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PERIODIC SOLUTION FOR HIGH-ORDER DIFFERENTIAL SYSTEM*

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Abstract Sufficient conditions are presented for the existence and stability of periodic solutions for a high-order differential system. Besides, an example is given to illustrate the result.

Keywords High-order, differential system, periodic solution, existence, stability.

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1. Introduction

In recent years, there are some works on periodic solutions for differential systems, see [3]– [10] and references therein. For example, by using the continuation theorem, Zhang & Tang [10] give the existence of positive periodic solutions of a first-order differential system

$$\begin{cases} x'(t) = x(t)F_1(t, x(t), y(t)) - h_1(t), \\ y'(t) = y(t)F_2(t, x(t), y(t)) - h_2(t), \end{cases} \tag{1.1}$$

and apply the result to a competition Lotka-Volterra population model. Wang & Lu [7] study a neutral functional differential system with delay

$$(x(t) + cx(t-\sigma))' = A(t, x(t))x(t) + f(t, x(t), x(t-\tau)), \tag{1.2}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^\top$, and obtain the existence, uniqueness and global attractivity of periodic solution for the system. Lu & Ge [6] observe a second-order neutral differential systems with deviating arguments

$$\frac{d^2}{dt^2}(x(t) + Cx(t-r)) + \frac{d}{dt}gradF(x(t)) + gradG(x(t-\tau(t))) = p(t), \tag{1.3}$$

by means of the generalized continuation theorem, they get a new result on the existence of periodic solutions. Afterwards, by employing the Deimling fixed point index theory, Wu & Wang [8] consider the following second-order nonlinear differential system with two paraments,

$$\begin{cases} u''(t) + a_1(t)u(t) = \lambda b_1(t)f_1(u(t), v(t)), \\ v''(t) + a_2(t)v(t) = \mu b_2(t)f_2(u(t), v(t)), \qquad t \in \mathbb{R}, \end{cases} \tag{1.4}$$

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and present the existence of periodic solutions for it. Recently, Liu etc. [4] get the solvability of anti-periodic solutions for the third order differential systems

$$x''' + Ax'' + \frac{d}{dt}\nabla F(x) + G(t,x) = E(t), \tag{1.5}$$

here x is a vector.

In general, most of the existing results are concentrated on lower order differential system, and studies on high-order differential systems are rather infrequent, especially for the research on stability for high-order differential system. Motivated by the problem, we consider the following high-order functional differential system

$$(x_i(t))^{(n)} = F_i(t, x(t), x'(t), \cdots, x^{(n-1)}(t)) + e_i(t), \quad i = 1, \ldots, m. \tag{1.6}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_m(t))$, F_i is a continuous function defined on $\mathbb{R} \times \mathbb{R}^{m\times n}$ and is periodic to t, i.e., $F_i(t, \cdot, \cdots, \cdot) = F_i(t+T, \cdot, \cdots, \cdot)$, $e_i(t)$ is a continuous function defined on $\mathbb{R}$ and is periodic to t with $e_i(t+T) = e_i(t)$ and $\int_0^T e_i(t)dt = 0$, $F_i(t, c, \theta, \cdots, \theta) + e_i(t) \not\equiv 0$, here c is any given constant $m-$order vector and θ is zero $m-$order vector.

The rest of this paper is organized as follows. In section 2, we give some Lemmas. In section 3, by using Mawhin's coincidence degree theorem, some sufficient conditions are obtained for the existence of periodic solutions of system (1.6). Moreover, by the construction of a Lyapunov function, we verify Lyapunov stability of periodic solution for system (1.6). Finally, an example is given to illustrate the result.

2. Preparation

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $Im\ L$ is closed in Y and $\dim Ker\ L = \dim(Y/Im\ L) < +\infty$. Consider supplementary subspaces X_1, Y_1, of X, Y respectively, such that $X = Ker\ L \oplus X_1$, $Y = Im\ L \oplus Y_1$, and let $P : X \to Ker\ L$ and $Q : Y \to Y_1$ denote the natural projections. Clearly, $Ker\ L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Let K denote the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1. *(Gaines and Mawhin [1]) Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(1) $Lx \neq \lambda Nx,\ \forall\ x \in \partial\Omega \cap D(L),\ \lambda \in (0,1)$;

(2) $Nx \notin Im\ L,\ \forall\ x \in \partial\Omega \cap Ker\ L$;

(3) $\deg\{JQN, \Omega \cap Ker\ L, 0\} \neq 0$, *where* $J : Im\ Q \to Ker\ L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

Lemma 2.2. *([11]) If $\omega \in C^1(\mathbb{R},\mathbb{R})$ and $\omega(0)=\omega(T)=0$, then*

$$\int_0^T |\omega(t)|^p dt \le \left(\frac{T}{\pi_p}\right)^p \int_0^T |\omega'(t)|^p dt,$$

where p is a fixed real number with $p>1$, and $\pi_p = 2\int_0^{(p-1)/p} \frac{ds}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$.

Lemma 2.3. *If $y \in C^n(\mathbb{R},\mathbb{R})$ and $y(t+T)\equiv y(t)$, then*

$$\int_0^T |y^{(r)}(t)|^p dt \le \left(\frac{T}{\pi_p}\right)^{p(n-r)} \int_0^T |y^{(n)}(t)|^p dt, \tag{2.1}$$

where $r = 1,\ 2,\ \ldots,\ n-1$.

Proof. If $y \in C^n(\mathbb{R},\mathbb{R})$ and $y(t+T)\equiv y(t)$, then $y, y', \cdots, y^{n-1}$ satisfy the assumptions of Lemma 2.2. Applying Lemma 2.2 repeatedly, we can get (2.1). □

Remark 2.1. If $p=2$, then $\pi_2 = 2\int_0^{(2-1)/2} \frac{ds}{(1-\frac{s^2}{2-1})^{1/2}} = \frac{2\pi(2-1)^{1/2}}{2\sin(\pi/2)} = \pi$. Therefore Eq.(2.1) is transformed into $\int_0^T |x^{(r)}(t)|^2 dt \le (\frac{T}{\pi})^{2(n-r)} \int_0^T |x^{(n)}(t)|^2 dt$.

Now set

$$X = \{x|\ x \in C^{n-1}(\mathbb{R}^m,\mathbb{R}^m),\quad x(t+T)\equiv x(t)\},$$
$$Y = \{x|\ x \in C^{0}(\mathbb{R}^m,\mathbb{R}^m),\quad x(t+T)\equiv x(t)\}$$

with

$$\text{norm}|x|_0 = \max\{|x| = \left(\textstyle\sum_{i=1}^m x_i^2(t)\right)^{\frac{1}{2}}\},$$
$$\text{norm}\|x\| = \max\{|x|_0, |x'|_0, \cdots, |x^{(n-1)}|_0\}.$$

Obviously, X and Y are Banach spaces. Define $L : D(L) = \{x \in C^n(\mathbb{R}^m,\mathbb{R}^m) : x(t+T) = x(t)\} \subset X \to Y$ by $Lx = x^{(n)}$, and $N : X \to Y$ by

$$Nx^\top = F(t, x(t), x'(t)\cdots, x^{(n-1)}(t)) + e(t), \qquad i = 1,\ldots,m, \tag{2.2}$$

where $F = (F_1, F_2, \ldots, F_m)^\top$, $e(t) = (e_1(t), e_2(t), \cdots, e_m(t))^\top$. Then system (1.6) can be converted to the abstract equation $Lx = Nx$. From the definition of L, one can easily see that $Ker\ L = \mathbb{R}^m$, $Im\ L = \{x : x \in X,\ \int_0^T x(s)ds = 0\}$. Let $P : X \to Ker\ L$ and $Q : Y \to Im\ Q$ be defined by

$$Px = \frac{1}{T}\int_0^T x(s)ds; \quad Qy = \frac{1}{T}\int_0^T y(s)ds.$$

It is easy to see that $Ker\ L = Im\ Q = \mathbb{R}^m$. Moreover, for all $y \in Y$, we have $\int_0^T y^*(s)ds = 0$ if $y^* = y - Q(y)$, which means $y^* \in Im\ L$. That is to say $Y = Im\ Q \oplus Im\ L$ and then $\dim(Y/Im\ L) = \dim Im\ Q = \dim Ker\ L$. So, L is a Fredhold operator with index zero. Let K denote the inverse of $L|_{Ker\ p\ \cap D(L)}$, we have

$$[Ky_i](t) = \sum_{j=1}^{n-1} \frac{1}{j!} x_i^{(j)}(0)t^j + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} y_i(s)ds, \tag{2.3}$$

where $x_i^{(j)}(0)\quad (j=1,2,\cdots,n-1)$ are defined by the equation $Bx_i=C$,

$$B=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ b_1 & 1 & 0 & \cdots & 0 & 0\\ b_2 & b_1 & 1 & \cdots & 0 & 0\\ \cdots & & & & & \\ b_{n-3} & b_{n-4} & b_{n-5} & \cdots & 1 & 0\\ b_{n-2} & b_{n-3} & b_{n-4} & \cdots & b_1 & 0 \end{pmatrix}_{(n-1)\times(n-1)}.$$

$$X^\top=(x^{(n-1)}(0),\cdots,x''(0),x'(0)),$$

$$C^\top=(C_1,C_2,\cdots,C_{n-1}),$$

$$C_j=-\frac{1}{j!T}\int_0^T (T-s)^j y_i(s)ds,$$

$$b_k=\frac{T^k}{(k+1)!},\qquad k=1,2,\cdots,n-2.$$

From (2.2) and (2.3), it is clearly that QN and $K(I-Q)N$ are continuous, $QN(\overline{\Omega})$ is bounded and then $K(I-Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega\subset X$ which means N is L-compact on $\bar{\Omega}$.

3. Main Results

For the sake of convenience, we let $Z_k=\{z_{k1},z_{k2},\ldots,z_{km}\}$,$U_k=\{u_{k1},u_{k2},\ldots,u_{km}\}$, $V_k=\{v_{k1},v_{k2},\ldots,v_{km}\}$, and z_{ki} be the $i-th$ component of m-order vector Z_k, $i=1,\cdots,m,\quad k=1,\ldots,n$. We give some assumptions:

(H_1) There exists a positive constant D such that

$$z_{1i}F_i(t,Z_1,Z_2,\cdots,Z_n)>0\qquad (\text{or } z_{1i}F_i(t,Z_1,Z_2,\cdots,Z_n)<0),$$

for all $(t,Z_1,Z_2,\cdots,Z_n)\in[0,T]\times\mathbb{R}^{m\times n}$ with $|z_{1i}|>D$;

(H_2) There exists a positive constant M such that

$$|F_i(t,Z_1,Z_2,\cdots,Z_n)|\le M,$$

for all $(t,Z_1,Z_2,\cdots,Z_n)\in[0,T]\times\mathbb{R}^{m\times n}$;

(H_3) There exist non-negative constant vectors $\Lambda_1=(\alpha_{11},\alpha_{12},\cdots,\alpha_{1m})$, $\Lambda_2=(\alpha_{21},\alpha_{22},\cdots,\alpha_{2m})$, $\cdots$, $\Lambda_n=(\alpha_{n1},\alpha_{n2},\cdots,\alpha_{nm})$, $P=(p_1,p_2,\ldots,p_m)$ such that

$$|F_i(t,Z_1,Z_2,\cdots,Z_n)|\le \alpha_{1i}|z_{1i}|+\alpha_{2i}|z_{2i}|+\cdots+\alpha_{ni}|z_{ni}|+p_i,$$

for all $(t,Z_1,Z_2,\cdots,Z_n)\in[0,T]\times\mathbb{R}^{m\times n}$;

(H_4) There exist non-negative constant vectors $\Upsilon_1=(\gamma_{11},\gamma_{12},\cdots,\gamma_{1m})$, $\Upsilon_2=(\gamma_{21},\gamma_{22},\cdots,\gamma_{2m})$, $\cdots$, $\Upsilon_n=(\gamma_{n1},\gamma_{n2},\cdots,\gamma_{nm})$, such that

$$\begin{aligned}&|F_i(t,U_1,U_2,\cdots,U_n)-F_i(t,V_1,V_2,\cdots,V_n)|\\ \le\ & \gamma_{1i}|u_{1i}-v_{1i}|+\gamma_{2i}|u_{2i}-v_{2i}|+\cdots+\gamma_{ni}|u_{ni}-v_{ni}|,\end{aligned}$$

for all $(t,U_1,U_2,\cdots,U_n)$, $(t,V_1,V_2,\cdots,V_n)\in[0,T]\times\mathbb{R}^{m\times n}$.

Theorem 3.1. *If (H_1) and (H_2) hold, then system* (1.6) *has at least one non-constant T-periodic solution.*

Proof. Consider the equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1),$$

i.e.,

$$(x_i(t))^{(n)} = \lambda F_i(t, x(t), x'(t), \cdots, x^{(n-1)}(t)) + \lambda e_i(t), \quad i = 1, \ldots, m. \tag{3.1}$$

Let $\Omega_1 = \{x : Lx = \lambda Nx,\ \lambda \in (0,1)\}$, and $x(t) \in \Omega_1$. We first claim that there exists a constant $\xi \in \mathbb{R}$ such that

$$|x_i(\xi)| \leq D. \tag{3.2}$$

Integrating system (3.1) over $[0, T]$, we have

$$\int_0^T F_i(t, x(t), x'(t), \cdots, x^{(n-1)}(t))dt = 0, \qquad i = 1, 2, \cdots, m.$$

Then from the continuity of F_i, we know there exists a $\xi \in [0, T]$ such that

$$F_i(\xi, x(\xi), \cdots, x^{(n-1)}(\xi)) = 0, \qquad i = 1, 2, \cdots, m.$$

From assumption (H_1) we get (3.2). As a consequence, we have

$$|x_i(t)| = \left| x_i(\xi) + \int_\xi^t x_i'(s)ds \right| \leq D + \int_0^T |x_i'(s)|ds. \tag{3.3}$$

On the other hand, multiplying both sides of the (3.1) by $x_i^{(n)}(t)$ and integrating over $[0, T]$, and in view to (H_2), we have

$$\begin{aligned} & \int_0^T |x_i^{(n)}(t)|^2 dt \\ = \ & \lambda \int_0^T F_i(t, x(t), x'(t), \cdots, x^{(n-1)}(t)) x_i^{(n)}(t)dt + \lambda \int_0^T e_i(t) x_i^{(n)}(t)dt \\ \leq \ & \int_0^T |F_i(t, x(t), x'(t), \cdots, x^{(n-1)}(t))||x_i^{(n)}(t)|dt + \int_0^T |e_i(t)||x_i^{(n)}(t)|dt \\ \leq \ & M \int_0^T |x_i^{(n)}(t)|dt + \max_{t \in [0,T]} |e_i(t)| \int_0^T |x_i^{(n)}(t)|dt \\ \leq \ & (M + |e|_0)\, T^{1/2} \Big(\int_0^T |x_i^{(n)}(t)|^2 dt \Big)^{1/2}, \end{aligned}$$

where $|e|_0 = \max_{t \in [0,T]}\{|e_1(t)|, \ldots, |e_m(t)|\}$.
It is easy to see that there exists a constant $(M + |e|_0)^2\, T$ (independent of λ) such that

$$\int_0^T |x_i^{(n)}(t)|^2 dt \leq (M + |e|_0)^2\, T.$$

From $x_i^{(n-2)}(0) = x_i^{(n-2)}(T)$, there exists a point $t_1 \in [0, T]$ such that $x_i^{(n-1)}(t_1) = 0$, and by applying Hölder's inequality, we have

$$|x_i^{(n-1)}(t)| \leq \int_0^T |x_i^{(n)}(t)|dt \leq T^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} \leq (M + |e|_0)\, T.$$

From $x_i^{(n-3)}(0) = x_i^{(n-3)}(T)$, there exists a point $t_2 \in [0,T]$ such that $x_i^{(n-2)}(t_2) = 0$, we have

$$|x_i^{(n-2)}(t)| \le \int_0^T |x_i^{(n-1)}(t)|dt \le T\,(M+|e|_0)\,T = (M+|e|_0)\,T^2.$$

Similarly,

$$|x_i^{(n-3)}(t)| \le T\,(M+|e|_0)\,T^2 = (M+|e|_0)\,T^3.$$

Continuing this way for $x_i^{(n-4)},\dots,x_i'$, we get

$$|x_i'(t)| \le T\,(M+|e|_0)\,T^{n-2} = (M+|e|_0)\,T^{n-1}.$$

Meanwhile, from Eq.(3.2), we can get

$$|x_i(t)| \le D + \int_0^T |x_i'(t)|dt \le D + T\,(M+|e|_0)\,T^{n-1} = D + (M+|e|_0)\,T^n.$$

Take

$$M_0 = \begin{cases} \sqrt{m}D + \sqrt{m}(M+|e|_0)T^n, & \text{if } \ T > 1; \\ \sqrt{m}D + \sqrt{m}(M+|e|_0)T, & \text{if } \ T \le 1. \end{cases}$$

Obviously, $|x|_0 \le M_0, \ \ |x'|_0 \le M_0, \ \cdots, \ \ |x^{(n-1)}|_0 \le M_0$.

Let $\Omega_2 = \{x \in Ker\ L : Nx \in Im\ L\}$. If $x \in \Omega_2$, then $x \in Ker\ L$ which means $x =$ constant vector, $x' = x'' = \cdots = x^{(n-1)} = \theta$, and $QNx = 0$. Integrating (3.1) over $[0,T]$, we have $F_i(t,x,\theta,\cdots,\theta) = 0$ which yields that $|x_i| \le D$.

Now take $\Omega = \{x \in X : |x|_0 < M_0+1,\ |x'|_0 < M_0+1,\dots,\ |x^{(n-1)}|_0 < M_0+1\}$. By the analysis of the above, it is easy to see that $\overline{\Omega}_1 \subset \Omega$, $\overline{\Omega}_2 \subset \Omega$ and conditions (1) and (2) of Lemma 2.1 are satisfied.

Next we show that condition (3) of Lemma 2.1 is also satisfied. Define the isomorphism

$$J : ImQ \to Ker\ L \text{by} J(x) = (J(x_1),\cdots,J(x_m))^\top \ \text{ and}$$
$$J(x_i) = \begin{cases} x_i, & \text{if } z_{1i}F_i(t,Z_1,Z_2,\cdots,Z_n) > 0, \ \text{ for } \ |z_{1i}| > D; \\ -x_i, & \text{if } z_{1i}F_i(t,Z_1,Z_2,\cdots,Z_n) < 0, \ \text{ for } \ |z_{1i}| > D, \end{cases}$$
$$\text{for } \ i = 1,2,\cdots,m.$$

Let $H(\mu,x) = (H(\mu,x_1),\cdots,H(\mu,x_m))^\top$ and $H(\mu,(x_i)) = \mu x_i + (1-\mu)JQNx_i$, $(\mu,x_i) \in [0,1]\times\Omega$, then $\forall\ (\mu,x_i) \in (0,1)\times(\partial\Omega \cap Ker\ L)$,

$$H(\mu,x_i) = \begin{cases} \mu x_i + \frac{1-\mu}{T}\int_0^T [F_i(t,x,\theta,\cdots,\theta) + e_i(t)]\,dt, \\ \quad \text{if } z_{1i}F_i(t,Z_1,Z_2,\cdots,Z_n) > 0, \ \text{for } |z_{1i}| > D; \\ \mu x_i - \frac{1-\mu}{T}\int_0^T [F_i(t,x,\theta,\cdots,\theta) + e_i(t)]dt, \\ \quad \text{if } z_{1i}F_i(t,Z_1,Z_2,\cdots,Z_n) < 0, \ \text{for } |z_{1i}| > D, \end{cases} \tag{3.4}$$

for $i = 1,2,\cdots,m$. Since $\int_0^T e_i(t)dt = 0$, (3.4) is transformed into

$$H(\mu,x_i) = \begin{cases} \mu x_i + \frac{1-\mu}{T}\int_0^T F_i(t,x,\theta,\cdots,\theta)dt, \\ \quad \text{if } z_{1i}F_i(t,Z_1,Z_2,\cdots,Z_n) > 0, \ \text{ for } \ |z_{1i}| > D; \\ \mu x_i - \frac{1-\mu}{T}\int_0^T F_i(t,x,\theta,\cdots,\theta)dt, \\ \quad \text{if } z_{1i}F_i(t,Z_1,Z_2,\cdots,Z_n) < 0, \ \text{ for } \ |z_{1i}| > D. \end{cases}$$

From (H_1), it is obvious that $x_iH(\mu,x_i)>0,\ \forall\ (\mu,x_i)\in(0,1)\times(\partial\Omega\cap Ker\ L)$. Therefore,

$$\begin{aligned}\deg\{JQN,\Omega\cap Ker\ L,0\}&=\deg\{H(0,x),\Omega\cap Ker\ L,0\}\\&=\deg\{H(1,x),\Omega\cap Ker\ L,0\}\\&=\deg\{I,\Omega\cap Ker\ L,0\}\neq 0,\end{aligned}$$

which means condition (3) of Lemma 2.1 is also satisfied. By applying Lemma 2.1, we conclude that equation $Lx=Nx$ has a solution x^* on $\bar{\Omega}$, i.e., system (1.6) has a T-periodic solution $x^*(t)$ with $\|x^*\|<M_0+1$.

Finally, observe that $x^*(t)$ is not constant. Otherwise, suppose $x^*(t)\equiv c$ (constant vector), then from system (1.6) we have $F_i(t,c,\theta,\cdots,\theta)+e_i(t)\equiv 0$, which contradicts to assumption $F_i(t,c,\theta,\cdots,\theta)+e_i(t)\not\equiv 0$, so the proof is complete. □

Theorem 3.2. *If (H_1) and (H_3) hold, then system (1.6) has at least a non-constant T-periodic solution if $(\alpha_{1_0}T+\alpha_{2_0})\left(\frac{T}{\pi}\right)^{n-1}+\alpha_{3_0}\left(\frac{T}{\pi}\right)^{n-2}+\cdots+\alpha_{n_0}\left(\frac{T}{\pi}\right)<1$, where $\alpha_{k_0}=\max\{\alpha_{k1},\alpha_{k2},\ldots,\alpha_{km}\}, k=1,2,\ldots,n$.*

Proof. Let Ω_1 be defined as in Theorem 3.1. If $x(t)\in\Omega_1$, then from the proof of Theorem 3.1 we see that

$$|x_i|_0\le D+\int_0^T|x_i'(s)|ds,\qquad i=1,\ldots,m. \tag{3.5}$$

We claim that $|x_i^{(n-1)}|_0$ is bounded.

Multiplying both sides of (3.1) by $x_i^{(n)}(t)$ and integrating over [0,T], by using assumption (H_3), we have

$$\begin{aligned}&\int_0^T|x_i^{(n)}(t)|^2dt\\
=\ &\lambda\int_0^T F_i(t,x(t),x'(t),\cdots,x^{(n-1)}(t))x_i^{(n)}(t)dt+\lambda\int_0^T e_i(t)x_i^{(n)}(t)dt\\
\le\ &\int_0^T|F_i(t,x(t),x'(t),\cdots,x^{(n-1)}(t))||x_i^{(n)}(t)|dt+\int_0^T|e_i(t)||x_i^{(n)}(t)|dt\\
\le\ &\alpha_{1i}\int_0^T|x_i(t)||x_i^{(n)}(t)|dt+\alpha_{2i}\int_0^T|x_i'(t)||x_i^{(n)}(t)|dt\\
&+\cdots+\alpha_{ni}\int_0^T|x_i^{(n-1)}(t)||x_i^{(n)}(t)|dt+p_i\int_0^T|x_i^{(n)}(t)|dt+\int_0^T|e_i(t)||x_i^{(n)}(t)|dt\\
\le\ &\alpha_{1i}|x_i|_0\int_0^T|x_i^{(n)}(t)|dt+\alpha_{2i}\int_0^T|x_i'(t)||x_i^{(n)}(t)|dt\\
&+\cdots+\alpha_{ni}\int_0^T|x_i^{(n-1)}(t)||x_i^{(n)}(t)|dt+p_i\int_0^T|x_i^{(n)}(t)|dt+\int_0^T|e_i(t)||x_i^{(n)}(t)|dt\\
\le\ &\alpha_{1i}\left(D+\int_0^T|x_i'(t)|dt\right)\int_0^T|x_i^{(n)}(t)|dt+\alpha_{2i}\int_0^T|x_i'(t)||x_i^{(n)}(t)|dt\\
&+\cdots+\alpha_{ni}\int_0^T|x_i^{(n-1)}(t)||x_i^{(n)}(t)|dt+(p_0+|e|_0)\int_0^T|x_i^{(n)}(t)|dt,\end{aligned}$$

where $p_0 = \max\{p_1, p_2, \dots, p_m\}$. By applying Hölder's inequality, we have

$$\begin{aligned}
& \int_0^T |x_i^{(n)}(t)|^2 dt \\
\le\ & \alpha_{1i} \left(D + T^{1/2} \left(\int_0^T |x_i'(t)|^2 dt \right)^{1/2} \right) T^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} \\
& + \alpha_{2i} \left(\int_0^T |x_i'(t)|^2 dt \right)^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} + \alpha_{3i} \left(\int_0^T |x_i''(t)|^2 dt \right)^{1/2} \\
& \cdot \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} + \cdots + \alpha_{ni} \left(\int_0^T |x_i^{(n-1)}(t)|^2 dt \right)^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} \\
& + (p_0 + |e|_0) T^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} \\
\le\ & (\alpha_{1i} T + \alpha_{2i}) \left(\int_0^T |x_i'(t)|^2 dt \right)^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} \\
& + \alpha_{3i} \left(\int_0^T |x_i''(t)|^2 dt \right)^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} + \cdots + \alpha_{ni} \left(\int_0^T |x_i^{(n-1)}(t)|^2 dt \right)^{1/2} \\
& \cdot \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} + (p_0 + |e|_0 + \alpha_{1i} D) T^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2}.
\end{aligned} \tag{3.6}$$

By using Lemma 2.3 and (3.6), we can get

$$\begin{aligned}
& \int_0^T |x_i^{(n)}(t)|^2 dt \\
\le\ & (\alpha_{1i} T + \alpha_{2i}) \left(\tfrac{T}{\pi} \right)^{n-1} \int_0^T |x_i^{(n)}(t)|^2 dt + \alpha_{3i} \left(\tfrac{T}{\pi} \right)^{n-2} \int_0^T |x_i^{(n)}(t)|^2 dt \\
& + \cdots + \alpha_{ni} \left(\tfrac{T}{\pi} \right) \int_0^T |x_i^{(n)}(t)|^2 dt + (\alpha_{1i} D + p_0 + |e|_0) T^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} \\
\le\ & \left[(\alpha_{1_0} T + \alpha_{2_0}) \left(\tfrac{T}{\pi} \right)^{n-1} + \alpha_{3_0} \left(\tfrac{T}{\pi} \right)^{n-2} + \cdots + \alpha_{n_0} \left(\tfrac{T}{\pi} \right) \right] \int_0^T |x_i^{(n)}(t)|^2 dt \\
& + (\alpha_{1_0} D + p_0 + |e|_0) T^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2},
\end{aligned}$$

where $\alpha_{k_0} = \max\{\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{km}\}$, $k = 1, 2, \dots, n$. Since $(\alpha_{1_0} T + \alpha_{2_0}) \left(\frac{T}{\pi} \right)^{n-1} + \alpha_{3_0} \left(\frac{T}{\pi} \right)^{n-2} + \cdots + \alpha_{n_0} \left(\frac{T}{\pi} \right) < 1$, it is easy to see that there exists a constant $M' > 0$ (independent of λ) such that

$$\int_0^T |x_i^{(n)}(t)|^2 dt \le M'.$$

From $x_i^{(n-2)}(0) = x_i^{(n-2)}(T)$, there exists a point $t_1 \in [0, T]$ such that $x_i^{(n-1)}(t_1) = 0$, By applying Hölder's inequality, we have

$$|x_i^{(n-1)}(t)| \le \int_0^T |x_i^{(n)}(t)| dt \le T^{1/2} \left(\int_0^T |x_i^{(n)}(t)|^2 dt \right)^{1/2} \le T^{1/2} M'^{1/2} := M.$$

This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 3.1. □

Next, we will construct some suitable function to study the Lyapunov stability of the periodic solution of system (1.6).

Theorem 3.3. *Assume* (H_4) *hold. If system* (1.6) *has* T*-periodic solution, then the* T*-periodic solution is Lyapunov stable.*

Proof. Let

$$x_i(t) = z_{1i}(t),\quad x_i'(t) = z_{2i}(t),\quad \cdots,\quad x_i^{(n-1)}(t) = z_{ni}(t),\qquad i = 1, 2, \cdots, m,$$

then system (1.6) be transformed into

$$\begin{cases} z_{1i}'(t) &= z_{2i}(t), \\ z_{2i}'(t) &= z_{3i}(t), \\ \quad\cdots \\ z_{ni}'(t) &= F_i(t, Z_1(t), Z_2(t), \cdots, Z_n(t)) + e_i(t), \end{cases} \qquad i = 1, 2, \cdots, m, \tag{3.7}$$

where $Z_k = \{z_{k1}, \dots, z_{km}\},\quad k = 1, \dots, n.$
Assume system (3.7) has T-periodic solution $z_i^*(t) = (z_{1i}^*(t), z_{2i}^*(t), \cdots, z_{ni}^*(t))^\top,\ i = 1, \dots, m$. Suppose $z_i(t) = (z_{1i}(t), z_{2i}(t), \cdots, z_{ni}(t))^\top,\quad i = 1, \dots, m$, is any arbitrary solution of (1.6). Let $w_{ki}(t) = z_{ki}(t) - z_{ki}^*(t),\quad k = 1, \dots, n$, then it follows from (3.7) that

$$\begin{cases} w_{1i}'(t) &= w_{2i}(t), \\ w_{2i}'(t) &= w_{3i}(t), \\ \quad\cdots \\ w_{ni}'(t) &= F_i(t, Z_1(t), Z_2(t), \cdots, Z_n(t)) - F_i(t, Z_1^*(t), Z_2^*(t), \cdots, Z_n^*(t)). \end{cases} \tag{3.8}$$

And we can get

$$\begin{cases} |w_{1i}'(t)| &= |w_{2i}(t)|, \\ |w_{2i}'(t)| &= |w_{3i}(t)|, \\ \quad\cdots \\ |w_{ni}'(t)| &= |F_i(t, Z_1(t), Z_2(t), \cdots, Z_n(t)) - F_i(t, Z_1^*(t), Z_2^*(t), \cdots, Z_n^*(t))|. \end{cases} \tag{3.9}$$

Let $y_{ki}^{(l)}(t) = |w_{ki}^{(l)}(t)|,\quad l = 0, 1,\quad k = 1, 2, \dots, n$, then

$$\begin{cases} y_{1i}'(t) &= y_{2i}(t), \\ y_{2i}'(t) &= y_{3i}(t), \\ \quad\cdots \\ y_{ni}'(t) &= |F_i(t, Z_1(t), Z_2(t), \cdots, Z_n(t)) - F_i(t, Z_1^*(t), Z_2^*(t), \cdots, Z_n^*(t))|. \end{cases} \tag{3.10}$$

Take $\beta = \max\{\gamma_{1_0}, \gamma_{2_0} + 1, \dots, \gamma_{n_0} + 1\} + 1$, here $\gamma_{k_0} = \{\gamma_{k1}, \gamma_{k2}, \cdots, \gamma_{km}\},\quad k = 1, 2, \cdots, n$. And define a function $V(\cdot)$

$$V(t, y_1, \cdots, y_n) = e^{-\beta t} \sum_{k=1}^{n} \sum_{i=1}^{m} y_{ki}(t). \tag{3.11}$$

There exists a sufficiently small positive constant ε such that $e^{-\beta t} \geq \varepsilon$. Take $U(y_1, \cdots, y_n) = \sum_{k=1}^{n} \sum_{i=1}^{m} \varepsilon y_{ki}(t)$, it is obvious that $V(t, y_1, \cdots, y_n) > 0$ and $V(t, y_1, \cdots, y_n) \geq U(y_1, \cdots, y_n) > 0$. Calculating the derivatives of V, from (H_4), we get

$$\begin{aligned}
& \dot{V}(t, y_1, \cdots, y_n) \\
= & -\beta e^{-\beta t}\left(\sum_{k=1}^{n}\sum_{i=1}^{m} y_{ki}(t)\right) + e^{-\beta t}\textstyle\sum_{i=1}^{m}(y_{2i}(t)+\cdots+y_{ni}(t)) \\
& +e^{-\beta t}\sum_{i=1}^{m}\left|F_i(t, Z_1(t), Z_2(t), \cdots, Z_n(t)) - F_i(t, Z_1^*(t), Z_2^*(t), \cdots, Z_n^*(t))\right| \\
\le & -\beta e^{-\beta t}\left(\sum_{k=1}^{n}\sum_{i=1}^{m} y_{ki}(t)\right) + e^{-\beta t}\sum_{i=1}^{m}(y_{2i}(t)+\cdots+y_{ni}(t)) \\
& +e^{-\beta t}\sum_{i=1}^{m}(\gamma_{1i}|z_{1i}(t)-z_{1i}^*(t)| + \cdots + \gamma_{ni}|z_{ni}(t)-z_{ni}^*(t)|) \\
= & -\beta e^{-\beta t}\left(\sum_{k=1}^{n}\sum_{i=1}^{m} y_{ki}(t)\right) + e^{-\beta t}\sum_{i=1}^{m}(y_{2i}(t)+\cdots+y_{ni}(t)) \\
& +e^{-\beta t}\sum_{i=1}^{m}(\gamma_{1i}|w_{1i}(t)| + \cdots + \gamma_{ni}|w_{ni}(t)|) \\
\le & -\beta e^{-\beta t}\left(\sum_{k=1}^{n}\sum_{i=1}^{m} y_{ki}(t)\right) + e^{-\beta t}\sum_{i=1}^{m}(y_{2i}(t)+\cdots+y_{ni}(t)) \\
& +e^{-\beta t}\sum_{i=1}^{m}(\gamma_{1_0} y_{1i}(t) + \cdots + \gamma_{n_0} y_{ni}(t)) \\
= & (-\beta+\gamma_{1_0})\sum_{i=1}^{m} y_{1i}(t)e^{-\beta t} + \sum_{k=2}^{n}(-\beta+1+\gamma_{k_0})\sum_{i=1}^{m} y_{ki}(t)e^{-\beta t} \\
< & \ 0.
\end{aligned}$$

From the above, we know V is a Lyapunov function for nonautonomous system (1.6)(P50, [2]), and then the T-periodic solution of system (1.6) is Lyapunov stable. □

Finally, we present an example to illustrate our result.

Example 3.1. Consider the three-order differential system

$$\begin{cases} x_1'''(t) = \frac{1}{12\pi}x_1(t) + \frac{1}{8}\sin x_1'(t) + \frac{1}{8}\cos x_2'(t)\sin t + \frac{1}{8}\sin x_3''(t) + \frac{1}{8}\sin t \\ x_2'''(t) = \frac{1}{12\pi}x_2(t) + \frac{1}{8}\sin x_2'(t) + \frac{1}{8}\cos x_3''(t)\cos t + \frac{1}{8}\sin x_1''(t) + \frac{1}{4}\cos t \\ x_3'''(t) = \frac{1}{12\pi}x_3(t) + \frac{1}{8}\sin x_3'(t) + \frac{1}{8}\cos x_1''(t)\sin t + \frac{1}{8}\sin x_2''(t) + \frac{1}{8}\cos t. \end{cases} \tag{3.12}$$

It is clear that $n = 3, \quad T = 2\pi$,

$$F(t, x(t), x'(t), x''(t)) = \begin{pmatrix} \frac{1}{12\pi}x_1(t) + \frac{1}{8}\sin x_1'(t) + \frac{1}{8}\cos x_2'(t)\sin t + \frac{1}{8}\sin x_3''(t) \\ \frac{1}{12\pi}x_2(t) + \frac{1}{8}\sin x_2'(t) + \frac{1}{8}\cos x_3''(t)\cos t + \frac{1}{8}\sin x_1''(t) \\ \frac{1}{12\pi}x_3(t) + \frac{1}{8}\sin x_3'(t) + \frac{1}{8}\cos x_1''(t)\sin t + \frac{1}{8}\sin x_2''(t) \end{pmatrix},$$

$$e(t) = \begin{pmatrix} \frac{1}{8}\sin t \\ \frac{1}{4}\cos t \\ \frac{1}{8}\cos t \end{pmatrix}, \ F(t, c, \theta, \theta) + e(t) = \begin{pmatrix} \frac{1}{12\pi}c_1 + \frac{1}{8}\sin t + \frac{1}{8}\sin t \\ \frac{1}{12\pi}c_2 + \frac{1}{8}\cos t + \frac{1}{4}\cos t \\ \frac{1}{12\pi}c_3 + \frac{1}{8}\sin t + \frac{1}{8}\cos t \end{pmatrix} \not\equiv 0,$$

$c = (c_1, c_2, c_3)$ and θ is zero 3−order vector. Choose $D = 12\pi$ such that (H_1) holds. It is obvious that (H_2) is not satisfied here. Now we consider the assumption (H_3).

Since

$$|F_i(t, x(t), x'(t), x''(t))| \leq \frac{1}{12\pi}|x_i(t)| + 3, \quad i = 1, 2, 3,$$

(H_3) holds with $\alpha_{1_0} = \frac{1}{12\pi}$, $\alpha_{2_0} = 0$, $\alpha_{3_0} = 0$, $p_i = 3$.

$$\begin{aligned} & \left[(\alpha_{1_0}T + \alpha_{2_0})\left(\frac{T}{\pi}\right)^{n-1} + \alpha_{3_0}\left(\tfrac{T}{\pi}\right)^{n-2} + \cdots + \alpha_{n_0}\left(\frac{T}{\pi}\right)\right] \\ = & \ (\frac{1}{12\pi} \times 2\pi + 0) \times \left(\frac{2\pi}{\pi}\right)^{3-1} + 0 + 0 \\ = & \ \frac{1}{6} \times 4 = \frac{2}{3} < 1. \end{aligned}$$

So by Theorem 3.2, we know system (3.12) has at least one nonconstant 2π-periodic solution.

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