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EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF A NONLINEAR SYSTEM

N.H. Zhao[1], Y. Xia [1,†], Wenxia Liu[1]
P. J. Y. Wong [2] and R.T. Wang [1]

Abstract This paper considers the existence of almost periodic solutions of a N-dimension non-autonomous Lotka-Volterra model with delays. The method is based on exponential dichotomy and Schauder fixed point theorem. The obtained results generalize some previously known ones.

Keywords Fixed point theorem, exponential dichotomy, periodic solutions, delayed.

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1. Introduction

The existence of almost periodic solutions of ordinary differential equations has been discussed extensively in theory and in practice. Many useful methods have been developed for the study of almost periodic differential equations, we refer the reader to classical references such as Hale [5], Fink [3], Yoshizawa [10]. These methods are useful in studying the existence of almost periodic solutions of differential equations, and thus have many applications in specific systems arising form biology, neural networks, physics, chemistry and engineering. One of the powerful approaches are the combination between exponential dichotomy and fixed point theory. The use of *different* fixed point theorems will yield *different* sufficient (necessary) conditions for the existence of almost periodic solutions. For example, if Banach fixed point theorem is employed, we need $r < 1$ (where $\|Tx - Ty\| \leq r\|x - y\|$). However, this restriction would not be necessary if one uses the Schauder fixed point theorem (e.g. see [2]). There are many fixed point theorems including Banach fixed point, Brower fixed point, generalized fixed point, Leray-Schauder fixed point, Horn fixed point, Schauder fixed point and so on. In this paper, we shall employ Schauder fixed point theorem to study a nonlinear system arising from the biological system - Lotka Volterra model.

[†]the corresponding author. Email address: yhxia@zjnu.cn;xiadoc@163.com(Y. Xia)
[1]Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, China
[2]School of Electrical and Electronic Engineering, Nanyang Technological University, 639798, Singapore
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In this paper, we consider the almost periodic Lotka-Voterra system as follows:

$$\begin{aligned}\dot{x_i}(t)) = \quad & x_i(t)\Big[r_i(t) - \sum_{j=1}^{N} a_{ij}(t)x_j(t) - \sum_{j=1}^{N} b_{ij}(t)x_j(t-\tau(t)) \\ & - \sum_{j=1}^{N} c_{ij}(t)\int_{-\infty}^{0} K_i(t-s)x_j(s)ds\Big].\end{aligned} \tag{1.1}$$

For system (1.1), many results have been obtained in regard to permanence, extinction, bifurcation, chaos, asymptotic stability and periodic solutions. Some authors have employed the hull system theory and asymptotic stability theory to obtain the existence of almost periodic solutions of (1.1). However, by such methods, they have obtained relatively strict conditions on the parameters in order to guarantee the asymptotic stability. The reader may refer to [1, 4, 6–9]. To obtain weaker sufficient conditions for the existence of almost periodic solutions of (1.1), we shall employ the exponential dichotomy theory and Schauder fixed point theorem to system (1.1). The proof and results are much different from the previous literature.

The structure of this paper is as follows. In Section 2, we shall introduce some definitions and lemmas. In Section 3, the main results are presented.

2. Some Definitions and Lemmas

In this section, we recall some definitions and lemmas which can be found in [3,5,10]. Suppose that $f:\mathbb{R}\longrightarrow\mathbb{R}$ is an almost periodic function. Define the *mean value* of $f(t)$ by

$$M[f] = \lim_{T\to\infty}\frac{1}{T}\int_{t}^{t+T} f(s)ds.$$

If $f(t)$ is a T-periodic function, then

$$M[f] = \frac{1}{T}\int_{0}^{T} f(s)ds.$$

The notation $mod(f)$ is used to denote the *modulus* of the almost periodic function $f(t)$. Let $f(t), g(t)$ be almost periodic functions with $\inf_{t\in\mathbb{R}}|f(t)| > 0$, then $g(t)/f(t)$ is also almost periodic and $mod(g/f) \subset mod(f, g)$. For the continuous and bounded function $f(t)$, denote

$$\|f\| = \sup_{t\in\mathbb{R}}|f(t)|, \qquad \overline{f} = \sup_{t\in\mathbb{R}} f(t), \qquad \underline{f} = \inf_{t\in\mathbb{R}} f(t).$$

Lemma 2.1. *Suppose that $\{f_n(t)\}$ is an almost periodic function sequence and $f(t)$ is an almost periodic function. If $mod(f_n) \subset mod(f)$, $n = 1, 2, \cdots$, and f_n is locally uniformly convergent on $\mathbb{R}$, then $\{f_n\}$ is uniformly convergent on $\mathbb{R}$.*

Lemma 2.2. *Suppose that $f(t, x)$ is a uniform almost periodic function, and $\varphi(t), g(t)$ are almost periodic functions. If $mod(f) \subset mod(g)$ and $mod(\varphi) \subset mod(g)$, then $mod(f(t, \varphi(t))) \subset mod(g)$.*

Lemma 2.3. *(Schauder fixed point theorem) Suppose that* $\mathbf{B}$ *is a Banach Space,* $\mathbf{C}$ *is a closed convex subset of* $\mathbf{B}$. *If* $\mathcal{T}$: $\mathbf{C} \to \mathbf{C}$ *is a continuous and compact operator, then* $\mathcal{T}$ *has a fixed point in* $\mathbf{C}$.

Lemma 2.4. *Suppose that the linear system* $\dot{x} = A(t)x$ *has an exponential dichotomy on* $\mathbb{R}$, *i.e., there exist constants* $K, \alpha > 0$ *and a fundamental matrix* $X(t)$ *satisfying*

$$\begin{cases} \|X(t)PX^{-1}(s)\| \le K\exp\{-\alpha(t-s)\}, & t \ge s, \\ \|X(t)(I-P)X^{-1}(s)\| \le K\exp\{\alpha(t-s)\}, & t \le s, \end{cases}$$

then the almost periodic system $\dot{x} = A(t)x + f(t)$ *has a unique almost periodic solution* $x(t)$ *with* $mod(x) \subset mod(A, f)$, *which can be represented as*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)f(s)ds.$$

3. Results and Proofs

Throughout the paper, we always take $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, N$, unless otherwise stated. Throughout the paper, we assume the following:

(H_1) $r_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t) : \mathbb{R} \longrightarrow \mathbb{R}$ are almost periodic functions with $M[r_i] > 0$.

(H_2) $K_i(s)$ is a nonnegative continuous function defined on $(-\infty, 0)$ with $\int_0^\infty K_i(s)ds = 1$.

(H_3) $\dfrac{1}{m}\displaystyle\int_{-\infty}^{t} e^{-\int_s^t r_i(\sigma)d\sigma} \sum_{j=1, j\ne i}^{N} [a_{ij}(s) + b_{ij}(s) + c_{ij}(s)]ds < 1$, where

$$m = \min\{m_1, m_2, \cdots, m_N\} > 0$$

and

$$m_i = \inf_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t r_i(\sigma)d\sigma}[a_{ii}(s) + b_{ii}(s) + c_{ii}(s)]ds.$$

Making the change of variables

$$u_i(t) = \frac{1}{x_i(t)},$$

the system (1.1) becomes

$$\begin{aligned} \dot{u_i}(t) = \quad & -r_i(t)u_i(t) + a_{ii}(t) + \sum_{j=1, j\ne i}^{N} a_{ij}(t)\frac{u_i(t)}{u_j(t)} \\ & + b_{ii}(t) + \sum_{j=1, j\ne i}^{N} b_{ij}(t)\frac{u_i(t)}{u_j(t-\tau(t))} \\ & + c_{ii}(t) + \sum_{j=1, j\ne i}^{N} c_{ij}(t)\int_{-\infty}^{0} K_i(t-\sigma)\frac{u_i(\sigma)}{u_j(\sigma)}d\sigma \end{aligned} \tag{3.1}$$

Theorem 3.1. *Suppose that $r_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t)$ are almost periodic functions. If $(H_1), (H_2), (H_3)$ hold, then system (3.1) has almost periodic solutions $u_i(t)$ with $\inf_{t\in\mathbb{R}} u_i(t) > 0$, i.e., system (1.1) has almost periodic solutions $x_i(t)$ with $\inf_{t\in\mathbb{R}} x_i(t) > 0$.*

Proof. Let $\mathbf{S} = \{\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_N(t)) \mid \varphi_i(t) : \mathbb{R}\longrightarrow\mathbb{R}$ is almost periodic function with $\inf_{t\in\mathbb{R}} \varphi_i(t) \geq m$, $mod(\varphi_i(t)) \subset mod(r_i, a_{ij} + b_{ij} + c_{ij})\}$. For any $\varphi(t) \in \mathbf{S}$, consider the following auxiliary equation

$$\begin{aligned} \dot{u_i}(t) = \quad & -r_i(t)u_i(t) + a_{ii}(t) + \sum_{j=1,j\neq i}^{N} a_{ij}(t)\frac{\varphi_i(t)}{\varphi_j(t)} \\ & +b_{ii}(t) + \sum_{j=1,j\neq i}^{N} b_{ij}(t)\frac{\varphi_i(t)}{\varphi_j(t-\tau(t))} \\ & +c_{ii}(t) + \sum_{j=1,j\neq i}^{N} c_{ij}(t)\int_{-\infty}^{0} K_i(t-\sigma)\frac{\varphi_i(\sigma)}{\varphi_j(\sigma)}d\sigma. \end{aligned} \tag{3.2}$$

Let $u^{\varphi}(t) = \left(u_1^{\varphi}(t), u_2^{\varphi}(t), \cdots, u_N^{\varphi}(t)\right)^T$ be a solution of system (3.2). In view of $M[r_i] > 0$, the linear system $\dot{u_i}(t) = -r_i(t)u_i(t)$ has an exponential dichotomy. From Lemma 2.4, system (3.2) has a unique almost periodic solution satisfying

$$\begin{aligned} u_i^{\varphi}(t) = \quad & \int_{-\infty}^{t} e^{-\int_s^t r_i(\sigma)d\sigma}\Big[a_{ii}(s) + \sum_{j=1,j\neq i}^{N} a_{ij}(s)\frac{\varphi_i(s)}{\varphi_j(s)} \\ & +b_{ii}(s) + \sum_{j=1,j\neq i}^{N} b_{ij}(s)\frac{\varphi_i(s)}{\varphi_j(s-\tau(s))} \\ & +c_{ii}(s) + \sum_{j=1,j\neq i}^{N} c_{ij}(s)\int_{-\infty}^{0} K_i(t-\sigma)\frac{\varphi_i(\sigma)}{\varphi_j(\sigma)}d\sigma\Big]ds, \end{aligned} \tag{3.3}$$

and $mod(u_i(\varphi_1(t), \varphi_2(t), \cdots, \varphi_N(t))) \subset mod(r_i, a_{ij} + b_{ij} + c_{ij})$. Obviously, $u_i^{\varphi}(t) \geq m$.

To prove that system (3.2) has a unique almost periodic solution, we shall apply Lemma 2.3 (Schauder fixed point theorem). To this end, we proceed in three steps.

Step 1: We shall show that for any $\varphi \in \mathbf{S}$ with $|\varphi_i(t)| < n_0$, there exists a sufficiently large n_0 such that system (3.3) has a unique almost periodic solution $u^{\varphi}(t) = (u_1^{\varphi}(t), u_2^{\varphi}(t), \cdots, u_N^{\varphi}(t))^T$ with $|u_i^{\varphi}(t)| \leq n_0$.

By way of contradiction, suppose the above does not hold, then there exists a sequence $\varphi^k(t) = \left(\varphi_1^k(t), \varphi_2^k(t), \cdots, \varphi_N^k(t)\right) \in \mathbf{S}$, with $|\varphi_i^k(t)| < n$, such that

$$\max\{u_1^{\varphi^k}(t), u_2^{\varphi^k}(t), \cdots, u_N^{\varphi^k}(t)\} \geq n. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned} 1 \quad & \leq \frac{\max\{u_1^{\varphi^k}(t), u_2^{\varphi^k}(t), \cdots, u_N^{\varphi^k}(t)\}}{n} \\ & \leq \max_{1\leq i\leq N}\Big\{\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\int_s^t r_i(\sigma)d\sigma}\Big[\frac{a_{ii}(s) + b_{ii}(s) + c_{ii}(s)}{n} \end{aligned}$$

$$+\frac{1}{n}\sum_{j=1,j\neq i}^{N}\Big(a_{ij}(s)\frac{u_i^k(s)}{u_j^k(s)}+b_{ij}(s)\frac{u_i^k(s)}{u_j^k(s-\tau(s))}+c_{ij}(s)\int_{-\infty}^{0}K_i(t-\sigma)\frac{u_i^k(\sigma)}{u_j^k(\sigma)}d\sigma\Big)\Big]ds\Big\},$$

for any $u_i^k(t)\in\mathbf{S}$. As $\inf\limits_{t\in\mathbb{R}} u_i^k(t)\geq m$, then $u_i^k(t)\geq m$ and $u_i^k(t-\tau(t))\geq m$. Using (H_2), we have

$$\begin{aligned}1\leq\ &\max_{1\leq i\leq N}\Big\{\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\int_s^t r_i(\sigma)d\sigma}\Big[\frac{a_{ii}(s)+b_{ii}(s)+c_{ii}(s)}{n}\\&+\frac{1}{m}\sum_{j=1,j\neq i}^{N}(a_{ij}(s)+b_{ij}(s)+c_{ij}(s))\Big]ds\Big\}.\end{aligned}$$

Let $n\to\infty$, in view of (H_3), we get

$$1\leq\max_{1\leq i\leq N}\Big\{\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\int_s^t r_i(\sigma)d\sigma}\frac{1}{m}\sum_{j=1,j\neq i}^{N}(a_{ij}(s)+b_{ij}(s)+c_{ij}(s))ds\Big\}<1,$$

which is a contradiction. Hence, there exists n_0 (we can choose $n_0>m$) such that for any $\varphi(t)\in\mathbf{S}$, system (3.2) has a unique almost periodic solution $u^{\varphi}(t)$ with $\|u_i^{\varphi}(t)\|\leq n_0$, as $\varphi_i(t)\leq n_0$.

Step 2: To apply Lemma 2.3, we need to construct the sets **C** and **B** and the operator $\mathcal{T}$. Take $\mathbf{B}=\{\varphi(t)=(\varphi_1(t),\varphi_2(t),\cdots,\varphi_N(t))\mid\varphi_i(t)$ are almost periodic functions on $\mathbb{R}$ and $mod\big(\varphi_1(t),\varphi_2(t),\cdots,\varphi_N(t)\big)\subset mod(r_i,a_{ij}+b_{ij}+c_{ij})\}$. Denote $\|\varphi\|=\max\limits_{1\leq i\leq N}\{\|\varphi_i(t)\|\}$, then **B** is a Banach space. Let

$$\mathbf{C}=\{(\varphi_1(t),\varphi_2(t),\cdots,\varphi_N(t))\mid(\varphi_1(t),\varphi_2(t),\cdots,\varphi_N(t))\in\mathbf{B},\ m\leq\varphi_i(t)\leq n_0\},$$

then **C** is a closed convex subset of **B**. Define a operator $\mathcal{T}:\mathbf{C}\to\mathbf{B}$ as

$$\mathcal{T}\varphi(t)=u^{\varphi}(t),\quad\varphi\in\mathbf{C}.\tag{3.5}$$

Step 3: By employing Schauder fixed point theorem, we shall show that $\mathcal{T}$ has a fixed point in **C**.

First, we prove that $\mathcal{T}$ is a continuous operator. In fact, for any $\varphi(t)=(\varphi_1(t),\varphi_2(t),\cdots,\varphi_N(t))$, $\psi(t)=(\psi_1(t),\psi_2(t),\cdots,\psi_N(t))\in\mathbf{C}$, we have

$$\begin{aligned}&\|\mathcal{T}_i(\varphi_1(t),\varphi_2(t),\cdots,\varphi_N(t))-\mathcal{T}_i(\psi_1(t),\psi_2(t),\cdots,\psi_N(t))\|\\ \leq\ &\int_{-\infty}^{t}e^{-\int_s^t r_i(\sigma)d\sigma}\sum_{j=1,j\neq i}^{N}[a_{ij}(s)+b_{ij}(s)+c_{ij}(s)]\Big[\frac{1}{m}\|\varphi_i(t)-\psi_i(t)\|\\&+\frac{n_0}{m^2}\|\varphi_j(t)-\psi_j(t)\|\Big]ds\\ =\ &\Big[\frac{1}{m}\|\varphi_i(t)-\psi_i(t)\|+\frac{n_0}{m^2}\|\varphi_j(t)-\psi_j(t)\|\Big]\\&\int_{-\infty}^{t}e^{-\int_s^t r_i(\sigma)d\sigma}\sum_{j=1,j\neq i}^{N}[a_{ij}(s)+b_{ij}(s)+c_{ij}(s)]ds\\ \leq\ &\|\varphi_i(t)-\psi_i(t)\|+\frac{n_0}{m}\|\varphi_j(t)-\psi_j(t)\|.\end{aligned}$$

Therefore, $\mathcal{T}$ is a continuous operator.

Next, we show that $\mathcal{T}$ is a compact operator. In fact, for any $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_N(t)) \in \mathbf{C}$ with $\|\varphi_i(t)\| \le M$, it suffices to prove that there exists subsequence $\{(\varphi_1^k(t), \varphi_2^k(t), \cdots, \varphi_N^k(t))\}$ such that

$$\Big\{\big(\mathcal{T}_1(\varphi_1^k(t), \varphi_2^k(t), \cdots, \varphi_N^k(t))\big), \cdots, \big(\mathcal{T}_N(\varphi_1^k(t), \varphi_2^k(t), \cdots, \varphi_N^k(t))\big)\Big\}$$

is convergent in $\mathbf{B}$. In fact, it follows from (3.3) and (3.5) that

$$\begin{aligned}\|\mathcal{T}_i(\varphi(t))\| \le \quad & \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t r_i(\sigma)d\sigma}\Big[a_{ii}(s) + \sum_{j=1,j\neq i}^{N} a_{ij}(s)\frac{n_0}{m} \\ & + b_{ii}(s) + \sum_{j=1,j\neq i}^{N} b_{ij}(s)\frac{n_0}{m} + c_{ii}(s) + \sum_{j=1,j\neq i}^{N} c_{ij}(s)\frac{n_0}{m}\Big]ds,\end{aligned}$$

which implies that $\{(\mathcal{T}_1(\varphi(t)), \cdots, \mathcal{T}_N(\varphi(t)))\}$ is uniformly bounded on $\mathbb{R}$, thus it follows from (3.2) that $\{(\dot{\mathcal{T}}_1(\varphi(t)), \cdots, \dot{\mathcal{T}}_N(\varphi(t)))\}$ is uniformly bounded on $\mathbb{R}$. It follows that $l\{(\mathcal{T}_1(\varphi(t))), \cdots, (\mathcal{T}_N(\varphi(t)))\}$ is equi-continuous. From Lemma 2.1, it is uniformly convergent on $\mathbb{R}$, i.e, it is convergent on $\mathbf{B}$. Since $\mathbf{C} \subset S$, from Step 1, it is easy to see that $\mathcal{T}\mathbf{C} \subseteq \mathbf{C}$. By Lemma 2.3, $\mathcal{T}$ has a fixed point in $\mathbf{C}$. Therefore, system (3.1) has an almost periodic solution in $\mathbf{C}$ with $m \le \varphi_i(t) \le n_0$. This completes the proof of Theorem 3.1. □

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