 0.002, \beta = 0.04, \beta_1 = 0.005, \delta = 0.02, \alpha = 0.9$ and $M = 5$.

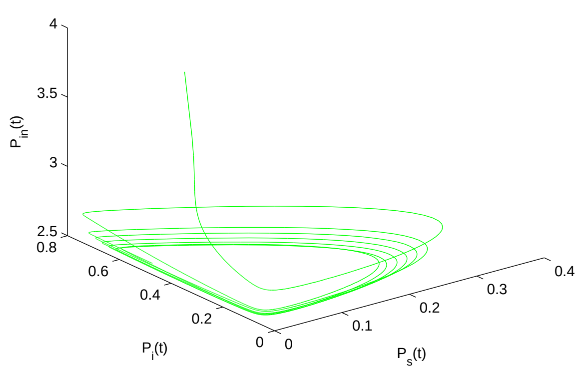

Figure 9. Phase plane of PM at $K = 50, \upsilon = 50, r = 8, \delta_1 = 0.002, \beta = 0.04, \beta_1 = 0.005, \delta = 0.02, \alpha = 0.95$ and $M = 5$.