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THE CYCLICITY OF THE PERIOD ANNULUS OF TWO CLASSES OF CUBIC ISOCHRONOUS SYSTEMS*

Yi Shao[1,†] and Kuilin Wu[2]

Abstract In this paper, we investigate the cyclicity of the period annulus of two classes of cubic isochronous systems. By using the Chebyshev criterion, we prove that the two systems have respectively at most three and four limit cycles produced from the period annulus around the isochronous center under cubic perturbations.

Keywords Isochronous center, limit cycle, cubic perturbations.

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1. Introduction

The problem of finding the maximal number of limit cycles for polynomial differential systems is the second part of the Hilbert's 16th problem. Until now the problem still remains unsolved even for planar polynomial systems of degree 2. Thereby, a weak version of this problem is proposed by Arnold [1] to study the zeros of Abelian integrals obtained integrating polynomial 1-forms over ovals of polynomial Hamiltonian, that is the weak Hilbert's 16th problem.

Consider the perturbed system of a Hamiltonian system

$$\begin{aligned} \dot{x} &= H_y(x,y) + \varepsilon P(x,y), \\ \dot{y} &= -H_x(x,y) + \varepsilon Q(x,y), \end{aligned} \tag{1.1}$$

where $H(x,y)$ is a polynomial of degree $n+1$, ε is a small parameter, $P(x,y)$ and $Q(x,y)$ are polynomials of degree m in the plane. Suppose that system $(1.1)_{\varepsilon=0}$ has at least one center surrounded by the compact connected component Γ_h of real algebraic curve $H(x,y)=h, h\in(a,b)$. We can define the displacement map $d(h,\varepsilon)$ of system (1.1) on a section to the flow, which is parameterized by the Hamiltonian value h. Then, the displacement function (see [15]) of system (1.1) is

$$d(h,\varepsilon) = \varepsilon(I(h) + O(\varepsilon)),$$

where $I(h) = \oint_{\Gamma_h} Pdy - Qdx$, which is called Abelian integral of system (1.1)(or the first order Melnikov function) and $O(\varepsilon)$ is a higher-order term of ε. The weak

†the corresponding author. Email address: mathsyishao@126.com(Y. Shao)
1Department of Mathematics, Zhaoqing University, Guangdong, 526061, China
2Department of Mathematics, Guizhou University, Guizhou, 550025, China
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Hilbert's 16th problem is to find the least upper bound of the number of real zeros of the Abelian integral $I(h)$. If $I(h)$ is not identically zero, then the number of real zeros of $I(h)$ provides an upper bound for the number of limit cycles of system (1.1) which bifurcate from the period annulus around the center.

If the unperturbed system is integrable but non-Hamiltonian, one has to use a integrating factor, say $M(x, y)$. Then the Abelian integral can be defined as

$$I(h) = \oint_{\Gamma_h} M(x,y)P(x,y)dy - M(x,y)Q(x,y)dx. \tag{1.2}$$

By the same mechanisms the integral I(h) gives the first approximation of the displacement function. If $I(h) \not\equiv 0$, then $I(h)$ is called sometimes *the generating function* and the upper bound of the number limit cycles which bifurcate from the period annulus is called *the cyclicity of the period annulus*.

Many authors have studied the bifurcation of limit cycles in planar quadratic systems under quadratic perturbations, in particular for quadratic systems with centers of genus one. We refer to [2,4–6], [9,12], [17,18,20,21], the survey paper [8] and references therein. The paper [10] estimated the number of limit cycles that bifurcate from the periodic orbits of cubic reversible isochronous centers having all their orbits formed by conics inside the class of all polynomial systems of degree n. In the paper [11], the authors study the bifurcation of limit cycles from the periodic orbits of certain centers of planar systems which are the sum of a linear center and homogeneous nonlinearities. The authors of [3] obtained an upper bound for the number of limit cycles from cubic Pleshkan's isochronous system S_1^* under a small polynomial perturbation of degree n. Wu & Zhao [19] study the number of limit cycles that bifurcate from the periodic orbits of a cubic reversible isochronous center under cubic perturbations.

In this paper we will study the cyclicity of the period annulus of two classes of cubic Pleshkan's isochronous systems

$$S_3^* : \begin{cases} \dot{x} = -y + 3x^2y, \\ \dot{y} = x - 2x^3 + 9xy^2, \end{cases} \qquad \bar{S}_3^* : \begin{cases} \dot{x} = -y - 3x^2y, \\ \dot{y} = x + 2x^3 - 9xy^2. \end{cases} \tag{1.3}$$

Pleshkan has proved in the paper [16] that the origin is an isochronous center of systems S_3^* and $\bar{S}_3^*$. It is easy to know that system S_3^* has a first integral

$$H_1(x,y) = \frac{(x-2x^3)^2 + y^2}{2(1-3x^2)^3} = h_1 \tag{1.4}$$

with the integrating factor $M_1(x,y) = 1/(1-3x^2)^4$. There is an unbounded period annulus between the invariant lines $x = -\sqrt{3}/3$ and $x = \sqrt{3}/3$. Since H_1 has a local minimum 0 at the origin, $h_1 \in (0, +\infty)$. System $\bar{S}_3^*$ has a first integral

$$H_2(x,y) = \frac{(x+2x^3)^2 + y^2}{2(1+3x^2)^3} = h_2 \tag{1.5}$$

with the integrating factor $M_2(x,y) = 1/(1+3x^2)^4$. Similarly, H_2 has a local minimum 0 at the origin and there exists an unbounded period annulus surrounding the origin. Then $h_2 \in (0, \frac{2}{27})$.

The main purpose in this paper is to show the cyclicity of the period annulus of systems S_3^* and $\bar{S}_3^*$ under cubic perturbations. We consider the following perturbations of systems (1.3):

$$\begin{cases} \dot{x} = -y + 3x^2y + \varepsilon f(x,y), \\ \dot{y} = x - 2x^3 + 9xy^2 + \varepsilon g(x,y), \end{cases} \qquad \begin{cases} \dot{x} = -y - 3x^2y + \varepsilon f(x,y), \\ \dot{y} = x + 2x^3 - 9xy^2 + \varepsilon g(x,y), \end{cases} \tag{1.6}$$

where

$$f(x,y) = \sum_{i+j=1}^{3} a_{ij}(\varepsilon)x^i y^j, \qquad g(x,y) = \sum_{i+j=1}^{3} b_{ij}(\varepsilon)x^i y^j$$

with $a_{ij}(\varepsilon)$ and $b_{ij}(\varepsilon)$ being analytic functions vanishing at $\varepsilon = 0$. By (1.2) we know that the Abelian integrals of systems (1.6) are

$$\bar{I}_k(h_k) = \oint_{\Gamma_{h_k}} M_k(x,y)f(x,y)dy - M_k(x,y)g(x,y)dx, \quad k = 1,2, \tag{1.7}$$

where Γ_{h_k} is the compact component of $H_k = h_k$ defined by (1.4) and (1.5), respectively.

The main result of this paper is following theorem.

Theorem 1.1. *For cubic perturbed systems of S_3^* and $\bar{S}_3^*$, when $h_1 \in (0, +\infty)$ and $h_2 \in (0, \frac{2}{27})$, the maximal number of zeros (taking into account the multiplicity) of the Abelian integral $\bar{I}_k(h_k)$ in* (1.7) *is equal to three and four, respectively.*

To prove Theorem 1.1, we introduce some definitions of Chebyshev property and lemmas that we shall use in Section 2. In Section 3, we shall change the Abelian integral $\bar{I}_k(h_k)(k = 1,2)$ in (1.7) to a linear combination of four integrals and use Chebyshev criterion to prove that the four integrals form an extended complete Chebychev system. Accordingly, we obtain the number of zeros of the generating function by some purely algebraic computations.

Since $\bar{I}_k(h_k)$ are not identically zero, Theorem 1.1 immediately implies the following result.

Theorem 1.2. *The cyclicity of the period annulus around the isochronous center at the origin in cases S_3^* and $\bar{S}_3^*$ under small cubic perturbations is three and four, respectively.*

2. Criterion of Chebyshev systems

In order to study the cyclicity of period annulus of systems S_3^* and $\bar{S}_3^*$ under cubic perturbations, we will use Chebyshev criterion of certain functions (see [4]) to study the maximal number of zeros of Abelian integrals $\bar{I}_k(h_k)$ by some purely algebraic computations. For this purpose we will firstly introduce some definitions and criterion of Chebyshev systems. The reader is referred to [4] in detail.

Definition 2.1. Let $f_0(x), f_1(x), \cdots, f_{n-1}(x)$ be analytic functions on an open interval L of $\mathbb{R}$.

(a) $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is a Chebyshe system (for short, a T-system) on L if any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros for $x \in L$.

(b) $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is a complete Chebyshev system (for short, a CT-system) on L if $(f_0(x), f_1(x), \cdots, f_{k-1}(x))$ is a T-system for all $k = 1, 2, \cdots, n$.

(c) $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is an extend complete Chebyshev system (for short, an ECT-system) on L if for all $k = 1, 2, \cdots, n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on L counted with multiplicities.

Remark 2.1. If $(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is an ECT-system on L, then, for each $k = 1, 2, \cdots, n-1$, there exists a linear combination with exactly k simple zeros on L (see [13] and [7] for instance).

Definition 2.2. Let $f_0(x), f_1(x), \cdots, f_{k-1}(x)$ be analytic functions on an open interval L of $\mathbb{R}$. The continuous Wronskian of $(f_0(x), f_1(x), \cdots, f_{k-1}(x))$ at $x \in L$ is

$$W[f_0, f_1, \cdots, f_{k-1}](x) = Det(f_j^{(i)}(x))_{0 \le i,j \le k-1} = \begin{vmatrix} f_0(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & \cdots & f_{k-1}'(x) \\ \cdots & \cdots & \cdots \\ f_0^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

Lemma 2.1. *$(f_0(x), f_1(x), \cdots, f_{n-1}(x))$ is an ECT-system on L if and only if, for each $k = 1, 2, \cdots, n$,*

$$W[f_k](x) \neq 0 \quad \text{for all } x \in L.$$

Now we consider the first integral of system with the form

$$H(x, y) = A(x) + B(x) y^{2m}, \tag{2.1}$$

where $H(x, y)$ is an analytic function in some open subset of the plane that has a local minimum at the origin. We fix that $H(0, 0) = 0$, then there exists a period annulus by the set of ovals $\Gamma_h \in \{(x, y) | H(x, y) = h\}$, which is parameterized by the Hamiltonian value $h \in (0, h_0)$ for some $h_0 \in (0, +\infty]$. It is easy to verify that $xA'(x) > 0$ for any $x \in (x_l, x_r) \backslash 0$, where (x_l, x_r) is the projection of the period annulus Γ_h on the x-axis. Thus, there exists an analytic involution $\sigma(x)$ ($\sigma \circ \sigma = Id$ and $\sigma \neq Id$) such that

$$A(x) = A(\sigma(x)) \quad \text{for all } x \in (x_l, x_r)$$

and $\sigma(0) = 0$.

Lemma 2.2. *([4]) Let us consider the Ablian integrals*

$$I_i(h) = \int_{\Gamma_h} f_i(x) y^{2s-1} dx, \quad i = 0, 1, \cdots, n-1,$$

where, for each $h \in (0, h_0)$, Γ_h is the oval surrounding the origin inside the level curve $\{A(x) + B(x) y^{2m} = h\}$. Let σ be the involution associated to A, and we define

$$l_i(x) = \left(\frac{f_i}{A' B^{\frac{2s-1}{2m}}} \right)(x) - \left(\frac{f_i}{A' B^{\frac{2s-1}{2m}}} \right)(\sigma(x)).$$

Then $(I_0, I_1, \cdots, I_{n-1})$ is an ECT-system on $(0, h_0)$ if $s > m(n-2)$ and $(l_0, l_1, \cdots, l_{n-1})$ is a CT-system on $(0, x_r)$.

3. Proof of Theorem 1.1

In this section we will estimate the number of zeros of corresponding Abelian integrals $I_k(h_k)$ by using Chebyshev criterion (Lemma 2.2) as in section 2.

Lemma 3.1. *(i) The generating function $\bar{I}_1(h_1)$ defined by (1.7) can be rewritten as*

$$\bar{I}_1(h_1) = \alpha_0 I_0(h_1) + \alpha_1 I_1(h_1) + \alpha_2 I_2(h_1) + \alpha_3 I_3(h_1), \tag{3.1}$$

where

$$I_0(h_1) = \int_{\Gamma_{h_1}} \frac{y^3}{(1-3x^2)^4}dx, \quad I_1(h_1) = \int_{\Gamma_{h_1}} \frac{x^2 y}{(1-3x^2)^4}dx,$$

$$I_2(h_1) = \int_{\Gamma_{h_1}} \frac{y}{(1-3x^2)^4}dx, \quad I_3(h_1) = \int_{\Gamma_{h_1}} \frac{y^3}{(1-3x^2)^5}dx,$$

$\alpha_0, \alpha_1, \alpha_2$ and α_3 are arbitrary constants.

(ii) The generating function $\bar{I}_2(h_2)$ defined by (1.7) can be rewritten as

$$\bar{I}_2(h_2) = \mu_0 \tilde{I}_0(h_2) + \mu_1 \tilde{I}_1(h_2) + \mu_2 \tilde{I}_2(h_2) + \mu_3 \tilde{I}_3(h_2), \tag{3.2}$$

where

$$\tilde{I}_0(h_2) = \int_{\Gamma_{h_2}} \frac{y^3}{(1+3x^2)^4}dx, \quad \tilde{I}_1(h_2) = \int_{\Gamma_{h_2}} \frac{x^2 y}{(1+3x^2)^4}dx,$$

$$\tilde{I}_2(h_2) = \int_{\Gamma_{h_2}} \frac{y}{(1+3x^2)^4}dx, \quad \tilde{I}_3(h_2) = \int_{\Gamma_{h_2}} \frac{y^3}{(1+3x^2)^5}dx,$$

μ_0, μ_1, μ_2 and μ_3 are arbitrary constants.

Proof. (i) It follows from (1.4) that $H_1(-x, y) = H_1(x, y) = H_1(x, -y)$. Then, from (1.7) we have that

$$\int_{\Gamma_{h_1}} \frac{x^i y^j}{(1-3x^2)^4}dy = 0 \qquad \text{for} \quad i = 2m,\ m \in \mathbb{N},$$

$$\int_{\Gamma_{h_1}} \frac{x^i y^j}{(1-3x^2)^4}dx = 0 \qquad \text{for} \quad j = 2m,\ m \in \mathbb{N}.$$

On the other hand, using the formula of integration by parts we have

$$\int_{\Gamma_{h_1}} \frac{xy}{(1-3x^2)^4}dy = \int_{\Gamma_{h_1}} \frac{(1+21x^2)y^2}{2(1-3x^2)^5}dx = 0.$$

Finally, by applying Green's formula and direct computation, we obtain the result (3.1).

(ii) Using the same argument as above, we can get the expression (3.2). □

In what following, we are going to apply Lemma 2.2 to prove that $(I_0(h_1), I_1(h_1), I_2(h_1), I_3(h_1))$ in (3.1) is an ECT-system on $(0, +\infty)$. However, we discover that $m = 1, n = 4$ and $s = 1, 2$, so that the conditions $s > m(n-2)$ is not satisfied. To solve this problem, we give the following lemma.

Lemma 3.2. *([4]) Let* Γ_h *be an oval inside the level curve* $\{A(x)+B(x)y^{2m}=h\}$ *and we consider a function* F *such that* $\frac{F}{A'}$ *is analytic at* $x=0$. *Then, for any* $k\in\mathbb{N}$,

$$\int_{\Gamma_h} F(x)y^{k-2}dx = \int_{\Gamma_h} G(x)y^k dx,$$

where $G(x)=\frac{2}{k}(\frac{BF}{A'})'(x)-(\frac{B'F}{A'})(x)$.

3.1. Proof of the case S_3^*

We write (1.4) as

$$H_1(x,y)=A_1(x)+B_1(x)y^2=h_1,\quad h_1\in(0,+\infty), \tag{3.3}$$

where

$$A_1(x)=\frac{(x-2x^3)^2}{2(1-3x^2)^3},\quad B_1(x)=\frac{1}{2(1-3x^2)^3}. \tag{3.4}$$

Denote by (x_l,x_r) the projection of the unbound period annulus Γ_{h_1} around the origin on the x-axis. It is easy to see that $H_1(0,0)=0$ and $(x_l,x_r)=(-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3})$. Then we can find an involution $\sigma_1(x)$ such that $A_1(x)=A_1(\sigma_1(x))$ for all $x\in(-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3})$.

By using Lemma 3.2, we can rewrite $I_1(h_1)$ and $I_2(h_1)$ in (3.1) as

$$I_1(h_1)=\int_{\Gamma_{h_1}}\frac{(1-10x^2+12x^4)y^3}{3(-1+2x^2)^2(-1+3x^2)^4}dx,$$
$$I_2(h_1)=\int_{\Gamma_{h_1}}\frac{-y^3}{3x^2(-1+2x^2)^2(-1+3x^2)^4}dx.$$

It follows from (3.3) that

$$\begin{aligned}I_0(h_1)&=\frac{1}{h_1}\int_{\Gamma_{h_1}}\frac{(A_1(x)+B_1(x)y^2)y^3}{(1-3x^2)^4}dx\\&=\frac{1}{h_1}\int_{\Gamma_{h_1}}\frac{A_1(x)y^3+B_1(x)y^5}{(1-3x^2)^4}dx.\end{aligned}$$

Applying Lemma 3.2 to $I_0(h_1)$, we obtain

$$I_0(h_1)=\frac{1}{h_1}J_0(h_1)=\frac{1}{h_1}\int_{\Gamma_{h_1}}f_0(x)y^5dx, \tag{3.5}$$

where

$$f_0(x)=\frac{-3(1-3x^2+6x^4)}{5(-1+3x^2)^7}.$$

In the same way, we get

$$I_i(h_1)=\frac{1}{h_1}J_i(h_1)=\frac{1}{h_1}\int_{\Gamma_{h_1}}f_i(x)y^5dx,\quad i=1,2,3, \tag{3.6}$$

where

$$f_1(x) = \frac{-3+45x^2-166x^4+300x^6-216x^8}{15(-1+2x^2)^2(-1+3x^2)^7},$$
$$f_2(x) = \frac{2}{15x^2(-1+2x^2)^2(-1+3x^2)^7},$$
$$f_3(x) = \frac{3(1-2x^2+4x^4)}{5(-1+3x^2)^8}.$$

We can see that $(I_0(h_1), I_1(h_1), I_2(h_1), I_3(h_1))$ is an ECT-system on $(0,+\infty)$ if and only if $(J_0(h_1), J_1(h_1), J_2(h_1), J_3(h_1))$ is an ECT-system. Therefore we apply Lemma 2.2 to $(J_0(h_1), J_1(h_1), J_2(h_1), J_3(h_1))$ with $m = 1, n = 4$ and $s = 3$. Denote by σ the involution associated to $A_1(x)$(i.e. $A_1(x) = A_1(\sigma_1(x))$. We have

$$l_i(x) = g_i(x) - g_i(z) = \left(\frac{f_i}{A_1' B_1^{\frac{5}{2}}}\right)(x) - \left(\frac{f_i}{A_1' B_1^{\frac{5}{2}}}\right)(z), \quad i = 0,1,2,3, \tag{3.7}$$

where

$$g_i(x) = \frac{4\sqrt{2}(1-3x^2)^{\frac{23}{2}} f_i(x)}{x(1-2x^2)}. \tag{3.8}$$

Since system S_3^* is symmetrical with respect to x-axis and y-axis, it is easy to see that

$$z = \sigma_1(x) = -x. \tag{3.9}$$

Now we apply Lemma 2.1 to check that $(l_0(x), l_1(x), l_2(x), l_3(x))$ is an ECT-system on $(0, \frac{\sqrt{3}}{3})$. Therefore, we shall compute Wronskian of $(l_0(x), l_1(x), l_2(x), l_3(x))$ for $x \in (0, \frac{\sqrt{3}}{3})$. The whole computation will be fulfilled by Mathematica software.

Lemma 3.3. *$(l_0(x), l_1(x), l_2(x), l_3(x))$ is an ECT-system on $(0, \frac{\sqrt{3}}{3})$.*

Proof. By the definition of the Wronskian of $(l_0(x), l_1(x), l_2(x), l_3(x))$, we need to prove that

$$W[l_k](x) \neq 0, \quad \text{for } x \in (0, \tfrac{\sqrt{3}}{3}) \text{ and for each } k = 0,1,2,3.$$

It follows from (3.5)-(3.9) that

$$u(x)l_0(x) = v_0(x) = 9x^2(1-3x^2)(-1+2x^2)^2(1-3x^2+6x^4),$$
$$u(x)l_1(x) = v_1(x) = x^2(1-3x^2)(3-45x^2+166x^4-300x^6+216x^8),$$
$$u(x)l_2(x) = v_2(x) = 2(-1+3x^2),$$
$$u(x)l_3(x) = v_3(x) = 9x^2(-1+2x^2)^2(1-2x^2+4x^4),$$

where

$$u(x) = \frac{-8\sqrt{2}(1-3x^2)^{\frac{7}{2}}}{15x^3(-1+2x^2)^3}.$$

Note that $u(x)$ has no zeros on $(0, \frac{\sqrt{3}}{3})$. Therefore, it is easy to see that system $(l_0(x), l_1(x), l_2(x), l_3(x))$ is an ECT-system on $(0, \frac{\sqrt{3}}{3})$ if and only if system $(v_0(x), v_1(x), v_2(x), v_3(x))$ is an ECT-system on $(0, \frac{\sqrt{3}}{3})$.

It is clear that

$$W[v_0](x) = v_0(x).$$

We can see that $v_0(x) \neq 0$ for all $x \in (0, \frac{\sqrt{3}}{3})$. Similarly, by straightforward computation, we get

$$\begin{aligned} W[v_0, v_1](x) &= -72x^5(-1+2x^2)(-1+3x^2)^2(-6+38x^2-111x^4 \\ &\quad +162x^6-156x^8+72x^{10}), \\ W[v_0, v_1, v_2](x) &= -3456x^3(-1+2x^2)^2(-1+3x^2)^3(2-17x^2+63x^4 \\ &\quad -120x^6+120x^8), \end{aligned}$$

and

$$\begin{aligned} W[v_0, v_1, v_2, v_3](x) = &-995328x^6(-1+2x^2)^3(5-18x^2-147x^4+1338x^6 \\ &-4635x^8+8910x^{10}-9720x^{12}+6480x^{14}). \end{aligned}$$

By using Sturm's Theorem, we can check that $W[v_0, v_1](x) \neq 0, W[v_0, v_1, v_2](x) \neq 0$ and $W[v_0, v_1, v_2, v_3](x) \neq 0$ for all $x \in (0, \frac{\sqrt{3}}{3})$. This finishes the proof of Lemma 3.3. □

Theorem 1.1 in case S_3^* follows from Lemmas 3.1, 3.3 and Lemma 2.1.

3.2. Proof of the case $\bar{S}_3^*$

We write the first integral (1.5) as

$$H_2(x, y) = A_2(x) + B_2(x)y^2 = h_2, \quad h_2 \in (0, \frac{2}{27}), \tag{3.10}$$

where

$$A_2(x) = \frac{(x+2x^3)^2}{2(1+3x^2)^3}, \quad B_2(x) = \frac{1}{2(1+3x^2)^3}. \tag{3.11}$$

The projection of the unbound period annulus Γ_{h_2} around the origin on the x-axis is $(\tilde{x}_l, \tilde{x}_r) = (-\infty, +\infty)$. It is clear that $H_2(0, 0) = 0$. Then we can also find a involution $\sigma_2(x)$ such that $A_2(x) = A_2(\sigma_2(x))$ for all $x \in (-\infty, +\infty)$.

We find that $(\tilde{I}_0(h_2), \tilde{I}_1(h_2), \tilde{I}_2(h_2), \tilde{I}_3(h_2))$ in (3.2) is not an ECT-system by similar computations to the case S_3^* above. Hence, to prove Theorem 1.1 for the case $\bar{S}_3^*$, we first give the following lemma.

Lemma 3.4. *(see [14] for instance) Consider the Ablian integrals*

$$I_i(h) = \int_{\Gamma_h} f_i(x)y^{2s-1}dx, \quad i = 0, 1, \cdots, n-1,$$

where, for each $h \in (0, h_0)$, Γ_h is the oval surrounding the origin inside the level curve $\{A(x)+B(x)y^{2m} = h\}$. Let σ be the involution associated to A, and we define

$$l_i(x) = \frac{1}{2}\left(\frac{f_i}{A'B^{\frac{2s-1}{2m}}}\right)(x) - \frac{1}{2}\left(\frac{f_i}{A'B^{\frac{2s-1}{2m}}}\right)(\sigma(x)).$$

If the following conditions are verified:

(a) $W[l_1, l_2, \cdots, l_i]$ *is non-vanishing on* $(0, x_r)$ *for* $i = 0, 1, \cdots, n-2$,

(b) $W[l_1, l_2, \cdots, l_{n-1}]$ *has* k *zeros on* $(0, x_r)$ *counted with multiplicities, and*

(c) $s > m(n+k-2)$,

then any nontrivial linear combination of $I_0, I_1, \cdots, I_{n-1}$ *has at most* $n+k-1$ *zeros on* $(0, h_0)$ *counted with multiplicities.*

Applying Lemma 3.2, we express $\tilde{I}_1(h_2), \tilde{I}_2(h_2)$ in (3.2) as

$$\tilde{I}_1(h_2) = \int_{\Gamma_{h_2}} \frac{(1+10x^2+12x^4)y^3}{3(1+2x^2)^2(1+3x^2)^4} dx.$$

$$\tilde{I}_2(h_2) = \int_{\Gamma_{h_2}} \frac{-y^3}{3x^2(1+2x^2)^2(1+3x^2)^4} dx.$$

We find that the condition (c) in Lemma 3.4 is not satisfied. Hence by applying twice Lemma 3.2 to $\tilde{I}_i(h_2)(i = 0, 1, 2, 3)$ and combining with (3.11) we have

$$\begin{aligned}\tilde{I}_0(h_2) &= \frac{1}{h_2^2} \int_{\Gamma_{h_2}} \frac{[A_2(x) + B_2(x)y^2]^2 y^3}{(1+3x^2)^4} dx \\ &= \frac{1}{h_2^2} \int_{\Gamma_{h_2}} \frac{(A_2(x))^2 y^3 + 2A_2(x)B_2(x)y^5 + (B_2(x))^2 y^7}{(1+3x^2)^4} dx.\end{aligned}$$

By direct computation, we obtain

$$\tilde{I}_0(h_2) = \frac{1}{h_2^2} \tilde{J}_0(h_2) = \frac{1}{h_2^2} \int_{\Gamma_{h_2}} \tilde{f}_0(x) y^7 dx, \tag{3.12}$$

where

$$f_0(x) = \frac{6(2+12x^2+48x^4+93x^6+90x^8))}{35(1+3x^2)^{10}}.$$

The similar computations show that

$$\tilde{I}_i(h_2) = \frac{1}{h_2^2} \tilde{J}_i(h_2) = \frac{1}{h_2^2} \int_{\Gamma_{h_2}} \tilde{f}_i(x) y^7 dx, \quad i = 1, 2, 3, \tag{3.13}$$

where

$$\begin{aligned}\tilde{f}_1(x) &= \frac{2(2+40x^2+244x^4+855x^6+1786x^8+2100x^{10}+1080x^{12})}{35(1+2x^2)^2(1+3x^2)^{10}}, \\ \tilde{f}_2(x) &= \frac{-2}{35x^2(1+2x^2)^2(1+3x^2)^{10}}, \\ \tilde{f}_3(x) &= \frac{6(2+8x^2+29x^4+50x^6+48x^8)}{35(1+3x^2)^{11}}.\end{aligned}$$

Obviously, the number of zeros of any nontrivial linear combination of $\tilde{I}_0, \tilde{I}_1, \tilde{I}_2$ and $\tilde{I}_3$ is equal to that of $\tilde{J}_0, \tilde{J}_1, \tilde{J}_2$ and $\tilde{J}_3$.

It follows from Lemma 3.4 that

$$\tilde{l}_i(x) = \frac{1}{2}\left(\frac{\tilde{f}_i}{A_2' B_2^{\frac{7}{2}}}\right)(x) - \frac{1}{2}\left(\frac{\tilde{f}_i}{A_2' B_2^{\frac{7}{2}}}\right)(\sigma_2(x)), \quad i = 0, 1, 2, 3. \tag{3.14}$$

We can see that system $\bar{S}_3^*$ is also symmetrical with respect to x-axis and y-axis. Hence we have

$$z = \sigma_2(x) = -x. \tag{3.15}$$

Lemma 3.5. *Any nontrivial linear combination of $\tilde{I}_0(h_2), \tilde{I}_1(h_2), \tilde{I}_2(h_2)$ and $\tilde{I}_3(h_2)$ has at most 4 zeros on $(0, +\infty)$ counted with multiplicities.*

Proof. From (3.12)-(3.14) we obtain

$$\begin{aligned}
\tilde{u}(x)\tilde{l}_0(x) &= \tilde{v}_0(x) = 3x^2(1+3x^2)(1+2x^2)^2(2+12x^2+48x^4+93x^6+90x^8),\\
\tilde{u}(x)\tilde{l}_1(x) &= \tilde{v}_1(x)\\
&= x^2(1+3x^2)(2+40x^2+244x^4+855x^6+1786x^8+2100x^{10}+1080x^{12}),\\
\tilde{u}(x)\tilde{l}_2(x) &= \tilde{v}_2(x) = -1-3x^2,\\
\tilde{u}(x)\tilde{l}_3(x) &= \tilde{v}_3(x) = 3x^2(1+2x^2)^2(2+8x^2+29x^4+50x^6+48x^8),
\end{aligned}$$

where

$$\tilde{u}(x) = \frac{16\sqrt{2}(1+3x^2)^{\frac{7}{2}}}{35x^3(1+2x^2)^3}.$$

Since $\tilde{u}(x) \neq 0$ for all $x \in (0, +\infty)$. We only need to check the Wronskian of $(\tilde{v}_0(x), \tilde{v}_1(x), \tilde{v}_2(x), \tilde{v}_3(x))$ on $(0, +\infty)$.

It is clear that

$$W[\tilde{v}_0](x) = \tilde{v}_0(x) \neq 0, \quad \text{for} \quad x \in (0, +\infty).$$

By the same procedure as in the case S_3^* above, we obtain

$$\begin{aligned}
W[\tilde{v}_0, \tilde{v}_1](x) &= 24x^5(1+2x^2)(1+3x^2)^2(10+120x^2+723x^4+2708x^6+7091x^8\\
&\quad +13782x^{10}+20172x^{12}+20856x^{14}+13680x^{16}+4320x^{18}),\\
W[\tilde{v}_0, \tilde{v}_1, \tilde{v}_2](x) &= -384x^3(1+2x^2)^2(1+3x^2)^3(5+85x^2+693x^4+3498x^6\\
&\quad +12042x^8+28845x^{10}+46710x^{12}+46620x^{14}+22680x^{16}),
\end{aligned}$$

and

$$\begin{aligned}
& W[\tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3](x)\\
= \; & 276480x^6(1+2x^2)^3(-5-66x^2-171x^4+2936x^6+36342x^8\\
& +221052x^{10}+896577x^{12}+2652444x^{14}+5947911x^{16}+10235646x^{18}\\
& +13340376x^{20}+12582864x^{22}+7756560x^{24}+2449440x^{26}).
\end{aligned}$$

Applying Sturm's Theorem, we discover that both $W[\tilde{v}_0, \tilde{v}_1](x)$ and $W[\tilde{v}_0, \tilde{v}_1, \tilde{v}_2](x)$ are non-vanishing for all $x \in (0, +\infty)$ and $W[\tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3](x)$ has one zero on $(0, +\infty)$. Since $s = 4, m = 1$ and $n = 4$, this implies that any nontrivial linear combination of $\tilde{I}_0(h_2), \tilde{I}_1(h_2), \tilde{I}_2(h_2)$ and $\tilde{I}_3(h_2)$ has at most 4 zeros on $(0, +\infty)$ counted with multiplicities. Thus we have proved that Lemma 3.4 holds. □

From Lemmas 3.1, 3.3, 3.5 and Lemma 2.1 we have proved Theorem 1.1 completely.

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