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MOND-WEIR TYPE HIGHER ORDER MINIMAX MIXED INTEGER DUAL PROGRAMMING UNDER GENERALIZED (ϕ, α, ρ)-UNIVEXITY

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Abstract A new generalized class of higher order (ϕ, α, ρ)-univex function is introduced with an example and we formulate Mond-Weir type nondifferentiable higher order minimax mixed integer dual programs and symmetric duality theorems are established.

Keywords Higher order minimax integer dual program, higher order (ϕ, α, ρ)-univex function, Schwartz inequality, additively separable.

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1. Introduction

Duality theory has played an important role in the development of optimization theory. For nonlinear programming problems a number of duals have been suggested, among which the Wolfe dual proposed by Dorn [11] is well known. Symmetric duality in nonlinear programming was introduced in [11] by defining a symmetric dual problem for quadratic programs. Subsequently Dantzing et al [8] established symmetric duality results for convex/concave functions. Devi [8], Weir and Mond [22], Mond and Schechter [18] studied non differentiable symmetric duality for a class of optimization problem in which the objective function consist of support function.

Higher order duality concept was first introduced by Mangasarian [15]. Zhang [24] established various duality results between (P) and dual (MHD) and (MWHD) under higher order invexity assumptions. Later on Yang et al [23] discussed higher order duality results under generalized convexity assumption for multiobjective programming problems involving support functions. Mishra and Rueda [16] established duality results under higher order generalized invexity whereas they generalized the results of Zhang [24] to higher order type 1 function in their paper [17]. The case of higher order symmetric duality for nondifferentiable multiobjective programming problem was considered by Chen [8]. Ahmad et al [3] formulated a general Mond-Weir type higher order dual and established duality results under (F, α, ρ, d)-type 1 function. Higher order symmetric multiobjective duality involving generalized (F, ρ, γ, b)-convexity was given by Batatoresue et al [6]. Gulati et al [13] and Ahmad [1] established second order symmetric duality for nondifferentiable minimax mixed integer problem. A unified higher order dual for a nondifferentiable minimax programming problem is formulated by Ahmad et al [4].

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In recent years, several extension and generalization have been considered for classical convexity. A significant generalization of convex function is that of in-vex function introduced by Hanson [14]. Bector et al [7] have introduced the concept of pre-univex function, univex functions and pseudo-univex function as a generalization of in-vex function. Further development on the application of univex function and generalized univex function can be found in Mishra et al [16,17]. Ojha [19] established symmetric duality results for (ϕ, ρ)-univex function and Thakur et al [21] established second order symmetric duality results for second order (ϕ, ρ)-univex function .Very recently Ojha [20] established higher order duality for multiobjective programming involving (ϕ, ρ)-univex function.

In this paper, I have introduced a new generalized class of higher order (ϕ, α, ρ)-univex functions with an example. I have formulated a pair of higher order Mond-Weir type minimax mixed integer program. Based on these concepts, weak and strong duality theorems are established .

In section-2, some definitions are recalled and higher order (ϕ, α, ρ)-univex functions are introduced. In section-3, Mond-Weir type higher order minimax mixed nondifferentiable symmetric dual program are formulated and duality results are established under higher order (ϕ, α, ρ)-univexity assumption. In section-4, self duality theorem is established. In section-5, I concluded with conclusion.

2. Notations and Definitions

We denote by R^n the n-dimensional Euclidean space and by R^n_+ its nonnegative orthant. Let $X \subseteq R^n, Y \subseteq R^m$. The following conventions for vectors $x, u \in R^n$ will be followed throughout this paper: $x < u \Leftrightarrow x_i < u_i, i = 1, 2, .., n$ and $x \leq u \Leftrightarrow x_i < u_i, i = 1, 2, .., n$. Further for any vector, we denote $x^T u = \sum_{i=1}^n x_i u_i$.

Let $r \in R, \rho \in R$. ϕ_1 *and* ϕ_2 are real valued functions defined on $R^n \times R^n \times R^{n+1}$ and $R^m \times R^m \times R^{m+1}$ respectively such that $\phi_1(x, u, \cdot)$ and $\phi_2(v, y, \cdot)$ are convex on R^{n+1} and R_{m+1} respectively and $\phi_1(x, u, (0, r)) \geq 0$, $\phi_2(v, y, (0, r)) \geq 0$. b_1 and b_2 are non negative real valued function defined on $R^n \times R^n$ and $R^m \times R^m$. $\psi_1, \psi_2 : R \to R$, satisfying $\psi_i(a) \leq 0 \Rightarrow a \leq 0, i = 1, 2$ and $\psi_i(-a) = -\psi_i(a)$. Suppose $\alpha_1 : R^n \times R^n \to R$ and $\alpha_2 : R^m \times R^m \to R$.

Let $f_i : X \times Y \to R, g_i : X \times Y \times X \to R$ and $h_i : X \times Y \times Y \to R$ are twice differentiable functions.

Definition 2.1. The function $f_i(\cdot, y)$ is said to be higher order (ϕ, α, ρ)-univex at $u \in X$ with respect to g_i, if for b_1, ϕ_1, α_1 and ρ, we have

$$b_1(x, u)\psi_1[f_i(x, y) - f_i(u, y) - g_i(u, y, q_i) + q_i^T \nabla_{q_i} g_i(u, y, q_i)]$$
$$\geq \phi_1(x, u; \alpha_1(x, u)(\nabla_u f_i(u, y) + \nabla_{q_i} g_i(u, y, q_i), \rho)).$$

Definition 2.2. The function $f_i(x, \cdot)$ is said to be higher order (ϕ, α, ρ)-univex at $y \in Y$ with respect to h_i, if for b_2, ϕ_2, α_2 and ρ, we have

$$b_2(v, y)\psi_2[f_i(x, v) - f_i(x, y) - h_i(x, y, p_i) + p_i^T \nabla_{p_i} h_i(x, y, p_i)]$$
$$\geq \phi_2(v, y; \alpha_2(v, y)(\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i), \rho)).$$

Definition 2.3. The function $f_i(\cdot, y)$ is said to be higher order (ϕ, α, ρ)-pseudo univex at $u \in X$ with respect to g_i, if for b_1, ϕ_1, α_1 and ρ, we have

$$\phi_1(x, u; \alpha_1(x, u)(\nabla_u f_i(u, y) + \nabla_{q_i} g_i(u, y, q_i), \rho)) \geq 0$$
$$\Rightarrow b_1(x, u)\psi_1[f_i(x, y) - f_i(u, y) - g_i(u, y, q_i) + q_i^T \nabla_{q_i} g_i(u, y, q_i)] \geq 0.$$

Definition 2.4. The function $f_i(x, \cdot)$ is said to be higher order (ϕ, α, ρ)-pseudo univex at $y \in Y$ with respect to h_i, if for b_2, ϕ_2, α_2 and ρ, we have

$$\phi_2(v, y; \alpha_2(v, y)(\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i), \rho)) \geq 0$$
$$\Rightarrow b_2(v, y)\psi_2[f_i(x, v) - f_i(x, y) - h_i(x, y, p_i) + p_i^T \nabla_{p_i} h_i(x, y, p_i)].$$

Definition 2.5. The function f_i is said to be higher order (ϕ, α, ρ)-unicave and higher (ϕ, α, ρ)- pseudo unicave functions with respect to h_i, if $-f_i$ is higher order (ϕ, α, ρ)-univex and higher order (ϕ, α, ρ)-pseudo univex function with respect to $-h_i$.

Example 2.1. We present here a function which is higher order (ϕ, α, ρ)-univex function. We can proceed similarly for other classes of function which are introduced.

Let us consider $X = (0, \infty)$ and $f : X \to R$ defined by $f(x) = x \log x$ and $h : X \times R \to$ defined by $h(u, y) = -y \log u$.

Obviously f is not convex.

We have $\nabla_u f(u) = 1 + \log u$, $\nabla_{uu} f(u) = \frac{1}{u}$ and $\nabla_u h(u, y) = -\log u$. Let $\phi : X \times X \times R \to R$ defined by $\phi(x, y; (a, b)) = b(a + a^2)$. So ϕ is not sub linear.

Let $b : X \times X \to, \psi : R \to R$ and $\alpha : X \times \times X \to R$ defined by

$$b(x, y) = \frac{xu(1+xu)}{x \log x - u \log u} \text{ if } (x \log x - u \log u) > 0 \text{ and } 0 \text{ if } (x \log x - u \log u) \leq 0,$$
$$\psi(x) = 3x \text{ and } \alpha(x, u) = xu.$$

Since $\nabla_u f(u) + \nabla_y h(u, y) = 1$ and $h(u, y) - y\nabla_y h(u, y) = 0$, we have

$$\phi(x, u; (\alpha(x, u)(\nabla_u f(u) + \nabla_y h(u, y), \rho)) = \rho(xu + x^2u^2)$$

and the definition of higher order (ϕ, α, ρ)-univex becomes,

$$\left(\frac{xu(1+xu)}{x \log x - u \log u}\right) \times 3(x \log x - u \log u) \geq \rho(xu + x^2u^2), \text{ if } x \log x - u \log u > 0$$
$$\text{or } 0 \geq \rho(xu + x^2u^2, \text{ if } x \log x - u \log u \leq 0$$
$$\Rightarrow 1 \geq \rho, \text{ if } x \log x - u \log u > 0 \text{ or } 0 \geq \rho, \text{ if } x \log x - u \log u \leq 0.$$

It now follows that the function $f(x) = x \log x$ is higher order (ϕ, α, ρ)-univex at $u \in X$ with respect to $h(x, y) = -y \log u$ for $0 \geq \rho$.

We consider the following multiobjective programming problem:

MP(**Primal**): Minimize $f(x, y) = (f_1(x, y), f_2(x, y), ..., f_r(x, y))$
Subject to $g(x, y) \leq 0, x \in X \subseteq R^n, y \in Y \subseteq R^m$,
where $f : X \times Y \to R^r, g : X \times Y \to R^k$.

Let P_0 be the set of all feasible solution of problem (MP)

$$\text{i.e. } P_0 = (x \in X, y \in Y | g(x, y) \leq 0).$$

Definition 2.6. A vector $a \in P_0$ is said to be an efficient solution of problem (MP), if there exist no $(x, y) \in P_0$ such that $f(x, y) \leq f(a)$.

Definition 2.7. A vector $a \in P_0$ is said to be a weakly efficient solution of problem (MP), if there exist no $(x, y) \in P_0$ such that $f(x, y) < f(a)$.

Definition 2.8. (**Schwartz Inequality**) Let $x, y \in R^n$ and $A \in R^n \times R^n$ be a positive semi definite matrix, then $x^T Ay \leq (x^T Ax)^{\frac{1}{2}}(y^T Ay)^{\frac{1}{2}}$. Equality holds, if for some $\lambda \geq 0, Ax = \lambda Ay$.

Definition 2.9. Let $s^1, s^2, ..., s^p$ be elements of an arbitrary vector space. A vector function $F(s^1, s^2, ..., s^p)$ will be called additively separable with respect to s^1, if there exist vector function $F^1(s^1)$ independent of $s^2, s^3 ..., s^p$ and $F^2(s^2, s^3 ..., s^p)$ independent of s^1 such that $F(s^1, s^2, ..., s^p) = F^1(s^1) + F^2(s^2, s^3, ..., s^p)$.

3. Mond-Weir type higher order mini-max mixed integer programming

As in Balas [4] and Gulati et al. [11], we constrain some of the components of the vector variables $x \in R^n$ and $y \in R^m$ to belong to arbitrary set of integers $U \subset R^{n_1}$ and $V \subset R^{m_1}$ respectively, where $0 \leq n_1 \leq n$ and $0 \leq m_1 \leq m$. Therefore we write $(x, y) = (x^1, x^2, y^1, y^2)$, where $x^1 = (x_1, x_2, ..., x_{n_1}) \in U$ and $y^1 = (y_1, y_2, ..., y_{m_1}) \in V$. x^2 and y^2 being the remaining components of x and y respectively.

We consider the following pair of non differentiable higher order minimax mixed integer symmetric dual.

- **Primal(HMNSP)**:

$$Max_{x^1} Min_{x^2,y,w,p} H(x, y, w, p) = (H_i(x, y, w_i, p_i), i = 1, 2, ..., r),$$

where

$$\begin{aligned} H_i(x, y, w_i, p_i) = \quad & f_i(x, y) + ((x^2)^T B_i x^2)^{\frac{1}{2}} - (y^2)^T C_i w_i \\ & + h_i(x, y, p_i) - p_i^T \nabla_{p_i} h_i(x, y, p_i) \end{aligned}$$

Subject to

$$\sum_{i=1}^{r} \lambda_i [\nabla_{y^2} f_i(x, y) - C_i w_i + \nabla_{p_i} h_i(x, y, p_i)] \leq 0, \tag{3.1}$$

$$(y^2)^T \sum_{i=1}^{r} \lambda_i [\nabla_{y^2} f_i(x, y) - C_i w_i + \nabla_{p_i} h_i(x, y, p_i)] \geq 0, \tag{3.2}$$

$$w_i^T C_i w_i \leq 1, \quad i = 1, 2, .., r; \tag{3.3}$$

$$x^1 \in U, \ y^1 \in V, \ x^2 \geq 0, \tag{3.4}$$

$$\lambda > 0, \ \sum_{i=1}^{r} \lambda_i = 1. \tag{3.5}$$

- **Dual(HMNSD)**:

$$Min_{v^1} Max_{u,v^2,z,q} G(u, v, z, q) = (G_i(u, v, z_i, q_i), i = 1, 2, ..., r),$$

where

$$G_i(u,v,z_i,q_i) = \begin{array}{l} f_i(u,v) - ((v^2)^T C_i v^2)^{\frac{1}{2}} + (u^2)^T B_i z_i \\ +g_i(u,v,q_i) - q_i^T \nabla_{q_i} g_i(u,v,q_i) \end{array}$$

Subject to

$$\sum_{i=1}^{r} \lambda_i [\nabla_{u^2} f_i(u,v) + B_i z_i + \nabla_{q_i} g_i(u,v,q_i)] \geq 0, \tag{3.6}$$

$$(u^2)^T \sum_{i=1}^{r} \lambda_i [\nabla_{u^2} f_i(u,v) + B_i z_i + \nabla_{q_i} q_i(u,v,q_i)] \leq 0, \tag{3.7}$$

$$z_i^T B_i z_i \leq 1, \ i = 1,2,..,r; \tag{3.8}$$

$$u^1 \in U, \ v^1 \in V, \ v^2 \geq 0, \tag{3.9}$$

$$\lambda > 0, \ \sum_{i=1}^{r} \lambda_i = 1, \tag{3.10}$$

where $p_i \in R^{m-m_1}$, $q_i \in R^{n-n_1}$, $w_i \in R^{m-m_1}$, $z_i \in R^{n-n_1}$, $i = 1,2,..,r$. B_i and C_i are positive semi definite matrixes of order n and m respectively. Also $w = (w_1, w_2, ..., w_r)$, $z = (z_1, z_2, ..., z_r)$, $p = (p_1, p_2, ..., p_r)$, $q = (q_1, q_2, ..., q_r)$.

For the following theorems assume that $r \in R$, $\rho \in R$. ϕ_1 and ϕ_2 are a real valued function defined on $R^{n-n_1} \times R^{n-n_1} \times R^{n-n_1+1}$ and $R^{m-m_1} \times R^{m-m_1} \times R^{m-m_1+1}$ respectively with $\phi_1(x^2, u^2, (0,r)) \geq 0$, $\phi_2(v^2, y^2, (0,r)) \geq 0$. b_1 and b_2 are non negative real valued function defined on $R^n \times R^n$ and $R^m \times R^m$. $\psi_1, \psi_2 : R \times R$, satisfying $\psi_i(u) \leq 0 \Rightarrow u \leq 0$, $i = 1,2$ and $\psi_i(-u) = -\psi_i(u)$. $\alpha_1 : R^n \times R^n \to R$ and $\alpha_2 : R^n \times R^n \to R$.

Theorem 3.1. ***(Weak Duality)*** *Let (x, y, w, λ, p) be feasible solution for (HMNSP) and (u, v, z, λ, q) be feasible solution for (HMNSD). Assume that the following conditions are satisfied:*

(1) $f_i(x,y)$, $h_i(x,y,p_i)$ and $g_i(x,y,q_i)$ be additively separable with respect to x^1 or y^1 (say x^1) i.e.

$$f_i(x,y) = f_{i1}(x^1) + f_{i2}(x^2, y),$$
$$h_i(x,y,p_i) = h_{i1}(x^1) + h_{i2}(x^2, y, p_i),$$
$$g_i(x,y,q_i) = g_{i1}(x^1) + g_{i2}(x^2, y, q_i).$$

(2) $f_i(x,y), h_i(x,y,p_i)$ and $g_i(x,y,q_i)$ be twice differentiable in x^2 and y^2.

(3) $\sum_{i=1}^{r} \lambda_i [f_i(x,y) + (x^2)^T B_i z_i]$ be higher order (ϕ, α, ρ)-univex at x^2 for every (x^1, y, z) and $\sum_{i=1}^{r} \lambda_i [f_i(x,y) - (y^2)^T C_i w_i]$ be higher order (ϕ, α, ρ)-unicave at y^2 for every (x, y^1, w).

(4) $\phi_1(x^2, u^2; (\xi, \rho)) + (u^2)^T \xi \geq 0$, where

$$\xi = \sum_{i=1}^{r} \lambda_i [\nabla_{u^2} f_{i2}(u^2, v) + B_i z_i + \nabla_{q_i} g_{i2}(u^2, v, q_i)].$$

(5) $\phi_2(v^2, y^2, (\zeta, \rho)) + (y^2)^T \zeta \leq 0$, where

$$\zeta = \sum_{i=1}^{r} \lambda_i [\nabla_{y^2} f_{i2}(x^2, y) - C_i w_i + \nabla_{p_i} h_{i2}(x^2, y, p_i)].$$

Then $Inf[H(x, y, z, p)] \geq Sup[G(u, v, w, q)]$.

Proof. Let

$$\begin{aligned} s &= max_{x^1} min_{x^2,y,w,p}\{f_i(x, y) + ((x^2)^T B_i x^2)^{\frac{1}{2}} - (y^2)^T C_i w_i \\ &\quad + h_i(x, y, p_i - p_i^T \nabla_{p_i} h_i(x, y, p_i), i = 1, 2, ..r; (x, y, w, p) \in S\}, \\ t &= min_{v^1} max_{u,v^2,z,q}\{f_i(u, v) - ((v^2)^T C_i v^2)^{\frac{1}{2}} + (u^2)^T B_i z_i \\ &\quad + g_i(u, v, q_i - q_i^T \nabla_{q_i} g_i(u, v, q_i), i = 1, 2, ..r; (u, v, z, q) \in T\}, \end{aligned}$$

where S and T are feasible region of Primal and Dual respectively.

Since, $h_i(x, y, p_i)$ and $g_i(x, y, q_i)$ are additively separable with respect to x^1 or y^1 (say with respect to x^1) from definition 2.9, it follows that

$$\begin{aligned} f_i(x, y) &= f_{i1}(x^1) + f_{i2}(x^2, y), \\ h_i(x, y, p_i) &= h_{i1}(x^1) + h_{i2}(x^2, y, p_i), \\ g_i(x, y, q_i) &= g_{i1}(x^1) + g_{i2}(x^2, y, q_i). \end{aligned}$$

Therefore

$$\begin{aligned} \nabla_{y^2} f_i(x, y) &= \nabla_{y^2} f_{i2}(x^2, y), \\ \nabla_{p_i} h_i(x, y, p_i) &= \nabla_{p_i} h_{i2}(x^2, y, p_i), \\ \nabla_{q_i} g_i(x, y, q_i) &= \nabla_{q_i} g_{i2}(x^2, y, q_i). \end{aligned}$$

So s can be written as

$$\begin{aligned} s &= max_{x^1} min_{x^2,y,w,p}\{f_{i1}(x^1) + h_{i1}(x^1) + f_{i2}(x^2, y) + ((x^2)^T B_i x^2)^{\frac{1}{2}} \\ &-(y^2)^T C_i w_i + h_{i2}(x^2, y, p_i - p_i^T \nabla_{p_i} h_{i2}(x^2, y, p_i), i = 1, 2, ..r; (x, y, w, p) \in S\} \end{aligned}$$

Subject to

$$\begin{aligned} &\textstyle\sum_{i=1}^r \lambda_i[\nabla_{y^2} f_i(x, y) - C_i w_i + \nabla_{p_i} h_i(x, y, p_i)] \leq 0, \\ &(y^2)^T \textstyle\sum_{i=1}^r \lambda_i[\nabla_{y^2} f_i(x, y) - C_i w_i + \nabla_{p_i} h_i(x, y, p_i)] \geq 0, \\ &w_i^T C_i w_i \leq 1, i = 1, 2, .., r; \\ &x^1 \in U, y^1 \in V, x^2 \geq 0, \lambda > 0, \textstyle\sum_{i=1}^r \lambda_i = 1. \end{aligned}$$

or

$$s = max_{x^1} min_{y^1}(f_{i1}(x^1) + h_{i1}(x^1) + \varphi_i(y^1), i = 1, 2, ..., r; x^1 \in U, y^1 \in V),$$

where (**HMNSP0**):

$$\begin{aligned} \varphi_i(y^1) \quad &= min_{x^2,y^2,w,p} H_i(x^2, y, w, p) \\ &= \{f_{i2}(x^2, y) + ((x^2)^T B_i x^2)^{\frac{1}{2}} - (y^2)^T C_i w_i + h_{i2}(x^2, y, p_i) \\ &\quad - p_i^T \nabla_{p_i} h_{i2}(x^2, y, p_i)\} \end{aligned}$$

Subject to

$$\sum_{i=1}^{r} \lambda_i[\nabla_{y^2} f_{i2}(x^2, y) - C_i w_i + \nabla_{p_i} h_{i2}(x^2, y, p_i)] \leq 0, \tag{3.11}$$

$$(y^2)^T \sum_{i=1}^{r} \lambda_i[\nabla_{y^2} f_{i2}(x^2, y) - C_i w_i + \nabla_{p_i} h_{i2}(x^2, y, p_i)] \geq 0, \tag{3.12}$$

$$w_i^T C_i w_i \leq 1, \;\; i = 1, 2, .., r; \tag{3.13}$$

$$x^1 \in U, \;\; y^1 \in V, \;\; x^2 \geq 0, \tag{3.14}$$

$$\lambda > 0, \;\; \sum_{i=1}^{r} \lambda_i = 1. \tag{3.15}$$

Similarly,

$$t = min_{v^1} max_{u^1} \{f_{i1}(u^1) + g_{i1}(u^1) + \theta_i(v^1); u^1 \in U, v^1 \in V\},$$

where (**HMNSD0**):

$$\begin{aligned} \theta_i(v^1) \;\; &= max_{u^2,v^2,z,q_i} G_i(u^2, v, z, q_i) \\ &= \{f_{i2}(u^2, v) + ((u^2)^T B_i z_i) - ((v^2)^T C_i v^2)^{\frac{1}{2}} + g_{i2}(u^2, v, q_i) \\ &\quad - q_i^T \nabla_{q_i} g_{i2}(u^2, v, q_i)\} \end{aligned}$$

Subject to

$$\sum_{i=1}^{r} \lambda_i[\nabla_{u^2} f_{i2}(u^2, v) + B_i z_i + \nabla_{q_i} g_{i2}(u^2, v, q_{i2})] \geq 0, \tag{3.16}$$

$$(u^2)^T \sum_{i=1}^{r} \lambda_i[\nabla_{u^2} f_{i2}(u^2, v) + B_i z_i + \nabla_{q_i} q_{i2}(u^2, v, q_i)] \leq 0, \tag{3.17}$$

$$z_i^T B_i z_i \leq 1, i = 1, 2, .., r; \tag{3.18}$$

$$u^1 \in U, v^1 \in V, v^2 \geq 0, \tag{3.19}$$

$$\lambda > 0, \sum_{i=1}^{r} \lambda_i = 1. \tag{3.20}$$

In order to prove the theorem, it is sufficient to show that $\varphi_i(y^1) \geq \theta_i(v^1)$. From the hypothesis (4), we get

$$\phi_1(x^2, u^2; \alpha_1(x^2, u^2)(\textstyle\sum_{i=1}^{r} \lambda_i[\nabla_{u^2} f_{i2}(u^2, v) + B_i z_i + \nabla_{q_i} g_{i2}(u^2, v, q_i)], \rho))$$
$$+(u^2)^T \textstyle\sum_{i=1}^{r} \lambda_i[\nabla_{u^2} f_{i2}(u^2, v) + B_i z_i + \nabla_{q_i} g_{i2}(u^2, v, q_i)] \geq 0.$$

which in view of (3.17) gives,

$$\phi_1(x^2, u^2; \alpha_1(x^2, u^2)(\sum_{i=1}^{r} \lambda_i[\nabla_{u^2} f_{i2}(u^2, v) + B_i z_i + \nabla_{q_i} g_{i2}(u^2, v, q_i)], \rho)) \geq 0. \tag{3.21}$$

Since $\sum_{i=1}^{r} \lambda_i[f_{i2}(u^2, v) + (u^2)^T B_i z_i]$ is higher order (ϕ, α, ρ)-univex at u^2 with respect to $g_{i2}(u^2, v, q_i)$, $i = 1, 2, ..r$; then for

$$b_1 : R^{n-n_1} \times R^{n-n_1} \to R_+ \text{ and } \psi_1 : R \to R$$

satisfying the property $\psi_1(u) \leq 0 \Rightarrow u \leq 0,\ \psi_1(-u) = -\psi_1(u)$ and using (3.21), we get

$$\begin{aligned}
& b_1(x^2,u^2)\psi_1\{\textstyle\sum_{i=1}^r \lambda_i[f_{i2}(x^2,v) + (x^2)^T B_i z_i - f_{i2}(u^2,v) - (u^2)^T B_i z_i \\
& -g_{i2}(u^2,v,q_i)] + q_i^T \nabla_{q_i} g_{i2}(u^2,v,q_i)\} \\
& \geq \phi_1(x^2,u^2;\alpha_1(x^2,y)(\textstyle\sum_{i=1}^r \lambda_i[\nabla_{u^2} f_{i2}(u^2,v) + B_i z_i + \nabla_{q_i} g_i(u^2,v,q_i)],\rho)) \\
& \geq 0 \\
\Rightarrow\ & \textstyle\sum_{i=1}^r \lambda_i[f_{i2}(x^2,v) + (x^2)^T B_i z_i - f_{i2}(u^2,v) - (u^2)^T B_i z_i \\
& -g_{i2}(u^2,v,q_i) + q_i^T \nabla_{q_i} g_{i2}(u^2,v,q_i)] \geq 0.
\end{aligned} \tag{3.22}$$

Again from the hypothesis (5), we have

$$\begin{aligned}
& \phi_2(v^2,y^2;\alpha_2(v^2,y^2)(\textstyle\sum_{i=1}^r \lambda_i[\nabla_{y^2} f_{i2}(x^2,y) - C_i w_i + \nabla_{p_i} h_{i2}(x^2,y,p_i)],\rho)) \\
& +(y^2)^T \textstyle\sum_{i=1}^r \lambda_i[\nabla_{y^2} f_{i2}(x^2,y) - C_i w_i + \nabla_{p_i} h_{i2}(x^2,y,p_i)] \leq 0,
\end{aligned}$$

which in view of (3.12)

$$\phi_2(v^2,y^2;\alpha_2(v^2,y^2)(\sum_{i=1}^r \lambda_i[\nabla_{y^2} f_{i2}(x^2,y) - C_i w_i + \nabla_{p_i} h_{i2}(x^2,y,p_i)],\rho)) \leq 0. \tag{3.23}$$

Since $\sum_{i=1}^r \lambda_i[f_i(x^2,y) - (y^2)^T C_i w_i]$ is higher order (ϕ,α,ρ)-unicave at y^2 for every (x,y^1,w) with respect to $h_i(x,y^2,p_i),\ i = 1,2,...,r$; then for

$$b_2 : Y \times Y \to R_+,\ \psi_2 : R \to R,$$

we have

$$\begin{aligned}
& b_2(v^2,y^2)\psi_2\{\textstyle\sum_{i=1}^r \lambda_i[f_{i2}(x^2,v) - (v^2)^T C_i w_i - f_{i2}(x^2,y) + (y^2)^T C_i w_i \\
& -h_{i2}(x^2,y,p_i)] + p_i^T \nabla_{p_i} h_{i2}(x^2,y,p_i)\} \\
\leq\ & \phi_2(v^2,y^2;\alpha_2(v^2,y^2)(\textstyle\sum_{i=1}^r \lambda_i[\nabla_{y^2} f_{i2}(x^2,y) - C_i w_i + \nabla_{p_i} h_i(x^2,y,p_i)],\rho)) \leq 0,
\end{aligned}$$

which in view of (3.23) with the property of b_2 and ψ_2, gives

$$\begin{aligned}
& \textstyle\sum_{i=1}^r \lambda_i[f_{i2}(x^2,v) - (v^2)^T C_i w_i - f_{i2}(x^2,y) + (y^2)^T C_i w_i \\
& -h_{i2}(x^2,y,p_i) + p_i^T \nabla_{p_i} h_{i2}(x^2,y,p_i)] \leq 0.
\end{aligned} \tag{3.24}$$

Subtracting (3.24) from (3.22), we obtain

$$\begin{aligned}
& \textstyle\sum_{i=1}^r \lambda_i[(x^2)^T B_i z_i + (v^2)^T C_i w_i] \\
\geq\ & \textstyle\sum_{i=1}^r \lambda_i[f_{i2}(u^2,v) + (u^2)^T B_i z_i + g_{i2}(u^2,v,q_i) - q_i^T \nabla_{q_i} g_{i2}(u^2,v,q_i)] \\
& - \textstyle\sum_{i=1}^r \lambda_i[f_{i2}(x^2,y) - (y^2)^T C_i w_i + h_{i2}(x^2,y,p_i) - p_i^T \nabla_{p_i} h_{i2}(x^2,y,p_i)].
\end{aligned} \tag{3.25}$$

Applying Schwartz inequality, (3.13) and (3.18)in (3.25)

$$\begin{aligned}
& \textstyle\sum_{i=1}^r \lambda_i[f_{i2}(x^2,y) + ((x^2)^T B_i x^2)^{\frac{1}{2}} - (y^2)^T C_i w_i \\
& +h_{i2}(x^2,y,p_i) - p_i^T \nabla_{p_i} h_{i2}(x^2,y,p_i)] \\
\geq\ & \textstyle\sum_{i=1}^r \lambda_i[f_{i2}(u^2,v) - ((v^2)^T C_i v^2)^{\frac{1}{2}} + (u^2)^T B_i z_i \\
& +g_{i2}(u^2,v,q_i) - q_i^T \nabla_{q_i} g_{i2}(u^2,v,q_i)].
\end{aligned} \tag{3.26}$$

So $H_i(x^2,y,w,p) \geq G_i(u,v^2,z,q) \Rightarrow \varphi(y^1) \geq \theta(v^1)$. Hence the results holds. □

Theorem 3.2. ***(Strong Duality)*** *Let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ be weak efficient solution of (HMNSP). $f_i : R^n \times R^m \to R$ is twice differentiable at $(\bar{x},\bar{y})$, $h_i : R^n \times R^m \times R^{m-m_1} \to R$ and $g_i : R^n \times R^m \times R^{n-n_1} \to R$ are twice differentiable function at $(\bar{x},\bar{y},\bar{p})$,and $(\bar{x},\bar{y},\bar{q})$,respectively. Assume that all the hypotheses of theorem 3.1 are satisfied with addition to the following conditions:*

a) for all $i \in \{1, 2, ..., r\}$,

$$h_{i2}(\bar{x}^2,\bar{y},0)=0,\ \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},0)=0,\ \nabla_{y^2}h_{i2}(\bar{x}^2,\bar{y},0)=0,$$
$$g_{i2}(\bar{x}^2,\bar{y},0)=0,\ \nabla_{x^2}h_{i2}(\bar{x}^2,\bar{y},0)=\nabla_{q_i}g_{i2}(\bar{x}^2,\bar{y},0),$$

b) for all $i \in \{1, 2, ..., r\}$, the Hessian matrix $\nabla_{p_ip_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i)$ is positive definite or negative definite,

c) the set of vectors $\{\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i),\ i = 1, 2, ..., r\}$ is linearly independent,

d) for some $\beta \in R^r_+$ and $p_i \in R^{m-m_1},\ p_i \neq 0,\ i = 1, 2, ..., r$ implies that

$$\sum_{i=1}^{r}\beta_i\bar{p}_i[\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i)] \neq 0.$$

Then

(i) $\bar{p}_i = 0,\ i = 1, 2, ..., r$,

(ii) there exist $\bar{z} \in R^{n-n_1}$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{q}=0)$ is efficient solution for (HMNSD) and two objective values are equal.

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ is efficient solution for (HMNSP0), from the proof of theorem 3.1, it is clear that $(\bar{x}^2,\bar{y},\bar{\lambda},\bar{w},\bar{p})$ is efficient solution for (HMNSP0). So by Fritz-John optimality conditions stated by Mangasarian [15], there exist $\beta \in R^r,\ \theta \in R^{m-m_1},\ \gamma \in R,\ \delta \in R^r,\ \nu \in R^r,\ \xi \in R^{n-n_1}$ such that

$$\begin{aligned}&\sum_{i=1}^{r}\beta_i[\nabla_{x^2}f_{i2}(\bar{x}^2,\bar{y}) + B_i\bar{z}_i + \nabla_{x^2}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i)]\\&+\sum_{i=1}^{r}\lambda_i[(\nabla_{y^2x^2}f_{i2}(\bar{x}^2,\bar{y}))^T(\theta-\gamma\bar{y})]\\&+\sum_{i=1}^{r}(\nabla_{p_ix^2}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i))^T(\lambda_i\theta-\beta_ip_i-\lambda_i\gamma\bar{y})-\xi=0,\end{aligned}\tag{3.27}$$

$$\begin{aligned}&\sum_{i=1}^{r}\beta_i[\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{y^2}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i)]\\&+\sum_{i=1}^{r}\lambda_i[(\nabla_{y^2y^2}f_{i2}(\bar{x}^2,\bar{y}))^T(\theta-\gamma\bar{y})]\\&+(\nabla_{p_iy^2}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i))^T(\lambda_i\theta-\beta_i\bar{p}_i-\bar{\lambda}_i\gamma\bar{y})\\&-\gamma\sum_{i=1}^{r}\lambda_i[\nabla\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i)]=0,\end{aligned}\tag{3.28}$$

$$(\lambda_i\theta-\beta_i\bar{p}_i-\bar{\lambda}_i\gamma\bar{y})^T(\nabla_{p_ip_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i))=0,\tag{3.29}$$

$$\begin{aligned}&(\theta-\gamma\bar{y})^T(\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p}_i))-\delta_i=0,\\&i=1,2,..,r;\end{aligned}\tag{3.30}$$

$$\theta^T(\sum_{i=1}^{r}\lambda_i[\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p_i})]) = 0, \tag{3.31}$$

$$\gamma\bar{y}^T(\sum_{i=1}^{r}\lambda_i[\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p_i})]) = 0, \tag{3.32}$$

$$\beta_iC_i\bar{y} + (\theta-\gamma\bar{y})^T\lambda_iC_i = 2\nu_iC_i\bar{w}_i, \tag{3.33}$$

$$(\bar{x}^2)^TB_i\bar{w}_i = ((\bar{x}^2)^TB_i\bar{x}^2)^{\frac{1}{2}}, \tag{3.34}$$

$$\bar{z_i}^TC_i\bar{z_i} \leq 1, \tag{3.35}$$

$$\nu_i(\bar{w_i}^TC_i\bar{w}_i - 1) = 0, \tag{3.36}$$

$$\delta^T\lambda = 0, \tag{3.37}$$

$$\bar{x}^2\xi = 0, \tag{3.38}$$

$$(\beta,\theta,\gamma,\nu,\delta,\xi) \geq 0, \tag{3.39}$$

$$(\beta,\theta,\gamma,\nu,\delta,\xi) \neq 0. \tag{3.40}$$

From the hypothesis (b), (3.29) yields

$$\bar{\lambda}_i\theta - \beta_i\bar{p_i} - \lambda_i\gamma\bar{y} = 0 \Rightarrow \lambda_i(\theta-\gamma\bar{y}) = \beta_i\bar{p_i}. \tag{3.41}$$

We claim that $\beta = (\beta_1,\beta_2,....,\beta_r) \neq 0$.

Otherwise, if $\beta = 0$, then (3.41) gives $\theta = \gamma\bar{y}$ and (3.28) yields

$$\gamma\sum_{i=1}^{r}\lambda_i[\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p_i})] = 0.$$

Since $\{\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p_i}), i = 1,2,..,r\}$ is linearly independent and $\lambda > 0$, we have $\gamma = 0$ and so $\theta = 0$. Hence (3.27) gives $\xi = 0$ and (3.30) gives $\delta = 0$. Also (3.33) implies $\nu_i = 0$. Thus $(\beta,\theta,\gamma,\nu,\delta,\xi) = 0$, this contradicts (3.40). So, $\beta \neq 0$.

Since $\lambda_i > 0 \Rightarrow \beta_i > 0,\ i = 1,2,..,r$. Subtracting (3.32) from (3.31), we get

$$(\theta-\gamma\bar{y})^T(\sum_{i=1}^{r}\lambda_i[\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p_i})]) = 0. \tag{3.42}$$

Using (3.41) in (3.42), we get

$$\sum_{i=1}^{r}\beta_i\bar{p_i}[\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p_i}] = 0.$$

By hypothesis (d), we get $\bar{p_i} = 0, i = 1,2,...r$; and from (3.41), we get

$$\theta = \gamma\bar{y}. \tag{3.43}$$

Now using (3.43), $\bar{p_i} = 0$ and the hypothesis (a) in (3.27) and (3.28), we get

$$\sum_{i=1}^{r}\beta_i[\nabla_{x^2}f_{i2}(\bar{x}^2,\bar{y}) + B_i\bar{z_i} + \nabla_{x^2}h_{i2}(\bar{x}^2,\bar{y},\bar{p_i})] = 0 \tag{3.44}$$

and

$$\sum_{i=1}^{r}(\beta_i-\gamma\lambda_i)[\nabla_{y^2}f_{i2}(\bar{x}^2,\bar{y}) - C_i\bar{w}_i + \nabla_{p_i}h_{i2}(\bar{x}^2,\bar{y},\bar{p_i})] = 0. \tag{3.45}$$

Similarly from condition (c), (3.45) gives $\beta = \gamma\lambda$. Thus, from (3.44) and $\nu > 0$, it holds

$$\sum_{i=1}^{r} \lambda_i [\nabla_{x^2} f_{i2}(\bar{x}^2, \bar{y}) + B_i \bar{z}_i + \nabla_{x^2} h_{i2}(\bar{x}^2, \bar{y}, \bar{p}_i)] = 0,$$

and from condition (a), we have

$$\sum_{i=1}^{r} \lambda_i [\nabla_{x^2} f_{i2}(\bar{x}^2, \bar{y}) + B_i \bar{z}_i + \nabla_{q_i} g_{i2}(\bar{x}^2, \bar{y}, \bar{q}_i)] = 0.$$

Taking $B_i = C_i$, we get $(\bar{x}, \bar{y}, \bar{z}, \bar{p}_i = 0)$ satisfied all the constraint from (3.16) to (3.20). So it is feasible for dual.

Now let $\frac{2\nu_i}{\beta_i} = a$, then $a \geq 0$.

So from (3.33) and (3.43), we have $\beta_i C_i \bar{y} = 2\nu_i C_i \bar{w}_i \Rightarrow C_i \bar{y} = a C_i \bar{w}_i$. This is a condition for equality in Schwartz Inequality. Therefore

$$(\bar{y}^2)^T C_i \bar{w}_i = ((\bar{y}^2)^T C_i \bar{y}^2)^{\frac{1}{2}} (\bar{w}_i{}^T C_i \bar{w}_i)^{\frac{1}{2}}.$$

In case $\nu_i \geq 0$, (3.46) gives $\bar{w}_i{}^T C_i \bar{w}_i = 1$. So $(\bar{y}^2)^T C_i \bar{w}_i = ((\bar{y}^2)^T C_i \bar{y}^2)^{\frac{1}{2}}$. Hence

$$\begin{aligned} \phi_i(\bar{y}^1) &= f_{i2}(\bar{x}^2, \bar{y}) + ((\bar{x}^2)^T B_i \bar{x}^2)^{\frac{1}{2}} - (\bar{y}^2)^T C_i \bar{w}_i + h_{i2}(\bar{x}^2, \bar{y}, \bar{p} = 0) \\ &\quad - p_i^T \nabla_{p_i} h_{i2}(\bar{x}^2, \bar{y}, \bar{p} = 0) \\ &= f_{i2}(\bar{x}^2, \bar{y}) + (\bar{x}^2)^T B_i \bar{z}_i - ((\bar{y}^2)^T C_i \bar{y}^2)^{\frac{1}{2}} + g_{i2}(\bar{x}^2, \bar{y}, \bar{q} = 0) \\ &\quad - q_i^T \nabla_{q_i} g_{i2}(\bar{x}^2, \bar{y}, \bar{q} = 0) \\ &= \psi_i(\bar{y}^1) \text{ for each } i. \end{aligned}$$

So the optimal values of both (HMNSP0)and (HMNSD0) are same.

Now we claim that $(\bar{x}^2, \bar{y}, \bar{z}, \bar{q} = 0)$ is an efficient solution of (HMNSD0). If this would not be the case, then there would exist a feasible solution $(\bar{u}^2, \bar{v}, \bar{z}, \bar{q} = 0)$ of (HMNSD0) such that

$$\begin{aligned} & G_i(\bar{x}^2, \bar{y}, \bar{z}, \bar{q} = 0) \leq G_i(\bar{u}^2, \bar{v}, \bar{z}, \bar{q} = 0) \\ \Rightarrow \quad & H_i(\bar{x}^2, \bar{y}, \bar{z}, \bar{q} = 0) \leq G_i(\bar{u}^2, \bar{v}, \bar{z}, \bar{q} = 0) \\ \Rightarrow \quad & \phi(y^1) \leq \psi(v^1). \end{aligned}$$

This is a contradiction to theorem 3.1. Hence $(\bar{x}^2, \bar{y}, \bar{z}, \bar{q} = 0)$ is an efficient solution of (HMNSD0). □

4. Self Dual

A mathematical programming problem is said to be self dual if it is formally identical with its dual i.e. the dual is recast in the form of the primal and new program constructed is same as the primal problem.

We now prove the following self duality theorem for the primal and dual.

Theorem 4.1. ***(Self Duality):***
Assume $m = n, U = V, B_i = C_i, p_i = q_i, z_i = w_i$. If f_i, h_i and g_i are skew symmetric with respect to x and y with $h_i(\bar{x}, \bar{y}, \bar{p}_i) = g_i(\bar{x}, \bar{y}, \bar{q}_i = 0)$, then (HMNSP) is a self

dual. Furthermore, if (HMNSP) and (HMNSD) are dual programs and $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ *is an efficient solution for (HMNSP), then* $\bar{p}=0$ *and* $(\bar{y},\bar{x},\bar{\lambda},\bar{w},\bar{p})$ *is an efficient solution for (HMNSD). Also the values of two objective functions are equal to zero.*

Proof. Rewriting the dual as maxi-min problem, we have

$$Max_{v^1} Min_{u,v^2,z,q} G^*(u,v,z,q) = \begin{array}{l}\{-f_i(u,v)+(v^TC_iv)^{\frac{1}{2}}-u^TB_iz_i-g_i(u,v,q_i)\\ +q_i^T\nabla_{q_i}g_i(u,v,q_i),\ i=1,..,r\}\end{array}$$

Subject to

$$\begin{array}{l}\sum_{i=1}^r \lambda_i[-\nabla_{u^2}f_i(u,v)-B_iz_i-\nabla_{q_i}g_i(u,v,q_i)]\le 0,\\ (u^2)^T\sum_{i=1}^r \lambda_i[-\nabla_{u^2}f_i(u,v)-B_iz_i-\nabla_{q_i}g_i(u,v,q_i)]\ge 0,\\ z_i^TB_iz_i\le 1,\ i=1,2,...,r;\\ u^1\in U,\ v^1\in V,\ v^2\ge 0,\\ \lambda>0,\ \sum_{i=1}^r\lambda_i=1.\end{array}$$

Since $f_i(u,v)$ and $g_i(u,v,q_i)$ are skew symmetric with respect to u and v, we have

$$\begin{array}{ll}f_i(u,v)=-f_i(v,u), & g_i(u,v,q_i)=-g_i(v,u,q_i),\\ \nabla_u f_i(u,v)=-\nabla_u f_i(v,u), & \nabla_{q_i}g_i(u,v,q_i)=-\nabla_{q_i}g_i(v,u,q_i).\end{array}$$

Hence the above dual program becomes,

$$Max_{v^1} Min_{u,v^2,z,q} G^*(u,v,z,q) = \begin{array}{l}\{f_i(v,u)+(v^TC_iv)^{\frac{1}{2}}-u^TB_iz_i+g_i(v,u,q_i)\\ -q_i^T\nabla_{q_i}g_i(v,u,q_i),\ i=1,..,r\}\end{array}$$

Subject to

$$\begin{array}{l}\sum_{i=1}^r \lambda_i[\nabla_{u^2}f_i(v,u)-B_iz_i+\nabla_{q_i}g_i(v,u,q_i)]\le 0,\\ (u^2)^T\sum_{i=1}^r \lambda_i[\nabla_{u^2}f_i(v,u)-B_iz_i+\nabla_{q_i}g_i(v,u,q_i)]\ge 0,\\ z_i^TB_iz_i\le 1,\ i=1,2,...,r;\\ u^1\in U,\ v^1\in V,\ v^2\ge 0,\\ \lambda>0,\ \sum_{i=1}^r\lambda_i=1.\end{array}$$

Again since $B_i=C_i$, rewriting B_i as C_i and C_i as B_i and replacing v by x, u by y, z by w in the above problem it becomes the primal. Hence the (HMNSP) is self dual. Thus $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p_i})$ is an efficient solution for (HMNSP) implies $(\bar{y},\bar{x},\bar{\lambda},\bar{w},\bar{p_i})$ is efficient solution for (HMNSD). By similar argument $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{q_i})$ is efficient for (HMNSP) implies $(\bar{y},\bar{x},\bar{\lambda},\bar{w},\bar{p_i})$ is efficient for (HMNSD). If (HMNSP) and (HMNSD) are dual programs and $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p_i})$ is jointly efficient solution, the by theorem 3.2, there exist $z\in R^n$ such that

$$H_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0)=G_i(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{q}=0). \tag{4.1}$$

Now

$$H_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0)=f_i(\bar{x},\bar{y})+((\bar{x}^2)^TB_i\bar{x}^2)^{\frac{1}{2}}-(\bar{y}^2)^TC_i\bar{w}_i+h_i(\bar{x},\bar{y},0). \tag{4.2}$$

Using Schwartz Inequality and (3.3) in (4.2), we get

$$H_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0) \geq f_i(\bar{x},\bar{y}) + ((\bar{x}^2)^T B_i \bar{x}^2)^{\frac{1}{2}} - ((\bar{y}^2)^T C_i \bar{y}^2)^{\frac{1}{2}} + h_i(\bar{x},\bar{y},0). \quad (4.3)$$

Similarly

$$G_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0) = f_i(\bar{x},\bar{y}) + (\bar{x}^2)^T B_i \bar{z}_i - ((\bar{y}^2)^T C_i \bar{y}^2)^{\frac{1}{2}} + g_i(\bar{x},\bar{y},0)$$

or by hypothesis of this theorem,

$$G_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0) = f_i(\bar{x},\bar{y}) + (\bar{x}^2)^T B_i \bar{z}_i - ((\bar{y}^2)^T C_i \bar{y}^2)^{\frac{1}{2}} + h_i(\bar{x},\bar{y},0). \quad (4.4)$$

Using Schwartz Inequality and (3.8) in (4.4), we get

$$G_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0) \leq f_i(\bar{x},\bar{y}) + ((\bar{x}^2)^T B_i \bar{x}^2)^{\frac{1}{2}} - ((\bar{y}^2)^T C_i \bar{y}^2)^{\frac{1}{2}} + h_i(\bar{x},\bar{y},0). \quad (4.5)$$

From (4.1), (4.3) and (4.5), we have

$$\begin{aligned} &H_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0) = G_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0) \\ &= f_i(\bar{x},\bar{y}) + ((\bar{x}^2)^T B_i \bar{x}^2)^{\frac{1}{2}} - ((\bar{y}^2)^T C_i \bar{y}^2)^{\frac{1}{2}} + h_i(\bar{x},\bar{y},0). \end{aligned} \quad (4.6)$$

Again $(\bar{y},\bar{x},\bar{w},\bar{p}=0)$ is also a joint efficient solution of (HMNSP) and (HMNSD). This implies

$$\begin{aligned} &H_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0) = G_i(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0) \\ &= f_i(\bar{y},\bar{x}) + ((\bar{y}^2)^T B_i \bar{y}^2)^{\frac{1}{2}} - ((\bar{x}^2)^T C_i \bar{x}^2)^{\frac{1}{2}} + h_i(\bar{y},\bar{x},0). \end{aligned} \quad (4.7)$$

Adding (4.6) and (4.7), we get

$$\begin{aligned} &2H(\bar{x},\bar{y},\bar{w},\bar{p}=0) = 2G(\bar{x},\bar{y},\bar{z},\bar{q}=0) \\ &= f_i(\bar{x},\bar{y}) + ((\bar{x}^2)^T B_i \bar{x}^2)^{\frac{1}{2}} - ((\bar{y}^2)^T C_i \bar{y}^2)^{\frac{1}{2}} + h_i(\bar{x},\bar{y},0) \\ &+ f_i(\bar{y},\bar{x}) + ((\bar{y}^2)^T B_i \bar{y}^2)^{\frac{1}{2}} - ((\bar{x}^2)^T C_i \bar{x}^2)^{\frac{1}{2}} + h_i(\bar{y},\bar{x},0). \end{aligned}$$

Now for $B_i = C_i$ and the skew symmetric of f_i and h_i, we obtain

$$H(\bar{x},\bar{y},\bar{w},\bar{p}=0) = G(\bar{x},\bar{y},\bar{z},\bar{q}=0).$$

□

5. Conclusion

In this paper, I presented a new generalized class of higher order (ϕ,α,ρ)-univex function with example. We formulated Mond-Weir type nondifferentiable higher order minimax mixed integer dual programs and symmetric duality theorems are established under higher order (ϕ,α,ρ)-univex function. The results of this paper studied under higher order (ϕ,α,ρ)-univex function is more general than the class of sub linear function with respect to third argument. The present work can be further generalized to fractional programming case.

References

[1] I. Ahmad, *Second order symmetric duality for nondifferentiable minimax mixed integer problem*, Southeast Asian Bull. Math., 29 (2005), 843-849.

[2] I. Ahmad and Z. Husain, *Nondifferentiable second order symmetric duality*, Asia-Pasific J. Oper. Res., 22 (2005), 19-31.

[3] I. Ahmad, Z. Husain and S. Sharma, *Higher order duality in nondifferentiable multiobjective programming*, Numer. Funct. Anal. Optim., 28 (2007), 989-1002.

[4] I. Ahmad, Z. Husain and S. Sharma, *Higher order duality in nondifferentiable minimax programming with generalized type 1 function*, J. Optim. Theory Appl., 141 (2009), 1-12.

[5] E. Balas, *Minimax and duality for linear and nonlinear mixed integer programming*, Integer and Nonlinear Programming, North Holland, Amsterdam, (1970), 385-417.

[6] A. Batatorescu, Preda and M. Beldiman, *Higher order symmetric multiobjective duality involving generalized* (F, ρ, γ, b)*-convexity*, Rev. Roumaine Math. Pure Appl., 52 (2007), 619-630.

[7] C.R. Bector, S. Chandra, S. Gupta and S.K. Suneja, *Univex sets, functions and univex nonlinear programming*, Lect. Notes in Eco. Math. System, Springer Verlag, Berlin, 405 (1994), 1-8.

[8] X. Chen, *Higher order symmetric duality in nondifferentiable multiobjective programming problem*, Math. Anal. Appl., 290 (2004), 423-435.

[9] G.B. Dantzing, E. Eisenberg and R.W. Cottle, *Symmetric dual Nonlinear programs*, , Pacific J. Math., 15 (1965), 809-812.

[10] G. Devi, *Symmetric duality for nonlinear programming problem involving-bonvex function*, European J. Oper. Res., 104 (1998), 615-621.

[11] W.S. Dorn, *A symmetric dual theorem for quadratic programs*, J. Oper. Res. Soc. Japan, 2 (1960), 93-97.

[12] T.R. Gulati and S.K. Gupta, *Higher order nondifferentiable symmetric duality with generalized F-convexity*, J. Math. Anal. Appl., 329 (2007), 229-237.

[13] T.R. Gulati, I. Husain and I. Ahamad, *Symmetric duality for nondifferentiable minimax mixed integer programming problems*, Optimization, 39 (1997), 69-84.

[14] M.A. Hanson, *On sufficiency of the Kuhn -Tucker condition*, J. Math. Anal. Appl., 80 (1981), 845-850.

[15] O.L. Mangasarian, *Second order and Higher order duality in nonlinear programming*, J. Math. Anal. Appl., 51 (1975), 607-620.

[16] S.K. Mishra and N.G. Rueda, *Higher order generalized invexity and duality in mathematical programming*, J. Math. Anal. Appl., 247 (2000), 173-182.

[17] S.K. Mishra and N.G. Rueda, *Higher order generalized invexity and duality in nondifferentiable mathematical programming*, J. Math. Anal. Appl., 272 (2002), 496-506.

[18] B. Mond and M. Schecther, *Non-differentiable symmetric duality*, Bull. Aust. Math. Soc., 53 (1996), 177-188.

[19] B. Mond and J. Zhang, *Higher order invexity and duality in mathematical programming*, Generalized convexity, Generalized Monotoncity, Recent Results, Edited by J.P.Crouzeix et al, Kluwer Academic Pub., Dordrecht, 1998, 357-372.

[20] D.B. Ojha, *Higher order duality for multiobjective programming involving* (ϕ, ρ)*-univexity*, World Applied Programming, 1 (2011), 155-162.

[21] G.K. Thakur and B.B. Priya, *Second order duality for nondifferentiable multiobjective programming involving* (ϕ, ρ)*-invexity*, Kathmandu University J. Sc., Engg. and Tech., 7 (2011), 92-104.

[22] T. Weir and B. Mond, *Symmetric and self duality in multiobjective programming*, Asia-Pacific J. Oper.Res., 5 (1991), 75-87.

[23] X.M. Yang, K.L. Teo and X.Q. Yand, *Higher order generalized convexity and duality in nondifferentiable multiobjective mathematical programming*, Edited by J.A.Eberhard, R.Hill, D.Ralph and B.M.Glover, Kluwer Academic Publ. Dordrecht, Appl. Optimization, 30 (1999), 101-116.

[24] J.Zhang, *Higher order convexity and duality in multiobjective programming problem, In. Progress in Optimization, Contribution from Australasia* ,(Eds: A.Eberhard, R.Hill, D.Ralph, B.M.Glover), Kluwer Academic Publisher, Dordrecht-Boston-London, Appl. Optimization, 30 (1999), 101-116.